Publ. Math. Debrecen
74/3-4 (2009), 433–480

Compactness of the moduli space of solutions of the Seiberg–Witten equations over higher dimensional compact Kähler manifolds

By HAJIME URAKAWA (Sendai)

Abstract. The formulation of the Seiberg–Witten equations over higher dimensional compact Kähler manifolds is given, and compactness of the moduli space of solutions of the Seiberg–Witten equations over it is shown.

1. Introduction

After the Seiberg–Witten theory in mathematical physics was initiated by [20], [21], [36] in 1994, many mathematical works related to it have given great influences on 4-dimensional topology and geometry (cf. [11], [27], [28], [29], [30], [13]). However, the Seiberg–Witten theory for the higher dimensional manifolds have not been so much studied (cf. [19], [26]), even though many works have been done for the Yang-Mills theory for higher dimensional manifolds (cf. [10], [17], [18], [23], [33], [35]).

In this paper, we formulate the Seiberg–Witten equations over an arbitrary compact Kähler manifold of complex dimension $n \geq 2$, and show the compactness theorem of the moduli space of solutions of the Seiberg–Witten equations (Theorem 11.1).

In the theory of the compactness theorem of the moduli space, the works

Mathematics Subject Classification: Primary: 53C07; Secondary: 53C27.
Key words and phrases: Seiberg–Witten equations, moduli space, Kähler manifold, regularity, compactness.
Supported by the Grant-in-Aid for the Scientific Research, (B), No. 16340044, Japan Society for the Promotion of Science.

of K. Uhlenbeck in [33] are very important for us (see also [35]). The procedure to show the compactness theorem in four-dimensional manifolds explained in [16] does not work any more in the higher dimensional manifolds. But, in the gauge theory, Uhlenbeck ([33]) showed weak compactness for the connections with uniformly L^p-bounded curvatures, and K. Wehrheim ([35]) showed strong compactness for weak Yang-Mills connections with uniformly L^p-bounded curvatures in any dimensional manifolds. The ideas of these papers are useful for the Seiberg–Witten theory in higher dimensional mainifolds.

The outline of this paper and the flow of our proof of compactness of the moduli space of solutions of the Seiberg–Witten equations are as follows. In Sections 2, 3, and 4, we first prepare Spin^c-structures, the Seiberg–Witten functional and the Seiberg–Witten equations over higher dimensional compact Kähler manifolds. In Sections 5, by using the Green kernel K_G, for all smooth section φ of a vector bundle, we will estimate the supremum of the pointwise norm of $\nabla\varphi$ by the L_p-norm of the rough Laplacian of φ for all $p > \dim M$ (cf. Theorem 5.2). In Section 6, we first will give the C^∞-regularity theorem for any solution of the Seiberg–Witten equations (cf. Theorem 6.1) which will be proved in the Appendix by using the L^p_ℓ-gauge fixing lemma (cf. Theorem 7.3) and the ellipticity of the Seiberg–Witten equations. Then, we will give a priori estimates of solutions (A, ψ) of the Seiberg–Witten equations, that is, the C^0 boundedness of F_A and ψ (cf. Theorem 6.3), and the C^1 boundedness of ψ by making use of Theorem 5.2. In Section 7, we will show the L^p_ℓ-gauge fixing lemma (cf. Theorem 7.3) due to the harmonic theory. In Section 8, we will show the L^p_1-boundedness of F_A for the solutions (A, ψ) of the Seiberg–Witten equations (cf. Theorem 8.1), and L^p_3-gauge equivalence to a connection $A' = A_0 + \alpha$ with a fixed connection A_0, $\delta\alpha = 0$ and boundedness of the L^p_2 norm of α (cf. Corollary 8.4) by using Theorem 7.3. In Section 9, we will show L^p_ℓ-boundedness of solutions (A, ψ) of the Seiberg–Witten equation (cf. Theorem 9.1) by using Theorem 6.3 and Corollary 8.4. In Section 10, we will clarify the structure of the moduli space of solutions of the Seiberg–Witten equations. Finally, in Section 11, we will give the compactness theorem of the moduli space of solutions of the Seiberg–Witten equation (cf. Theorem 11.1), by using the structure theory in Section 10, and L^p_ℓ-boundedness of solutions (A, ψ) of the Seiberg–Witten equations (cf. Theorem 9.1).

We would like to express our sincere gratitude to Professors H. Kamada, H. Izeki and Y. Sakane, and Doctors T. Yamada and H. Yamagata who gave to us many useful comments and communications.

Table of Contents.

2. The Spin^c-structure on a Kähler manifold

2.1. Preliminary. In this section, following [16], we give materials which we need in the arguments in the sequal sections.

Let (M, g, J) be a compact almost Hermitian manifold of complex dimension $n \geq 2$, with an almost complex structure J, and a compatible Hermitian metric g. Let P be the orthonormal frame bundle over (M, g) which is a $SO(2n)$-principal bundle over (M, g). Let $\widetilde{P}$ be the Spin^c-structure over (M, g) which is a Spin$^c(2n)$-principal bundle over (M, g), and a natural lifting of $SO(2n)$-principal bundle P. Let $\mathcal{L}$ be the determinant line bundle of $\widetilde{P}$, and $S_{\mathbb{C}}(\widetilde{P})$ be the associated complex spinor bundle over (M, g), respectively. Let us recall the Spin^c-structure determined by the almost complex structure J ([16], p. 49, Corollary 3.4.5) i.e., there exists a natural Spin^c structure $\widetilde{P}_M$ whose determinant line bundle is isomorphic to K_M^{-1}, the inverse of the canonical line bundle of $(0, n)$-forms on M for J. The associated complex spin bundle $S_{\mathbb{C}}(\widetilde{P})$ is isomorphic to the complex exterior algebra of the complex tangent bundle $\bigwedge^* T^{\mathbb{C}}M$. The half-spinor bundles $S^+_{\mathbb{C}}(\widetilde{P})$, $S^-_{\mathbb{C}}(\widetilde{P})$ are isomorphic with $\bigwedge^{\text{even}} T^{\mathbb{C}}M$, $\bigwedge^{\text{odd}} T^{\mathbb{C}}M$, respectively.

Furthermore, $S_{\mathbb{C}}(\widetilde{P})$ is isomorphic to the direct sum over all q of the exterior algebra bundle of complex-valued $(0, q)$-forms. $S^+_{\mathbb{C}}(\widetilde{P})$ is identified with the bundle of $(0, 2*)$-forms and $S^-_{\mathbb{C}}(\widetilde{P})$ is identified with the bundle of $(0, 2*+1)$-forms. The Clifford multiplication by a vector field X on M on a $(0, q)$-form μ is given by

$$X \cdot \mu = \sqrt{2}(\pi^{0,1}(\omega_X) \wedge \mu - \pi^{0,1}(\omega_X) \angle \mu), \tag{1}$$

where ω_X is the dual one-form to X, $\pi^{0,1}$ denotes the projection onto $\bigwedge^{0,1} T^*M$ and $\angle$ is the contraction operator ([16], p. 51, Corollary 3.4.6).

Let us also recall Remark 3.4.7 in [16] that in view of differential forms, the action of k-forms on $S_{\mathbb{C}}(\widetilde{P})$ is given as follows.

Suppose $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^k$ with the α^j being orthonormal at each point. Then Clifford multiplication by α is the Clifford multiplication by the α^j and Clifford multiplication by α^j is given by

$$\alpha^j \cdot \mu = \sqrt{2}(\pi^{0,1}(\alpha^j) \wedge \mu - \pi^{0,1}(\alpha^j) \angle \mu).$$

2.2. Some calculations. In the following, we assume that (M, g, J) is a compact Kähler manifold of complex dimension n. Let ∇ be Levi–Civita connection of (M, g), and e_i $(i = 1, \ldots, 2n)$, a locally defined orthonromal frame field on (M, g) which is given by

$$e_i = X_i, \quad e_{n+i} = JX_i \quad (i = 1, \ldots, n)$$

where $\langle X_i, X_j \rangle = \delta_{ij}$, $\langle X_i, JX_j \rangle = 0$ $(i, j = 1, \ldots, n)$. We extend g by complex bilinearly, and define a Hermitian metric by $\langle Z, W \rangle = g(Z, \overline{W})$ for $Z, W \in T_x^{\mathbb{C}} M$ $(x \in M)$. We put $W_j = \frac{1}{\sqrt{2}}(X_j - \sqrt{-1} JX_j)$, $\overline{W}_j = \frac{1}{\sqrt{2}}(X_j + \sqrt{-1} JX_j)$, $(j = 1, \ldots, n)$. Then,

$$\langle W_j, W_k \rangle = \langle \overline{W}_j, \overline{W}_k \rangle = \delta_{jk}, \quad \langle W_j, \overline{W}_k \rangle = 0.$$

Let us denote by $\Gamma(\bigwedge^{p,q})$ the space of smooth (p, q)-forms on M, and define $\eta_j \in \Gamma(\bigwedge^{1,0})$ locally by $\eta_j(W_k) = \delta_{jk}$, $\eta_j(\overline{W}_k) = 0$, and $\overline{\eta}_j \in \Gamma(\bigwedge^{0,1})$ by $\overline{\eta}_j(Z_k) = 0$, $\overline{\eta}_j(\overline{W}_k) = \delta_{jk}$, respectively. Then,

$$\langle \eta_j, \eta_k \rangle = \langle \overline{\eta}_j, \overline{\eta}_k \rangle = \delta_{jk}, \langle \eta_j, \overline{\eta}_k \rangle = 0.$$

Furthermore, for $J = (j_1 \ldots j_q)$ with $j_1 < \cdots < j_q$, we put $\eta_J = \eta_{j_1} \wedge \cdots \wedge \eta_{j_q}$. We extend $\langle\, , \rangle$ to the $\bigwedge^{0,q}$, denoted by the same letter, by

$$\langle \eta_J, \eta_K \rangle = \delta_{JK} = \begin{cases} 1 & \text{if } j_t = k_t \ (t = 1, \ldots, q), \\ 0 & \text{otherwise}, \end{cases}$$

for $J = (j_1 \ldots j_q)$, $K = (k_1 \ldots k_q)$ with $j_1 < \cdots < j_q$ and $k_1 < \cdots < k_q$.

Then, Clifford multiplication can be calculated in terms of the above as follows.

Lemma 2.1. *For all $\sigma \in \Gamma(\bigwedge^{0,q})$, we have*

(i)
$$X_j \cdot \sigma = \overline{\eta}_j \wedge \sigma - \overline{\eta}_j \angle \sigma, \tag{2}$$
$$JX_j \cdot \sigma = \sqrt{-1}(\overline{\eta}_j \wedge \sigma + \overline{\eta}_j \angle \sigma). \tag{3}$$

(ii)
$$W_j \cdot \sigma = \sqrt{2}\,\overline{\eta}_j \wedge \sigma \tag{4}$$
$$\overline{W}_j \cdot \sigma = -\sqrt{2}\,\overline{\eta}_j \angle \sigma. \tag{5}$$

(iii)
$$W_j \wedge W_k \cdot \sigma = \overline{\eta}_j \wedge \overline{\eta}_k \wedge \sigma, \tag{6}$$
$$\overline{W}_j \wedge \overline{W}_k \cdot \sigma = \overline{\eta}_j \angle (\overline{\eta}_k \angle \sigma) \tag{7}$$
$$W_j \wedge \overline{W}_k \cdot \sigma = -\overline{\eta}_j \wedge (\overline{\eta}_k \angle \sigma) \tag{8}$$

PROOF. The proof is omitted. □

Definition 2.2. The Clifford multiplication of $\Gamma(\bigwedge^2)$ on $S_{\mathbb{C}}(\widetilde{P})$ preserves $S^{\pm}_{\mathbb{C}}(\widetilde{P})$ invariant. For 2-forms F and $G \in \Gamma(\bigwedge^2)$, one can define the Hermitian semi-inner product $\ll F, G \gg_{\pm}$ by

$$\ll F, G \gg_{+} = \sum_{\eta_J \in \bigwedge^{2*}} \langle F \cdot \eta_J, G \cdot \eta_J \rangle, \tag{9}$$
$$\ll F, G \gg_{-} = \sum_{\eta_J \in \bigwedge^{2*+1}} \langle F \cdot \eta_J, G \cdot \eta_J \rangle, \tag{10}$$

where the dot $\cdot$ is Clifford multiplication of $\Gamma(\bigwedge^2)$ on $S^{\pm}_{\mathbb{C}}(\widetilde{P})$ being identified with $\Gamma(\bigwedge^*)$ with the Hermitian metric $\langle\, , \rangle$ as in 2.1, and J runs over the set of all $(j_1 \dots j_q)$ with $j_1 < \cdots < j_q$ where q are even integers in $\{0, 1, \dots, n\}$. Here we put $\eta_J = 1$ for $J = \emptyset$ with $q = 0$.

We want to show the above $\ll\, , \gg_{\pm}$ is related to the Hermitian metric $\langle\, , \rangle$ as follows.

Theorem 2.3. *Let us decompose* $\bigwedge^2 = (\bigwedge^{2,0} \oplus \bigwedge^{0,2}) \oplus \bigwedge^{1,1}$.

(i) *For* F, $G \in \Gamma(\bigwedge^{2,0} \oplus \bigwedge^{0,2})$, *we have*

$$\ll F, G \gg_{+} = \begin{cases} \langle F, G \rangle & (n = 2), \\ 2^{n-3} \langle F, G \rangle & (n \geq 3). \end{cases}$$

(ii) *For* $F \in \Gamma(\bigwedge^{2,0} \oplus \bigwedge^{0,2})$ *and* $G \in \Gamma(\bigwedge^{1,1})$,

$$\ll F, G \gg_{+} = 0.$$

(iii) *For* F, $G \in \Gamma(\bigwedge^{1,1})$, *we have*

$$\ll F, G \gg_{+} = \begin{cases} \langle \Lambda F, \Lambda G \rangle & (n = 2), \\ 2^{n-3}(\langle \Lambda F, \Lambda G \rangle + \langle F, G \rangle) & (n \geq 3), \end{cases}$$

where $\Lambda F = \sum_{j=1}^{n} F(W_j, \overline{W}_j)$ *is the trace of* $F \in \Gamma(\bigwedge^2)$.

Proof. The proof is omitted. □

For the $\ll , \gg_{-}$, we have

Theorem 2.4. (i) *For* $F, G \in \Gamma(\bigwedge^{2,0} \oplus \bigwedge^{0,2})$, *we have*

$$\ll F, G \gg_{-} = \begin{cases} 0 & (n = 2), \\ 2^{n-3} \langle F, G \rangle & (n \geq 3). \end{cases}$$

(ii) *For* $F \in \Gamma(\bigwedge^{2,0} \oplus \bigwedge^{0,2})$ *and* $G \in \Gamma(\bigwedge^{1,1})$,

$$\ll F, G \gg_{-} = 0.$$

(iii) *For* $F, G \in \Gamma(\bigwedge^{1,1})$,

$$\ll F, G \gg_{-} = \begin{cases} \langle F, G \rangle & (n = 2), \\ 2^{n-3}(\langle \Lambda F, \Lambda G \rangle + \langle F, G \rangle & (n \geq 3). \end{cases}$$

Proof. The proof is omitted. □

Finally, we have immediately from Theorem 2.3,

Corollary 2.5. *For* $F \in \Gamma(\Lambda^2)$,

$$|c^{+}(F)|^2 = \begin{cases} |F^{+}|^2 + |\Lambda(F)|^2 & (n = 2), \\ 2^{n-3}\big(|F|^2 + |\Lambda(F)|^2\big) & (n \geq 3). \end{cases}$$

Here F^{+} *is the* $\Lambda^{2,0} \oplus \Lambda^{0,2}$*-component of* F *corresponding to the decomposition* $\Lambda^2 = (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \Lambda^{1,1}$.

3. The Seiberg–Witten energy functional

3.1. Endomorphisms on $S^+_{\mathbb{C}}(\widetilde{P})$.

Definition 3.1. For every $\alpha \in \bigwedge^2 T^*_x M$ $(x \in M)$, the endmorphism $c^+(\alpha)$ of $S^+_{\mathbb{C},x}(\widetilde{P})$ is defined by

$$c^+(\alpha)(\mu) = \alpha \cdot \mu \in S^+_{\mathbb{C},x}(\widetilde{P}) \quad \mu \in S^+_{\mathbb{C},x}(\widetilde{P}).$$

Lemma 3.2. *For every $\alpha \in \bigwedge^2 T^*_x M$ $(x \in M)$, $c^+(\alpha)$ is a trace free endomorphism of $S^+_{\mathbb{C},x}(\widetilde{P})$.*

Proof. The proof is omitted. □

Definition 3.3. For $\alpha, \beta \in \mathrm{End}(S^+_{\mathbb{C}}(\widetilde{P}))$, the pointwise Hermitian norm of β is defined by

$$\langle \alpha, \beta \rangle = \sum_{\overline{\eta}_J \in \bigwedge^{2*}} \langle \alpha(\overline{\eta}_J), \beta(\overline{\eta}_J) \rangle.$$

The norm of α is given by $|\alpha|^2 = \langle \alpha, \alpha \rangle$.

By definition, we have immediately

Lemma 3.4.

$$\langle c^+(F), c^+(G) \rangle = \ll F, G \gg_+, \quad (F, G \in \Gamma(\textstyle\bigwedge^2)). \tag{11}$$

Lemma 3.5. (i) *For each $\psi \in \Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$, the endmorphism $\psi \otimes \psi^*$ of $S^+_{\mathbb{C}}(\widetilde{P})$ defined $(\psi \otimes \psi^*)(\varphi) = \langle \varphi, \psi \rangle \psi$ $(\varphi \in \Gamma(S^+_{\mathbb{C}}(\widetilde{P})))$ satisfies the following:*

$$\langle c^+(\alpha), \psi \otimes \psi^* \rangle = \langle \alpha \cdot \psi, \psi \rangle, \tag{12}$$

$$|\psi \otimes \psi^*| = |\psi|^2 \tag{13}$$

(ii) *Let us denote by $\mathrm{tr}(\alpha)$ the trace of $\alpha \in \mathrm{End}(S^+_{\mathbb{C}}(\widetilde{P}))$. Then,*

$$\mathrm{tr}(\psi \otimes \psi^*) = |\psi|^2. \tag{14}$$

(iii) *Then, $\psi \otimes \psi^*$ is decomposed orthogonally into*

$$\psi \otimes \psi^* = (\psi \otimes \psi^*)_0 + \frac{1}{2^{n-1}} |\psi|^2 \ \mathrm{Id}, \tag{15}$$

where α_0 is the trace free part of $\alpha \in \mathrm{End}(S^+_{\mathbb{C}}(\widetilde{P}))$, and Id *is the identity operator.*

(iv) *Furthermore, we have*

$$|(\psi \otimes \psi^*)_0|^2 = \left(1 - \frac{1}{2^{n-1}}\right) |\psi|^4. \tag{16}$$

Proof. The proofs are omitted. □

3.2. Connections. Let us recall $\widetilde{P}$ admits the $\mathrm{Spin}^c(2n)$ action. Since $\mathrm{Spin}^c(2n) = \{e^{\sqrt{-1}\theta}x;\ x \in \mathrm{Spin}(2n),\ \theta \in \mathbb{R}\}$, $\widetilde{P}$ admits the actions of $U(1)$ and $\mathrm{Spin}(2n)$. Then, we may consider the orbit spaces $\widetilde{P}/U(1)$ and $\widetilde{P}/\,\mathrm{Spin}(2n)$ which are the two-fold covering of P and the $U(1)$-principal bundle associated to the determinant bundle $\mathcal{L}$, respectively.

Since $\widetilde{P}$ is a $\mathrm{Spin}^c(2n)$-principal bundle over (M,g), a connection form $\widetilde{\omega}_A$ on $\widetilde{P}$ is a $\mathfrak{spin}^c(2n)$-valued 1-form on $\widetilde{P}$ satisfying that

$$(*) \qquad \begin{cases} {R_a}^*\widetilde{\omega}_A = \mathrm{Ad}(a^{-1})\,\widetilde{\omega}_A, & (a \in \mathrm{Spin}^c(2n)), \\ \widetilde{\omega}_A(X^*) = X, & (X \in \mathfrak{spin}^c(2n)), \end{cases}$$

where $X^*_p = \frac{d}{dt}|_{t=0}\, p \cdot \exp(tX)$ $(p \in \widetilde{P})$. According to the splitting of the Lie algebra $\mathfrak{spin}^c(2n)$ into $\mathfrak{spin}(2n) = \mathfrak{so}(2n) \oplus \sqrt{-1}\mathbb{R}$, we may write

$$\widetilde{\omega}_A = \widetilde{\omega} + \widetilde{A}, \tag{17}$$

where $\widetilde{\omega}$ is a $\mathfrak{so}(2n)$-valued 1-from on $\widetilde{P}$ and $\widetilde{A}$ is a $\sqrt{-1}\mathbb{R}$-valued 1-form on it, respectively. The first condition of $(*)$ becomes that

$${R_a}^*\widetilde{\omega} = \mathrm{Ad}(a^{-1})\widetilde{\omega}, \quad {R_a}^*\widetilde{A} = \widetilde{A} \quad (a \in \mathrm{Spin}^c(2n)).$$

The second condition of $(*)$ corresponds to that

$$\widetilde{\omega}(Y^*) = Y \quad (Y \in \mathfrak{so}(2n)), \quad \widetilde{A}(Z^*) = Z \quad (Z \in \sqrt{-1}\mathbb{R}).$$

Then, the conditions about $\widetilde{A}$ is equivalent to that $\widetilde{A} = \pi^* A$ for some $\sqrt{-1}\mathbb{R}$-valued 1-form A on M, where $\pi;\, \widetilde{P} \to P$ is the natural projection.

Corresoponding to the connection form $\widetilde{\omega}_A$ on $\widetilde{P}$, the covariant differentiation D_A on the space $\widetilde{\Gamma}(\widetilde{P})$ is given by

$$D_A\widetilde{\varphi} = d\widetilde{\varphi} + \rho(\widetilde{\omega}_A)\widetilde{\varphi}, \quad (\widetilde{\varphi} \in \widetilde{\Gamma}(\widetilde{P})).$$

Here, d is the exterior differentiation on $\widetilde{P}$ and $\widetilde{\Gamma}(\widetilde{P})$ is the space of all smooth functions $\widetilde{\varphi}$ from $\widetilde{P}$ to $\Delta^+_{\mathbb{C}}$ satisfying that

$$\widetilde{\varphi}(pa) = \rho(a^{-1})\widetilde{\varphi}(p), \quad (a \in \mathrm{Spin}^c(2n),\ p \in \widetilde{P}),$$

where $\rho : \mathrm{Spin}^c(2n) \to GL(\Delta^+_{\mathbb{C}})$ is the complex half-spin representation of $\mathrm{Spin}^c(2n)$. $D_A\widetilde{\varphi}$ is a $\Delta^+_{\mathbb{C}}$-valued 1-form on $\widetilde{P}$ satisfying that

$$\begin{cases} {R_a}^*(D_A\widetilde{\varphi}) = \rho(a^{-1})D_A\widetilde{\rho}, & (a \in \mathrm{Spin}^c(2n)) \\ D_A\widetilde{\varphi}(X^*) = 0, & (X \in \mathfrak{spin}^c(2n)). \end{cases}$$

Since the half-spinor bundle $S^+_{\mathbb{C}}(\widetilde{P})$ is $\widetilde{P} \times_\rho \Delta^+_{\mathbb{C}} = (\widetilde{P} \times \Delta^+_{\mathbb{C}})/\sim$, where the equivalence relation $\sim$ is defined by $(p, v) \sim (pa, \rho(a^{-1})v$, $(p \in \widetilde{P},\ v \in \Delta^+_{\mathbb{C}},\ a \in \mathrm{Spin}^c(2n))$, the covariant differentiation $\widetilde{\nabla}^A$ on $S^+_{\mathbb{C}}(\widetilde{P})$ is defined by

$$\widetilde{\nabla}^A_X \varphi = p(D_A \widetilde{\varphi}(W_X)), \quad (X \in T_x(M), \varphi \in \Gamma(S^+_{\mathbb{C}}(\widetilde{P}))),$$

where $p \in \widetilde{P}$ is identified with the isomorphism, denoted by the same letter, $p : \Delta^+_{\mathbb{C}} \ni v \mapsto [p, v] \in S^+_{\mathbb{C}x}$ $(x = \pi(p) \in M)$, and $\Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$ is identified with the space $\widetilde{\Gamma}(\widetilde{P})$ by $\varphi(x) = p(\widetilde{\varphi}(p))$, $(x = \pi(p) \in M,\ p \in \widetilde{P})$. W_X is a vector at $\widetilde{P}$ lifting $X \in T_x M$. The above expression is independent on the choice of lifting. Then, we have

Lemma 3.6.

$$\widetilde{\nabla}^A_X \varphi = \widetilde{\nabla}_X \varphi + A(X)\varphi, \quad (X \in \mathfrak{X}(M),\ \varphi \in \Gamma(S^+_{\mathbb{C}}(\widetilde{P}))),$$

$\mathfrak{X}(M)$ stands for the space of smooth vector fields on M, and $\widetilde{\nabla}$ is the connection on $S^+_{\mathbb{C}}(\widetilde{P})$ corresponding to the connection form $\widetilde{\omega}$ given by

$$\widetilde{\nabla}_X \varphi = p(D_0 \widetilde{\varphi}(W_X)) = p(\{d\widetilde{\varphi} + \rho(\widetilde{\omega})\widetilde{\varphi}\}(W_X)).$$

PROOF. The proof is omitted. □

We may usually take $\widetilde{\omega}$ and $\widetilde{\nabla}$ corresponding to the Levi–Civita connection of (M, g). Because $\widetilde{\omega}$ is a $\mathfrak{so}(2n)$-valued 1-from on $\widetilde{P}$ satisfying

$$\begin{cases} {R_a}^* \widetilde{\omega} = \mathrm{Ad}(a^{-1})\widetilde{\omega}, & (a \in \mathrm{Spin}^c(2n)), \\ \widetilde{\omega}(Y^*) = Y, & (Y \in \mathfrak{so}(2n)). \end{cases}$$

It induces a connection form on $\widetilde{P}/U(1)$ which is a two-fold covering of P, so it corresponds to a connection form on P, and vice-versa.

On the other hand, the determinant line bundle $\mathcal{L}$ is the one over M associated to $\widetilde{P}$ corresponding to the 1-dimensional representation of $\mathrm{Spin}^c(2n)$, $\delta : \mathrm{Spin}^c(2n) \ni e^{\sqrt{-1}\theta} x \mapsto e^{2\sqrt{-1}\theta} \in U(1)$ $(\theta \in \mathbb{R},\ x \in \mathrm{Spin}(2n))$, i.e.,

$$\mathcal{L} = \widetilde{P} \times_\delta \mathbb{C} = (\widetilde{P} \times \mathbb{C})/\sim,$$

where the equivalence relation $\sim$ in $\widetilde{P} \times \mathbb{C}$ is given by $(pa, \delta(a^{-1})) \sim (p, b)$, $(a \in \mathrm{Spin}^c(2n),\ p \in \widetilde{P},\ b \in \mathbb{C})$. Then, for every $\sqrt{-1}\mathbb{R}$-valued 1-form A on M, let $\widetilde{A} = \pi^* A$. Let $\widetilde{\Gamma}_\delta(\widetilde{P})$ be the space of all smooth maps $\widetilde{u}$ of $\widetilde{P}$ into $\mathbb{C}$ satisfying that $\widetilde{u}(pa) = \delta(a^{-1})\widetilde{u}(p)$, $(p \in \widetilde{P},\ a \in \mathrm{Spin}^c(2n))$. For each $u \in \Gamma(\mathcal{L})$, let us define $\widetilde{u} \in \widetilde{\Gamma}_\delta(\widetilde{P})$ by $u(x) = p\ \widetilde{u}(p)$, $(x = \pi(p) \in M,\ p \in \widetilde{P})$. Then, the connection ∇^{2A} on $\mathcal{L}$ is defined by

$$\nabla^{2A}_X u = p(D_A \widetilde{u}(W_X)), \quad (X \in T_x(M),\ u \in \Gamma(\mathcal{L})),$$

where D_A is the covariant differentiation on the space $\widetilde{\Gamma}_\delta(\widetilde{P})$ given by

$$D_A \widetilde{u} = d\widetilde{u} + \delta(\widetilde{A})\widetilde{u}.$$

Then we have

Lemma 3.7.

$$\nabla^{2A}_X u = \nabla^0_X u + 2A(X)u, \quad (X \in \mathfrak{X}(M),\ u \in \Gamma(\mathcal{L})).$$

where ∇^0 is the connection of $\mathcal{L}$ defined by $\nabla^0_X u = p(d\widetilde{u}(W_X))$, $X \in \mathfrak{X}(M)$, $u \in \Gamma(\mathcal{L})$.

Proof. The proof is omitted. □

The curvature tensor fields are given by a standard way.

Lemma 3.8.

(1) *The curvature tensor field $R^{\widetilde{\nabla}^A}$ of a connection $\widetilde{\nabla}^A$ of $S^+_{\mathbb{C}}(\widetilde{P})$ is given by*

$$R^{\widetilde{\nabla}^A} = R^{\widetilde{\nabla}} + F_A,$$

where $R^{\widetilde{\nabla}}$ is the curvature tensor field of the Leve–Civita connection $\widetilde{\nabla}$ and $F_A = dA$ is the exterior differentiation of $\sqrt{-1}\mathbb{R}$-valued 1-form A on M.

(2) *The curvature tensor field R^{∇^A} of a connection ∇^A of $\mathcal{L}$ defined by $\nabla^A = \nabla^0 + A$ is given by*

$$R^{\nabla^A} = F_A = dA,$$

i.e., ∇^0 is the flat connection of $\mathcal{L}$.

Proof. The proof is omitted. □

One can define the Hermitian metrics $\langle\ ,\ \rangle$ on $\mathcal{L}$ and $S^{\pm}_{\mathbb{C}}(\widetilde{P})$ induced from (M, g), and the usual global Hermitian metrics $(\ ,\ )$ are defined by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle v_g, \quad (\varphi, \psi \in \Gamma(S^{\pm}_{\mathbb{C}}(\widetilde{P}))),$$

where v_g is the volume element of (M, g).

3.3. The dirac operator. Let us recall the Spin^c-Dirac operator which is defined as follows: For every $\sqrt{-1}\mathbb{R}$-valued 1-form A on M, the first order elliptic differential operator

$$\mathfrak{D}_A : \Gamma(S^{\pm}_{\mathbb{C}}(\widetilde{P})) \to \Gamma(S^{\mp}_{\mathbb{C}}(\widetilde{P}))$$

is given by

$$\mathfrak{D}_A\,\varphi = \sum_{i=1}^{2n} e_i \cdot \widetilde{\nabla}^A_{e_i}\varphi, \quad (\varphi \in \Gamma(S^{\pm}_{\mathbb{C}}(\widetilde{P}))),$$

where $\{e_i\}_{i=1}^{2n}$ is a locally defined orthonormal frame field on (M, g) and the dot $\cdot$ is the Clifford multiplication. Let

$$\mathfrak{D}_A{}^* : \Gamma(S^{\mp}_{\mathbb{C}}(\widetilde{P})) \to \Gamma(S^{\pm}_{\mathbb{C}}(\widetilde{P}))$$

be the L^2-adjoint operator of $\mathfrak{D}_A$. Then, the Weitzenböck formula is given as follows:

Lemma 3.9 (the Weitzenböck formula)**.** *For $\varphi \in \Gamma(S^{+}_{\mathbb{C}}(\widetilde{P}))$,*

$$\mathfrak{D}_A{}^*(\mathfrak{D}_A\varphi) = \widetilde{\nabla}^{A*}\widetilde{\nabla}^A\varphi + \frac{1}{4}\kappa\,\varphi + 2\,F_A \cdot \varphi, \tag{18}$$

where κ is the scalar curvature of (M, g), and F_A is the curvature tensor of a connection ∇^A of $\mathcal{L}$ (cf. Lemma 3.7).

Remark 3.10. Remark here that the constant factor in the third term is different from the one in [16], p. 73.

Proof. By a direct computation. The proof is omitted. □

3.4. Seiberg–Witten energy functional. In this subsection, we introduce the Seiberg–Witten energy functional over a compact Kähler manifold, and show a characterization theorem of the Seiberg–Witten equation.

Definition 3.11. For every $\sqrt{-1}\mathbb{R}$-valued 1-form A on M and $\psi \in \Gamma(S^{+}_{\mathbb{C}}(\widetilde{P}))$, let us define the Seiberg–Witten energy functional $E(A, \psi)$ by

$$\begin{aligned} E(A,\psi) = {} & \frac{1}{2^{n-1}}\|\widetilde{\nabla}^A\psi\|^2_{L_2} + \frac{1}{2^{n+1}}\int_M |\psi|^2\left(\kappa - 4\left(1 - \frac{1}{2^{n-1}}\right)|\psi|^2\right)v_g \\ & + \frac{1}{4}\left(\|F_A\|_{L_2}{}^2 + \|\Lambda F_A\|_{L_2}{}^2\right) \quad (n = 2), \end{aligned} \tag{19}$$

$$\begin{aligned} E(A,\psi) = {} & \frac{1}{2^{n-1}}\|\widetilde{\nabla}^A\psi\|_{L_2}{}^2 + \frac{1}{2^{n+1}}\int_M |\psi|^2\left(\kappa - 4\left(1 - \frac{1}{2^{n-1}}\right)|\psi|^2\right)v_g \\ & + \frac{1}{2}\|F_A^-\|_{L_2}{}^2, \quad (n \geq 3), \end{aligned} \tag{20}$$

where $\|\cdot\|_{L_2}$ is the L_2-norm with respect to the volume element v_g, and for $F_A \in \Gamma(\bigwedge^2)$, $\Lambda F_A = \sum_{j=1}^n F_A(W_j, \overline{W}_j)$ and F_A^- are the trace and the $\bigwedge^{1,1}$-component relative to the decomposition $\bigwedge^2 = (\bigwedge^{2,0} \oplus \bigwedge^{0,2}) \oplus \bigwedge^{1,1}$, respectively.

Then, we have

Theorem 3.12. *The following equality and inequality hold.*

$$E(A,\psi) = \frac{1}{2^{n-1}}\int_M |\mathfrak{D}_A\psi|^2\, v_g + \frac{1}{2^{n-1}}\int_M |c^+(F_A) - (\psi\otimes\psi^*)_0|^2\, v_g \\ - \pi^2\langle c_1(\mathcal{L})^2, [M]\rangle \geq -\pi^2\langle c_1(\mathcal{L})^2, [M]\rangle. \tag{21}$$

Equality holds for (21) *if and only if the Seiberg–Witten equations for* (A,ψ) *hold, i.e.,*

$$\begin{cases} \mathfrak{D}_A\psi = 0, \\ c^+(F_A) \qquad = (\psi\otimes\psi^*)_0. \end{cases} \tag{22}$$

Proof. By the Weitzenböck formula (18) in Lemma 3.9, we have

$$\frac{1}{2^{n-1}}\|\mathfrak{D}\psi\|_{L^2}{}^2 = \frac{1}{2^{n-1}}\|\widetilde{\nabla}_A\psi\|_{L^2}{}^2 + \frac{1}{2^{n+1}}\int_M \kappa\,|\psi|^2\, v_g \\ + \frac{2}{2^{n-1}}\int_M \langle F_A\cdot\psi, \psi\rangle\, v_g. \tag{23}$$

By Lemma 3.4 and Theorem 2.3,

$$\frac{1}{2^{n-1}}\int_M |c^+(F_A) - (\psi\otimes\psi^*)_0|^2\, v_g = \frac{1}{2^{n-1}}\|c^+(F_A)\|_{L^2}{}^2 \\ - \frac{2}{2^{n-1}}\int_M \langle F_A\cdot\psi,\psi\rangle\, v_g + \frac{1}{2^{n-1}}\|(\psi\otimes\psi^*)_0\|_{L^2}{}^2, \tag{24}$$

and due to Theorem 2.3, the first term of (23), $\frac{1}{2^{n-1}}\|c^+(F_A)\|_{L^2}{}^2$, is equal to

$$\begin{cases} \dfrac{1}{2}\left(\|F_A^+\|_{L^2}{}^2 + \|\Lambda F_A\|_{L^2}{}^2\right) \quad (n=2), \\ \dfrac{1}{4}\left(\|F_A\|_{L^2}{}^2 + \|\Lambda F_A\|_{L^2}{}^2\right) \quad (n\geq 3). \end{cases} \tag{25}$$

Due to Lemma 3.5 (iv), the third term of (23) is equal to

$$\frac{1}{2^{n-1}}\left(1 - \frac{1}{2^{n-1}}\right)\int_M |\psi|^4\, v_g. \tag{26}$$

Furthermore, we have (cf. [34])

$$-\pi^2\langle c_1(\mathcal{L})^2, [M]\rangle = -\frac{1}{4}\|F_A^+\|_{L^2}{}^2 - \frac{1}{4}\|\Lambda F_A\|_{L^2}{}^2 + \frac{1}{4}\|F_A^-\|_{L^2}{}^2. \tag{27}$$

Therefore, summing up (22), (23) and (26) all together,

$$\frac{1}{2^{n-1}}\int_M |\mathfrak{D}_A\psi|^2\, v_g + \frac{1}{2^{n-1}}\int_M |c^+(F_A) - (\psi\otimes\psi^*)_0|^2\, v_g - \pi^2\langle c_1(\mathcal{L})^2, [M]\rangle$$

coincides with

$$\frac{1}{2^{n-1}}\|\widetilde{\nabla}_A\psi\|_{L^2}{}^2 + \frac{1}{2^{n+1}}\int_M \kappa\,|\psi|^2\, v_g + \frac{1}{2^{n-1}}\left(1-\frac{1}{2^{n-1}}\right)\int_M |\psi|^4\, v_g$$
$$+\frac{1}{2^{n-1}}\int_M |c^+(F_A)|^2\, v_g - \frac{1}{4}\left(\|F_A^+\|_{L^2}{}^2 + \|\Lambda F_A\|_{L^2}{}^2\right) + \frac{1}{4}\|F_A^-\|_{L^2}{}^2. \quad (28)$$

Then, due to (24), the sum of the last three terms of (27) is equal to

$$\begin{cases} \frac{1}{4}\left(\|F_A\|_{L^2}{}^2 + \|\Lambda F_A\|_{L^2}{}^2\right) & (n=2), \\ \frac{1}{2}\|F_A^-\|_{L^2}{}^2 & (n\geq 3). \end{cases} \quad (29)$$

Therefore, we have the desired. □

Remark 3.13. Our Seiberg–Witten energy formula in (19) is slightly different from the usual formula up to (28).

4. The Seiberg–Witten equations

In this section, we calculate the Seiberg–Witten equations in terms of local holomorphic 1-forms η_j $(j=1,\dots,n)$ as in 2.2. We have

Theorem 4.1. *For (A,ψ), the equation $c^+(F_A) = (\psi\otimes\psi^*)_0$ holds if and only if the following two equations hold:*

$$|\langle\psi,\overline{\eta}_J\rangle|^2 - \frac{1}{2^{n-1}}|\psi|^2 = -\sum_{t=1}^{p} F_{j_t\,\overline{j_t}} \quad (30)$$

for all $J=(j_1\dots j_p)$ with $j_1<\cdots<j_p$. Furthermore, $\langle\overline{\eta}_J,\psi\rangle\,\langle\psi,\overline{\eta}_K\rangle$ coincides with (1) $-F_{k\,\overline{j_t}}$ if $J=(j_1\dots j_p)$, $K=(j_1\dots j_{t-1}\,k\,j_{t+1}\dots j_p)$, (2) $(-1)^{s+t+1}F_{k_s\,k_t}$ if $J=(k_1\dots k_{s-1}k_{s+1}\dots k_{t-1}k_{t+1}\dots k_p)$, $K=(k_1\dots k_{p+2})$, (3) $(-1)^{s+t}F_{\overline{j_s}\,\overline{j_t}}$ if $J=(j_1\dots j_{p+2})$, $K=(j_1\dots j_{s-1}j_{s+1}\dots j_{t-1}j_{t+1}\dots j_{p+2})$ and (4) 0 otherwise. Here we write F_A locally as

$$F_A = \sum_{i<j}\left(F_{ij}\,\eta_i\wedge\eta_j + F_{\overline{i}\,\overline{j}}\,\overline{\eta}_i\wedge\overline{\eta}_j\right) + \sum_{i,j=1}^{n} F_{i\,\overline{j}}\,\eta_i\wedge\overline{\eta}_j.$$

Corollary 4.2. *For (A,ψ), the equation $c^+(F_A) = (\psi \otimes \psi^*)_0$ holds if and only if the following hold:*

Case 1: $n = 2$.

$$\begin{cases} F_{12} = \langle 1,\psi\rangle\,\langle \psi,\overline{\eta}_1\wedge\overline{\eta}_2\rangle, \qquad F_{\overline{1}\,\overline{2}} = -\langle\psi,1\rangle\,\langle\overline{\eta}_1\wedge\overline{\eta}_2,\psi\rangle, \\ F_{1\,\overline{1}} + F_{2\,\overline{2}} = -|\langle\overline{\eta}_1\wedge\overline{\eta}_2,\psi\rangle|^2 + \frac{1}{2}|\psi|^2. \end{cases}$$

Case 2: $n \geq 3$.

(1) *(The $\Lambda^{2,0}\oplus\Lambda^{0,2}$-components) For $1 \leq i < j \leq n$,*

$$\begin{cases} F_{ij} = \langle 1,\psi\rangle\,\langle\psi,\overline{\eta}_i\wedge\overline{\eta}_j\rangle, \\ F_{\overline{i}\,\overline{j}} = (-1)^{i+j}\langle\overline{\eta}_1\wedge\cdots\wedge\overline{\eta}_n,\psi\rangle \\ \qquad \times\,\langle\psi,\overline{\eta}_1\wedge\cdots\wedge\overline{\eta}_{i-1}\wedge\overline{\eta}_{i+1}\wedge\cdots\wedge\overline{\eta}_{j-1}\wedge\overline{\eta}_{j+1}\wedge\cdots\wedge\overline{\eta}_n\rangle. \end{cases}$$

(2) *(The $\Lambda^{1,1}$-components)* (2-1) *For $1 \leq i < j \leq n$,*

$$F_{i\,\overline{j}} = -\langle\overline{\eta}_j\wedge\overline{\eta}_\ell,\psi\rangle\,\langle\psi,\overline{\eta}_i\wedge\overline{\eta}_\ell\rangle, F_{j\,\overline{i}} = -\langle\psi,\overline{\eta}_j\wedge\overline{\eta}_\ell\rangle\,\langle\overline{\eta}_i\wedge\overline{\eta}_\ell,\psi\rangle,$$

for all $\ell \in \{1,\ldots,i-1,i+1,\ldots,j-1,j+1,\ldots,n\}$.

(2-2-1) Case 2-1: $n = 3$.

$$F_{1\,\overline{1}} = \frac{1}{2}\left\{|\langle\overline{\eta}_2\wedge\overline{\eta}_3,\psi\rangle|^2 - |\langle\overline{\eta}_1\wedge\overline{\eta}_2,\psi\rangle|^2 - |\langle\overline{\eta}_1\wedge\overline{\eta}_3,\psi\rangle|^2 + \frac{1}{4}|\psi|^2\right\}$$

$$F_{2\,\overline{2}} = \frac{1}{2}\left\{|\langle\overline{\eta}_1\wedge\overline{\eta}_3,\psi\rangle|^2 - |\langle\overline{\eta}_1\wedge\overline{\eta}_2,\psi\rangle|^2 - |\langle\overline{\eta}_2\wedge\overline{\eta}_3,\psi\rangle|^2 + \frac{1}{4}|\psi|^2\right\}$$

$$F_{3\,\overline{3}} = \frac{1}{2}\left\{|\langle\overline{\eta}_1\wedge\overline{\eta}_2,\psi\rangle|^2 - |\langle\overline{\eta}_1\wedge\overline{\eta}_3,\psi\rangle|^2 - |\langle\overline{\eta}_2\wedge\overline{\eta}_3,\psi\rangle|^2 + \frac{1}{4}|\psi|^2\right\}.$$

(2-2-2) Case 2-2: $n \geq 4$ *and even. For $i = 1,2,\ldots,n-1$,*

$$F_{i\,\overline{i}} = -|\langle\overline{\eta}_i\wedge\overline{\eta}_n,\psi\rangle|^2 + \frac{1}{n-2}\left\{\sum_{j=1}^{n-1}|\langle\overline{\eta}_j\wedge\overline{\eta}_n,\psi\rangle|^2 - |\langle\overline{\eta}_1\wedge\cdots\wedge\overline{\eta}_n,\psi\rangle|^2\right\},$$

$$F_{n\,\overline{n}} = \frac{1}{2^{n-1}}|\psi|^2 - \frac{1}{n-2}\left\{\sum_{j=1}^{n-1}|\langle\overline{\eta}_j\wedge\overline{\eta}_n,\psi\rangle|^2 - |\langle\overline{\eta}_1\wedge\cdots\wedge\overline{\eta}_n,\psi\rangle|^2\right\}.$$

(2-2-3) Case 2-3: $n \geq 5$ *and odd. For $i = 1,2,\ldots,n-2,n$,*

$$F_{i\overline{i}} = -|\langle\overline{\eta}_i\wedge\overline{\eta}_{n-1},\psi\rangle|^2 + \frac{1}{n-|,3}\left\{\sum_{j=1}^{n-2}|\langle\overline{\eta}_j\wedge\overline{\eta}_{n-1},\psi\rangle|^2 - |\langle\overline{\eta}_1\wedge\cdots\wedge\overline{\eta}_{n-1},\psi\rangle|^2\right\},$$

$$F_{n-1\,\overline{n-1}} = \frac{1}{2^{n-1}}|\psi|^2 - \frac{1}{n-3}\left\{\sum_{j=1}^{n-2}|\langle\overline{\eta}_j\wedge\overline{\eta}_{n-2},\psi\rangle|^2 - |\langle\overline{\eta}_1\wedge\cdots\wedge\overline{\eta}_{n-1},\psi\rangle|^2\right\}.$$

For the proof of Theorem 4.1, notice that the equation $c^+(F_A) = (\psi \otimes \psi^*)_0$ as endomorphisms of $S^+_{\mathbb{C}}(\widetilde{P})$, is equivalent to that, for all $J = (j_1 \dots j_p)$ and $K = (k_1 \dots k_q)$ with even nonnegative integers p and q,

$$\langle F_A \cdot \overline{\eta}_J, \overline{\eta}_K \rangle = \langle (\psi \otimes \psi^*)_0(\overline{\eta}_J), \overline{\eta}_K \rangle, \tag{31}$$

The right hand side of (51) coincides with

$$\langle \overline{\eta}_J, \psi \rangle \, \langle \psi, \overline{\eta}_K \rangle - \frac{1}{2^{n-1}} |\psi|^2 \, \delta_{JK}, \tag{32}$$

where

$$\delta_{JK} = \langle \overline{\eta}_J, \overline{\eta}_K \rangle = \begin{cases} 1, & J = K, \\ 0, & \text{otherwise.} \end{cases}$$

By the same calculation of Lemma 2.1, we have

Lemma 4.3.

(1) *If $F \in \Gamma(\Lambda^{2,0})$, then for all K ($|K| = p+2$) and J ($|J| = p$), $\langle F \cdot \overline{\eta}_J, \overline{\eta}_K \rangle$ coincides with $(-1)^{s+t+1} F_{k_s\, k_t}$ if $K = (k_1 \dots k_{p+2})$, $J = (k_1 \dots k_{s-1} k_{s+1} \dots k_{t-1} k_{t+1} \dots k_{p+2})$, and 0 otherwise, respectively.*

(2) *If $F \in \Gamma(\Lambda^{0,2})$, then for all K ($|K| = p$) and J ($|J| = p+2$), $\langle F \cdot \overline{\eta}_J, \overline{\eta}_K \rangle$ coincides with $(-1)^{s+t} F_{\overline{j_s}\, \overline{j_t}}$, if $J = (j_1 \dots j_{p+2})$, $K = (j_1 \dots j_{s-1} j_{s+1} \dots j_{t-1} j_{t+1} \dots j_{p+2})$ and 0 otherwise, respectively.*

(3) *If $F \in \Gamma(\Lambda^{1,1})$, then for all K ($|K| = p$) and J ($|J| = p$), $\langle F \cdot \overline{\eta}_J, \overline{\eta}_K \rangle$ coincides with $-F_{k\, \overline{j_t}}$ if $J = (j_1 \dots j_p)$, $K = (j_1 \dots j_{t-1}\, k\, j_{t+1} \dots j_p)$ ($t = 1, \dots, p$), and 0 otherwise, respectively.*

Due to Lemmas 3.5 and 4.3 and a direct computation, we have immediately Theorem 4.1.

5. C^0, C^1 estimates of sections of vector bundles

In this section, we prepare C^0 and C^1 pointwise Korn–Lichitenstein type esimates (cf. [3], p. 91, Theorem 3.67) for sections of an arbitrary vector bundle E with the inner product h over a compact Riemannian manifold (M, g) which is necessary in the next section.

First, we give materials of our setting. Let (E, h) be a vector bundle over a compact Riemannian manifold (M, g), with the inner product h and a connection ∇ compatible to h, i.e.,

$$Xh(s,t) = h(\nabla_X s, t) + h(s, \nabla_X t), \quad X \in \mathfrak{X}(M), s,\, t \in \Gamma(E).$$

Let $\overline{\Delta} = \nabla^*\nabla$ be the rough Laplacian acting on $\Gamma(E)$, where ∇^* is the L^2-adjoint of ∇ with respect to the inner product given by

$$(s,t) = \int_M h(s,t)\, v_g, \quad s,\, t \in \Gamma(E).$$

Since $\overline{\Delta}$ is a selfadjoint elliptic operator acting on $\Gamma(E)$, the spectrum of $\overline{\Delta}$ consists of a countable set of eigenvalues with finite multiplicities. Let $\Gamma_\lambda(E) = \{s \in \Gamma(E); \overline{\Delta} s = \lambda s\}$ for some nonnegative real number λ, and $P_\lambda : \Gamma(E) \to \Gamma_\lambda(E)$, the projection, respectively. Let us denote also by $\mathcal{H} = \Gamma_0(E)$, the space of harmonic sections with respect to $\overline{\Delta}$, and $H = P_0$, the harmonic projection onto $\mathcal{H}$. The Green operator $G : \Gamma(E) \to \Gamma(E)$ is defined by $G = \sum_{\lambda > 0} \frac{1}{\lambda} P_\lambda$. Then, it holds that

$$I = H + \overline{\Delta}\, G = H + G\, \overline{\Delta} \quad \text{on} \quad \Gamma(E), \tag{33}$$

where I is the identity operator of $\Gamma(E)$. The Green operator G has the distributional kernel, called the Green kernel, $K_G \in \mathcal{D}'(E \otimes E')$, which satisfies that $\langle s', Gt\rangle = \int_M h(Gt, s) v_g = \langle K_G,\, s' \otimes t\rangle$. Here E' is the dual bundle of E, and the identification $\Gamma(E) \ni s \mapsto s' \in \Gamma(E')$ is given by $\langle s', t\rangle = \int_M h(t,s)\, v_g$.

Let $\{\lambda_i\}$ be a complete set of the eigenvalues of $\overline{\Delta}$ counted with their multiplicities, and let $\{\varphi_i\}$ be a complete orthonormal system of $L^2(E)$ which are the eigensections of $\overline{\Delta}$ corresponding to the eigenvalue λ_i, where $L^2(E)$ is the L^2 space of sections of E with respect to $(\ ,\ )$. Then, K_G can be expressed as

$$K_G = \sum_{\lambda_i > 0} \frac{1}{\lambda_i}\, \varphi_i \otimes {\varphi_i}' = \int_0^\infty (k_t - H) dt, \tag{34}$$

where $k_t \in \Gamma(E \otimes E')$ is the heat kernel of $\overline{\Delta}$ which is given by

$$k_t(x,y) = \sum_{\lambda_i \geq 0} e^{-\lambda_i\, t} \varphi_i(x) \otimes \varphi_i'(y) \quad (t > 0,\ x, y \in M). \tag{35}$$

Then, we have

Theorem 5.1 (cf. [25]).

(1) *The singular support of the Green kernel K_G is included in the diagonal set $\{(x,x);\ x \in M\}$ in $M \times M$.*

(2) *The pointwise norm $|K_G(x,y)|$ satisfies that*

$$|K_G(x,y)| \leq \begin{cases} \dfrac{C_1}{r(x,y)^{d-2}} & (x \neq y),\ d > 2, \\ C_2 \log \dfrac{1}{r(x,y)} + C_3 & (x \neq y),\ d = 2, \end{cases} \tag{36}$$

where $d = \dim M$ and $r(x,y)$ is the Riemannian distance between x and y.

(3) *The pointwise norms of* $\nabla_x K_G(x,y)$ *and* $\nabla_y K_G(x,y)$ *satisfies*

$$|\nabla_x K_G(x,y)|,\ |\nabla_y K_G(x,y)| \leq \frac{C_4}{r(x,y)^{d-1}} \quad (x \neq y),\ d \geq 2, \tag{37}$$

where we denote by the same symbol the connection on E' *induced from the connection* ∇ *on* E.

Proof. For a case of functions on M, see [3], p. 108, Theorem 4.13. For the case of $\Gamma(E)$, see [25], p. 30. □

Then, we have

Theorem 5.2. *Let* p *be a real number with* $p > d = \dim(M)$.

(1) *For all* $\varphi \in \Gamma(E)$,

$$\sup_{x\in M} |\nabla\varphi|_x \leq \sup_{x\in M} \|\,|\nabla_x K_G(x,\cdot)|\,\|_{L^{\frac{p}{p-1}}} \|\overline{\Delta}\varphi\|_{L^p}. \tag{38}$$

(2) *Assume that* $H\varphi = 0$ *and* $|\nabla\varphi| \in L^p$. *Then,*

$$\sup_{x\in M} |\varphi|_x \leq \sup_{x\in M} \|\,|\nabla_x K_G(x,\cdot)|\,\|_{L^{\frac{p}{p-1}}} \|\nabla\varphi\|_{L^p}. \tag{39}$$

Proof. The proof goes by a similar way as Theorem 3.67 in [3], p. 91 in the case of functions.

(1) Every $\varphi \in \Gamma(E)$ is decomposed into $\varphi = H\varphi + G\,\overline{\Delta}\varphi$. So, we have, since $\nabla H\varphi = 0$,

$$(\nabla\varphi)(x) = \nabla_x(H\varphi) + \nabla_x G\,\overline{\Delta}\varphi = \int_M \nabla_x K_G(x,y)\,(\overline{\Delta}\varphi)(y)\,v_g(y). \tag{40}$$

By Hölder inequality, we have

$$|\nabla\varphi(x)| \leq \int_M |\nabla_x K_G(x,y)|\,|\overline{\Delta}\varphi(y)|\,v_g(y) \leq \|\,|\nabla_x K_G(x,\cdot)|\,\|_q\,\|\overline{\Delta}\varphi\|_p, \tag{41}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. By Theorem 5.1 (3),

$$\|\,|\nabla_x K_G(x,\cdot)|\,\|_q \leq C_4 \left[\int_M r(x,\cdot)^{q(1-d)}\,v_g(\cdot)\right]^{1/q} \tag{42}$$

which is finite if and only if $p > d = \dim(M)$ since the volume element v_g is expressed locally as $C\,r(x,\cdot)^{d-1} dr(x,\cdot) d\omega$ in terms of the polar coordinate around x, $(r(x,\cdot),\omega) \in (0,c_x) \times S^{d-1}$. Here c_x is the injectivity radius from x and $d\omega$ is a canonical measure on the unit sphere S^{d-1}.

(2) Assume that $H\varphi = 0$ and $|\nabla\varphi| \in L^p$ with $p > d = \dim(M)$. Then, we have

$$|\varphi(x)| = |\int_M K_G(x,y)\overline{\Delta}\varphi(y)\, v_g(y)| = |\int_M \langle \nabla_y K_G(x,y), \nabla\varphi(y)\rangle\, v_g(y)|$$
$$\leq \|\, |\nabla K_G(x,\cdot)|\, \|_{L^q}\, \|\nabla\varphi\|_{L^p}, \tag{43}$$

where $q = \frac{p}{p-1}$. By the same reason as (1), $\|\, |\nabla K_G(x,\cdot)|\, \|_{L^q}$ is finite if and only if $p > d$.

□

Remark 5.3. This method does never work for the estimate of $|\nabla\nabla\varphi|$. Indeed, by the similar way, we have

$$|\nabla\nabla\varphi(x)| \leq \|\, |\nabla\nabla\, K_G(x,\cdot)|\, \|_q\, \|\overline{\Delta}\varphi\|_p$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Here, since we also have $|\nabla\nabla\, K_G(x,y)| \leq C r(x,y)^{-d}$, it should be concluded that $\|\, |\nabla\nabla\, K_G(x,\cdot)|\, \|_q$ is finite if and only if $q < 1$ which never happens because $1 > 1 - \frac{1}{p} = \frac{1}{q} > 1$.

6. A priori bounds of solutions to the Seiberg–Witten equations

In this section, we show *a priori* pointwise bounds of all solutions of the Seiberg–Witten equations.

We first have the regularity theorem on solutions of the Seiberg–Witten equations of which proof is given in the appendix. Indeed, our proof is different from the proof of Lemma 5.2.1 in [16] p. 77, since it seems that the proof in [16] would be incomplete.

Theorem 6.1. *Every solution to the Seiberg–Witten equations is gauge equivalent to a C^∞ solution.*

Then, we have *a priori* estimate of solutions of the Seiberg–Witten equations as follows.

Lemma 6.2. *Let (M, g) be a compact Kähler manifold of complex dimension $n \geq 2$. Assume that (A, ψ) is a solution of the Seiberg–Witten equation for a* $\mathrm{Spin}^c(2n)$*-structure $\widetilde{P}$ over (M, g). Then, we have a similar formula as in* [16], *p. 76. I.e.,*

$$\|\widetilde{\nabla}^A\psi\|_{L^2}{}^2 + \frac{1}{4}\langle\kappa, \psi\rangle_{L^2} + 2\left(1 - \frac{1}{2^{n-1}}\right)\|\psi\|_{L^4}{}^4 = 0.$$

In particular, if we set $\kappa_M^- = \sup_{x\in M}\max\{0, -\kappa(x)\}$*, where* $\kappa(x)$ *is scalar curvature of* (M, g) *at* $x \in M$*, then,*

$$\kappa_M^- \,\|\psi\|_{L^2}{}^2 \geq 8\left(1 - \frac{1}{2^{n-1}}\right)\,\|\psi\|_{L^4}{}^4.$$

Proof. Let us recall the Seiberg–Witten equations (22) in Theorem 3.12. Since $\mathfrak{D}_A\psi = 0$, due to Lemma 3.9 (Weitzenböck formula), we have

$$0 = \mathfrak{D}_A{}^*\mathfrak{D}_A\psi = \widetilde{\nabla}^{A*}\widetilde{\nabla}^A\psi + \frac{1}{4}\kappa\psi + 2F_A\cdot\psi. \tag{44}$$

Since $c^+(F_A) = (\psi\otimes\psi^*)_0$, due to Definition 3.1 and Lemma 3.5, we have

$$F_A\cdot\psi = \left(1 - \frac{1}{2^{n-1}}\right)\,|\psi|^2\,\psi. \tag{45}$$

We have

$$0 = \widetilde{\nabla}^{A*}\widetilde{\nabla}^A\psi + \frac{1}{4}\,\kappa\,\psi + 2\left(1 - \frac{1}{2^{n-1}}\right)\,|\psi|^2\,\psi. \tag{46}$$

Then,

$$\begin{aligned} 0 &= \int_M \left\{\langle\widetilde{\nabla}^{A*}\widetilde{\nabla}^A\psi, \psi\rangle + \frac{1}{4}\kappa\,|\psi|^2 + 2\left(1 - \frac{1}{2^{n-1}}\right)|\psi|^4\right\}\,v_g \\ &= \|\widetilde{\nabla}^A\psi\|_{L^2}{}^2 + \frac{1}{4}\int_M \kappa|\psi|^2\,v_g + 2\left(1 - \frac{1}{2^{n-1}}\right)\,\|\psi\|_{L^4}{}^4. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{\kappa_M^-}{4}\,\|\psi\|_{L^2}{}^2 &\geq \frac{1}{4}\int_M(-\kappa)\,|\psi|^2\,v_g = \|\widetilde{\nabla}^A\psi\|_{L^2}{}^2 + 2\,\left(1 - \frac{1}{2^{n-1}}\right)\,\|\psi\|_{L^4}{}^4 \\ &\geq 2\,\left(1 - \frac{1}{2^{n-1}}\right)\,\|\psi\|_{L^4}{}^4. \end{aligned}$$

We have the lemma. □

Theorem 6.3. *Let* (M, g) *be a compact Kähler manifold of complex dimension* $n \geq 2$*, and let* (A, ψ)*, a solution of the Seiberg–Witten equation. Then,*

(1) *If* $\kappa \geq 0$*, then* $\psi \equiv 0$*.*

(2) *We have*

$$\sup_{x\in M}|\psi(x)|^2 \;\leq\; \frac{2^{n-2}}{2^{n-1}-1}\,\kappa_M^-. \tag{47}$$

(3) *Let us decompose F_A into $F_A = F_A^+ + F_A^-$, where $F_A^+ \in \bigwedge^{2,0} \oplus \bigwedge^{0,2}$ and $F_A^- \in \bigwedge^{1,1}$, and let ΛF_A is the trace of F_A. Then, we have,*

$$\begin{aligned}
&\sup_{x\in M}\left\{|F_A|^2(x) + |\Lambda F_A|^2(x)\right\} \leq \frac{1}{2^{n-1}-1}\,(\kappa_M^-)^2 \quad (n \geq 3),\\
&\sup_{x\in M}\left\{|F_A^+|^2(x) + |\Lambda F_A|^2(x)\right\} \leq \frac{1}{2}\,(\kappa_M^-)^2 \quad (n = 2).
\end{aligned} \tag{48}$$

(4) *Assume that $p > 2n = \dim_{\mathbf{R}} M$. Then, we have*

$$\begin{aligned}
\sup_{x\in M} |\widetilde{\nabla}^A\psi|(x) &\leq \sup_{x\in M} \|\,|\nabla_x K_G(x,\cdot)|\,\|_{L^{\frac{p}{p-1}}}\\
&\quad\times\left(\frac{1}{4}|\kappa_M| + \kappa_M^-\right)\left\{\left(\frac{2^{n-2}}{2^{n-1}-1}\right)\kappa_M^-\right\}^{\frac{1}{2}}\mathrm{Vol}(M,g),
\end{aligned} \tag{49}$$

where $|\kappa_M| = \sup_{x\in M}|\kappa(x)|$, and $\mathrm{Vol}(M,g)$ is the volume of (M,g).

Proof. (1) We have

$$0 = \frac{\kappa_M^-}{4}\,\|\psi\|_{L^2}{}^2 \geq 2\left(1 - \frac{1}{2^{n-1}}\right)\|\psi\|_{L^2}{}^4 \geq 0.$$

Since $\kappa_M^- = 0$, we have $\psi = 0$.

(2) By (44) and (45), we have for every $x \in M$,

$$\langle \widetilde{\nabla}^{A*}\widetilde{\nabla}^A\psi, \psi\rangle\,(x) + \frac{1}{4}\,\kappa(x)\,|\psi(x)|^2 + 2\left(1 - \frac{1}{2^{n-1}}\right)|\psi(x)|^4 = 0. \tag{50}$$

In particular, the first term of the left hand side of (50) is a real number. Taking an orthonormal local frame field of (M,g) around x, $\{e_i\}_{i=1}^{2n}$ which is parallel, i.e., $\widetilde{\nabla} e_i = 0$ with respect to the Levi–Civita connection $\widetilde{\nabla}$ of (M,g), we have

$$\begin{aligned}
&-\sum_{i=1}^{2n} e_i{}^2\langle\psi,\psi\rangle\,(x)\\
&\quad= -\sum_{i=1}^{2n}\left\{\langle\widetilde{\nabla}^A{}_{e_i}\,\widetilde{\nabla}^A{}_{e_i}\psi,\psi\rangle + 2\,\langle\widetilde{\nabla}^A{}_{e_i}\psi,\widetilde{\nabla}^A{}_{e_i}\psi\rangle + \langle\psi,\widetilde{\nabla}^A{}_{e_i}\,\widetilde{\nabla}^A{}_{e_i}\psi\rangle\right\},\\
&\quad= \langle\widetilde{\nabla}^{A*}\widetilde{\nabla}^A\psi,\psi\rangle - 2\sum_{i=1}^{2n}|\widetilde{\nabla}^A{}_{e_i}\psi|^2 + \langle\psi,\widetilde{\nabla}^{A*}\widetilde{\nabla}^A\psi\rangle\\
&\quad= -2\sum_{i=1}^{2n}|\widetilde{\nabla}^A{}_{e_i}\psi|^2 + 2\,\langle\widetilde{\nabla}^{A*}\widetilde{\nabla}^A\psi\rangle,
\end{aligned} \tag{51}$$

which yields that

$$\Delta_g\,|\psi|^2 + 2\sum_{i=1}^{2n} |\widetilde{\nabla}^A{}_{e_i}\psi|^2 = 2\,\langle \widetilde{\nabla}^{A*}\widetilde{\nabla}^A\psi,\psi\rangle, \tag{52}$$

where Δ_g is the (positive) Laplacian of (M,g).

Now take a point $x_0 \in M$ satisfying that $|\psi(x_0)|^2 = \sup_{x\in M}|\psi(x)|^2$. Then,

$$\Delta_g\,|\psi|^2\,(x_0) \geq 0.$$

Together with (50), we have

$$0 \leq \langle \widetilde{\nabla}^{A*}\widetilde{\nabla}^A\psi,\psi\rangle\,(x_0) = -\frac{1}{4}\kappa(x_0)\,|\psi(x_0)|^2 - 2\left(1-\frac{1}{2^{n-1}}\right)\,|\psi(x_0)|^4,$$

which implies that

$$|\psi(x_0)|^2\left\{\frac{\kappa(x_0)}{4} + 2\left(1-\frac{1}{2^{n-1}}\right)\,|\psi(x_0)|^2\right\} \leq 0. \tag{53}$$

Thus, either

$$\psi(x_0) = 0 \tag{54}$$

or

$$\frac{\kappa(x_0)}{4} + 2\left(1-\frac{1}{2^{n-1}}\right)\,|\psi(x_0)|^2 \leq 0 \tag{55}$$

occur.

In the case (54), it holds that $\psi \equiv 0$.

In the case (55), it holds that

$$\sup_{x\in M}|\psi(x)|^2 = |\psi(x_0)|^2 \leq \frac{2}{2\left(1-\frac{1}{2^{n-1}}\right)}\left(-\frac{\kappa(x_0)}{4}\right) \leq \frac{2^{n-2}}{2^{n-1}-1}\kappa_M^-, \tag{56}$$

so that we have (2).

(3) By the equation $c^+(F_A) = (\psi\otimes\psi^*)_0$, we have by Lemma 3.5,

$$|c^+(F_A)|^2(x) = \left(1-\frac{1}{2^{n-1}}\right)\,|\psi(x)|^4. \tag{57}$$

By Definition 3.1, Lemma 3.4 and Theorem 2.3, we have

$$\begin{aligned} 2^{n-3}\left\{|F_A|^2(x) + |\Lambda F_A|^2(x)\right\} &= \left(1-\frac{1}{2^{n-1}}\right)|\psi(x)|^4, \quad (n\geq 3), \\ |F_A^+|^2(x) + |\Lambda F_A|^2(x) &= \frac{1}{2}|\psi(x)|^4, \quad (n=2) \end{aligned} \tag{58}$$

Then, by (57) together with (2), we have (3).

(4) By (46), we have

$$\widetilde{\nabla}^{A*}\widetilde{\nabla}^{A}\psi = f, \tag{59}$$

where

$$f = -\frac{1}{4}\,\kappa\,\psi - 2\left(1 - \frac{1}{2^{n-1}}\right)|\psi|^2\,\psi. \tag{60}$$

Now we apply Theorem 5.2 (1) to (59), (60). For all $p > 2n$, we have

$$\sup_{x\in M} |\widetilde{\nabla}^{A}\psi|(x) \leq C_p \|\widetilde{\nabla}^{A*}\widetilde{\nabla}^{A}\psi\|_{L^p} = C_p\|f\|_{L^p}, \tag{61}$$

where $C_p = \sup_{x\in M} \|\, |\nabla_x K_G(x,\cdot)|\,\|_{L^{\frac{p}{p-1}}}$, and

$$\begin{aligned}\|f\|_{L^p} &\leq \left(\int_M |\psi|^p \left(\frac{1}{4}|\kappa| + 2\left(1 - \frac{1}{2^{n-1}}\right)|\psi|^2\right)^p v_g\right)^{1/p} \leq \left(\frac{1}{4}|\kappa_M| + \kappa_M^-\right)\|\psi\|_{L^p} \\ &\leq \left(\frac{1}{4}|\kappa_M| + \kappa_M^-\right)\left\{\left(\frac{2^{n-2}}{2^{n-1}-1}\right)\kappa_M^-\right\}^{\frac{1}{2}} \mathrm{Vol}(M,g),\end{aligned} \tag{62}$$

by $2\big(1 - \frac{1}{2^{n-1}}\big)|\psi|^2 \leq \kappa_M^-$ and (2). We have (4). □

7. L^p_ℓ-gauge fixing lemma

7.1. Gauge transformation. Let us recall definition of $\mathcal{G}(\widetilde{P})$ (cf. [16]).

Definition 7.1. The gauge transformation group $\mathcal{G}(\widetilde{P})$ is defined by the set of all C^∞ bundle automorphisms σ of the principal $\mathrm{Spin}^c(2n)$-bundle $\widetilde{P}$ which cover the identity on the orthonormal frame bundle P over a compact Kähler manifold (M,g) of complex dimension $n \geq 2$. $\mathcal{G}(\widetilde{P})$ is isomorphic with

$$C^\infty(M, U(1)) = \{\sigma;\ M \to U(1), C^\infty \text{ maps}\}.$$

$\mathcal{G}(\widetilde{P})$ acts on $\Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$ and $\Gamma(\mathcal{L})$ as follows.

The action of $\mathcal{G}(\widetilde{P})$ on $\Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$ is given, for $\sigma \in \mathcal{G}(\widetilde{P})$ and $\psi \in \Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$,

$$(S^+(\sigma)\,\psi)(p) := \sigma(\pi(p))\widetilde{\psi}(p) = [p, \sigma(\pi(p))\widetilde{\psi}(p)], \quad (p \in \widetilde{P}), \tag{63}$$

where $\psi : \widetilde{P} \to M$ is the projection and $\psi \in \Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$ is regarded as a C^∞ mapping ψ from $\widetilde{P}$ to $\Delta^+_{\mathbb{C}}$ satisfying $\psi(pa) = \rho(a^{-1})\psi(p)$, ($a \in \mathrm{Spin}^c(2n)$, $p \in \widetilde{P}$), $\rho : \mathrm{Spin}^c(2n) \to GL(\Delta^+_{\mathbb{C}})$ is the complex half-spin representation as in 3.2, and the right hand side of (63) is the structure group action of $\mathrm{Spin}^c(2n) \supset U(1)$ on $\widetilde{P}$. We denote by $S^+(\sigma)\,\psi$ simply $\sigma\,\psi$, sometimes.

The action of $\mathcal{G}(\widetilde{P})$ on $\Gamma(\mathcal{L})$ is given by the following way: For $\sigma \in \mathcal{G}(\widetilde{P})$ and $u \in \Gamma(\mathcal{L})$, define $\det \sigma\, u \in \Gamma(\mathcal{L})$ by

$$(\det \sigma\, u)(x) := \sigma(x)^m\, u(x) \quad (x \in M), \tag{64}$$

where $m = 2^{n-1}$.

Then, $\mathcal{G}(\widetilde{P})$ acts on the spaces of connections on $S^+_{\mathbb{C}}(\widetilde{P})$ and $\mathcal{L}$, respectively, by usual way as follows: For $\sigma \in \mathcal{G}(\widetilde{P})$ and a connection on $S^+_{\mathbb{C}}(\widetilde{P})$, $\widetilde{\nabla}^A$, let us define a connection $\widetilde{\nabla}^{\sigma^* A}$ by

$$\widetilde{\nabla}^{\sigma^* A}{}_X \psi = \sigma^{-1}(\widetilde{\nabla}^A{}_X(\sigma\,\psi)), \quad (X \in \mathfrak{X}(M),\ \psi \in \Gamma(S^+_{\mathbb{C}}(\widetilde{P}))). \tag{65}$$

By a similar way, for $\sigma \in \mathcal{G}(\widetilde{P})$ and a connection on $\mathcal{L}$, ∇^A, let us define a connection $\nabla^{\sigma^* A}$ by

$$\nabla^{\sigma^* A}{}_X u = \sigma^{-1}(\nabla^A{}_X(\sigma\, u)), \quad (X \in \mathfrak{X}(M),\ u \in \Gamma(\mathcal{L}))), \tag{66}$$

and also, a connection $\nabla^{\det \sigma^* A}$ by

$$\nabla^{\det \sigma^* A}{}_X u = (\det \sigma)^{-1}(\nabla^A{}_X(\det \sigma\, u)), \quad (X \in \mathfrak{X}(M),\ u \in \Gamma(\mathcal{L}))), \tag{67}$$

respectively.

Since $\widetilde{\nabla}^A = \widetilde{\nabla} + A$ on $\Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$ (cf. Lemma 3.6), if we write as $\widetilde{\nabla}^{\sigma^* A} = \widetilde{\nabla} + \sigma^* A$, we have

$$\sigma^* A = A + \sigma^{-1} d\sigma. \tag{68}$$

Since $\nabla^A = \nabla^0 + A$ on $\Gamma(\mathcal{L})$ (cf. Lemma 3.7), if we write as $\nabla^{\sigma^* A} = \nabla^0 + \sigma^* A$, and $\nabla^{\det \sigma^* A} = \nabla^0 + \det \sigma^* A$, we have also

$$\begin{cases} \sigma^* A = A + \sigma^{-1} d\sigma, \\ \det \sigma^* A = A + \det \sigma^{\,-1} d\, \det \sigma = A + m\, \sigma^{-1} d\sigma, \end{cases} \tag{69}$$

where $m = 2^{n-1}$.

Definition 7.2 (cf. [16] pp. 57, 60). The configuration space $\mathcal{C}(\widetilde{P})$ is defined by the space of all pairs (A, ψ), where A is $\sqrt{-1}\mathbb{R}$-valued C^∞ 1-forms on M and $\psi \in \Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$. The gauge transformation group $\mathcal{G}(\widetilde{P})$ acts on the configuration space $\mathcal{C}(\widetilde{P})$ by

$$(A, \psi) \cdot \sigma = (\sigma^* A, S^+(\sigma^{-1})\psi), \quad (\sigma \in \mathcal{G}(\widetilde{P}),\ (A, \psi) \in \mathcal{C}(\widetilde{P})). \tag{70}$$

7.2. L^p_ℓ-gauge fixing lemma. In this subsection, let F be a vector bundle with a metric h and a smooth connection ∇ compatible to h over a compact Riemannian manifold (M,g). We denote the Sobolev L^p_ℓ-space of sections of F, by $L^p_\ell(F)=\{\varphi;\varphi \text{ is a section of } F,\ \|\varphi\|_{L^p_\ell}<\infty\}$, where the Sobelev L^p_ℓ-norm $\|\ \|_{L^p_\ell}$ is defined by

$$\|\varphi\|_{L^p_\ell}=\left(\sum_{k=0}^{\ell}\int_M \Big|\overbrace{\nabla\ldots\nabla}^{k}\varphi\Big|^p v_g\right)^{1/p},$$

for nonnegative integer ℓ and real number p with $p\geq 1$.

We also have the L^p_{k+1}-gauge group $\mathcal{G}^p_{k+1}(\widetilde{P})$ is isomorphic to the space $L^p_{k+1}(M,U(1))$. Then, we have

Theorem 7.3 (L^p_ℓ-gauge fixing lemma)**.** *Let (M,g) be a compact Kähler manifold of complex dimension $n\geq 2$ and $\widetilde{P}$ the principal* $\mathrm{Spin}^c(2n)$*-bundle over (M,g), and $\mathcal{L}$ its the determinant bundle, respectively. Assume that A_0 be a arbitrarily fixed $\sqrt{-1}\mathbb{R}$-valued C^∞ 1-form on M, i.e., ∇^{A_0} is a C^∞ connection on $\mathcal{L}$, and $p>2n=\dim_{\mathbb{R}} M$. Then, for every integer $\ell\geq 1$, there exist positive constants K and C depending only on (M,g), A_0 and ℓ such that, for every L^p_ℓ-unitary connection ∇^A on $\mathcal{L}$, there exists a $L^p_{\ell+1}$-gauge transformation σ of $\widetilde{P}$ such that*

$$\det\sigma^*A=A_0+\alpha \quad or \quad \sigma^*A=A_0+\alpha,$$

and $\alpha\in L^p_\ell(T^*M\otimes\sqrt{-1}\mathbb{R})$ satisfies that

$$\begin{cases}\delta\alpha=0,\\ \|\alpha\|_{L^p_\ell}\leq C\|F_A^+ +\dfrac{\Lambda(F_A)}{n}\,\Phi\|_{L^p_{\ell-1}}+K,\end{cases}\tag{71}$$

where Φ is the Kähler form of (M,g), F_A is the curvature of ∇^A and δ is the L^2-adjoint of the exterior differentiation d.

7.3. Proof of L^p_ℓ-gauge fixing lemma. The proof goes by a similar way as in [16].

- For every L^p_ℓ-unitary connection A on $\mathcal{L}$, and $\sigma\in\mathcal{G}^p_{\ell+1}(\widetilde{P})$, we write as

$$\det\sigma^*A=A+m\,\sigma^{-1}d\sigma,\tag{72}$$

where $m=2^{n-1}$, and also as

$$A=A_0+\alpha_0,$$

where $\alpha_0\in L^p_\ell(T^*M\otimes\sqrt{-1}\mathbb{R})$. $\delta\alpha_0\in L^p_{\ell-1}(M,\sqrt{-1}\mathbb{R})$ is L^2-orthogonal to the constant functions on M. Let us define the space $\mathcal{I}_{L^p_{\ell-1}}$ by

$$\mathcal{I}_{L^p_{\ell-1}}=\left\{f\in L^p_{\ell-1}(M,\sqrt{-1}\mathbb{R});\ \int_M f\,v_g=0\right\}.$$

Then, there exists a bounded linear operator

$$G = \Delta^{-1}; \quad \mathcal{I}_{L^p_{\ell-1}} \to \mathcal{I}_{L^p_{\ell+1}}.$$

So let us define

$$s_0 := -\frac{1}{m}\Delta^{-1}(\delta\alpha_0) \in \mathcal{I}_{L^p_{\ell+1}},$$

and define a $L^p_{\ell+1}$-gauge transformation σ by

$$\sigma := \exp(s_0) \in L^p_{\ell+1}(M, U(1)).$$

Put

$$\alpha_1 := \alpha_0 + m\, ds_0 \in L^p_\ell(T^*M \otimes \sqrt{-1}\mathbb{R}).$$

Then, we have

$$\begin{cases} \det \sigma^* A = A_0 + \alpha_1, \\ \delta\alpha_1 = 0. \end{cases} \tag{73}$$

Indeed, for the first equation, we have $d\sigma = \exp(s_0)\, ds_0$, so that $ds_0 = \sigma^{-1}d\sigma$. Then, we have

$$\det \sigma^* A = A + m\, ds_0 = A_0 + \alpha_0 + m\, ds_0 = A_0 + \alpha_1.$$

For the second equation, we have

$$\delta\alpha_1 = \delta\alpha_0 + m\,\delta ds_0 = \delta\alpha_0 + m\,\delta d\left(-\frac{1}{m}\Delta^{-1}(\delta\alpha_0)\right) = \delta\alpha_0 - \Delta\Delta^{-1}(\delta\alpha_0) = 0.$$

It is the same for the case $\alpha_1 = \alpha_0 + ds_0$. We have (73).

• Notice that $\delta : L^p_\ell(T^*M \otimes \sqrt{-1}\mathbb{R}) \to L^p_{\ell-1}(M, \sqrt{-1}\mathbb{R})$. Next, we consider the operator

$$d^+ := P_+ \circ d, \tag{74}$$

where

$$d : L^p_\ell\big(T^*M \otimes \sqrt{-1}\mathbb{R}\big) \to L^p_{\ell-1}\big(\textstyle\bigwedge^2 T^*M \otimes \sqrt{-1}\mathbb{R}\big),$$

and according to the decomposition (cf. [10], p. 247)

$$L^p_{\ell-1}\big(\textstyle\bigwedge^2 T^*M \otimes \sqrt{-1}\mathbb{R}\big) = L^p_{\ell-1}(B^2_+) \oplus L^p_{\ell-1}(B^2_-), \tag{75}$$

P_+ (resp. P_-) are the projections of $L^p_{\ell-1}(\bigwedge^2 T^*M \otimes \sqrt{-1}\mathbb{R})$ onto $L^p_{\ell-1}(B^2_+)$, (resp. $L^p_{\ell-1}(B^2_-)$) respectively, where $L^p_{\ell-1}(B^2_+)$ is the dirct sum of $L^p_{\ell-1}$-space of the pure imaginary valued forms in $\Lambda^{2,0} \oplus \Lambda^{0,2}$ and $L^p_{\ell-1}(M, \sqrt{-1}\mathbb{R})\,\Phi$, ($\Phi$ is the Kähler form of (M, g)) and $L^p_{\ell-1}(B^2_-)$ is $L^p_{\ell-1}$-space of pure imaginary valued forms in $\Lambda_0^{1,1} = \{\varphi \in \Lambda^{1,1}; \Lambda\varphi = 0\}$. Then, we have (cf. [10]) that, for all $b \in L^p_\ell(T^*M \otimes \sqrt{-1}\mathbb{R})$,

$$d^+b = 0 \iff db \in L^p_{\ell-1}(\Lambda_0^{1,1}), \tag{76}$$

$$d^+b = 0 \text{ and } \delta b = 0 \iff b \text{ is a harmonic 1-form.} \tag{77}$$

Indeed, if we decompose $b = b' + b''$, where $b' = \sum_i b'_i dz_i$ and $b'' = \sum_i b''_i d\overline{z_i}$. Then, we have $b' = -{}^t\overline{b''}$, and

$$db \in \Lambda^{1,1} \iff d''b'' = 0,$$

$$\Lambda(db) = 0,\ \delta b = 0 \iff \delta''b'' = 0,$$

so we have the equivalence of (77).

- For every $\alpha_1 \in L^p_\ell(T^*M \otimes \sqrt{-1}\mathbb{R})$, is decomposed into

$$\alpha_1 = h + \beta,$$

where h is a harmonic 1-form which is C^∞, and β is L^2-orthogonal for all harmonic 1-forms. Here, since $p\ell > \dim_{\mathbb{R}} M$, $L^p_\ell(T^*M \otimes \sqrt{-1}\mathbb{R})$ is contained in the space of continuous pure imaginary valued 1-forms on M, so we have the above decomposition.

Notice that (δ, d^+) is an elliptic operator (cf. [10], p. 247), because our case is the trivial bundle case, so Proposition (2.19) and Lemma (2.20) in [10] hold in this case.

Thus, there exists a positive constant C depending only on (M, g), ℓ and p such that

$$\|\beta\|_{L^p_\ell} \leq C\,\|(\delta\beta, d^+\beta)\|_{L^p_{\ell-1}} \leq C\|d^+\beta\|_{L^p_{\ell-1}}, \tag{78}$$

Because by (73) and h is harmonic, we have

$$0 = \delta\alpha_1 = \delta h + \delta\beta = \delta\beta.$$

Furthermore, since h is harmonic,

$$F_A = F_{\det\sigma^* A} = F_{A_0+\alpha_1} = F_{A_0} + d\alpha_1 = F_{A_0} + dh + d\beta = F_{A_0} + d\beta,$$

so we have

$$d^+\beta = F_A^+ - F_{A_0}^+ + \frac{1}{n}(\Lambda(F_A) - \Lambda(F_{A_0}))\,\Phi.$$

Therefore,

$$\|d^+\beta\|_{L^p_{\ell-1}} \le \|F_A^+ + \frac{1}{n}\Lambda(F_A)\,\Phi\|_{L^p_{\ell-1}} + K, \tag{79}$$

where $K := \|F_{A_0} + \frac{1}{n}\Lambda(F_{A_0})\,\Phi\|_{L^p_{\ell-1}}$ which is a constant. Thus, together with (78), we have

$$\|\beta\|_{L^p_\ell} \le C\|F_A^+ + \frac{1}{n}\Lambda(F_A)\,\Phi\|_{L^p_{\ell-1}} + K. \tag{80}$$

- (the harmonic part h of α_1) We need

Lemma 7.4. *For a pure imaginary valued harmonic* 1*-form* h_0 *on* (M, g) *with periods in* $2\pi\sqrt{-1}\mathbb{Z}$*, there exists a* $U(1)$*-valued harmonic function* φ *on* (M, g) *such that* $d\varphi = h_0$.

(cf. [16] p. 81, *Claim 5.3.2.*)

Since the quotient space {pure imaginary valued harmonic 1-forms on (M, g)}/ {pure imaginary valued harmonic 1-forms with periods $2\pi\sqrt{-1}\mathbb{Z}$} is a compact torus, there exists a positive constant K_2 depending only on (M, g), ℓ and p such that, for every pure imaginary valued harmonic 1-form h on (M, g), there exist a harmonic 1-form h_1 on (M, g) with L^p_ℓ-norm $\|h_1\|_{L^p_\ell} \le K_2$, and a harmonic 1-form h_2 on (M, g) with periods in $2\pi\sqrt{-1}\mathbb{Z}$ such that

$$h = h_1 + m\,h_2.$$

Let h be the harmonic part of α_1. Then, we can write

$$h = h_1 - m\,d\varphi,$$

where h_1 is a harmonic 1-form on (M, g) with L^p_ℓ-norm, $\|h_1\|_{L^p_\ell} \le K_2$, and a $U(1)$-valued harmonic function φ on (M, g). Then, $\varphi \in \Gamma(\mathcal{G}(\widetilde{P}))$, and we have

$$\begin{aligned}\det\varphi^* A &= \det\varphi^*(A_0 + \alpha_1) = A_0 + \alpha_1 + m\,d\varphi \\ &= A_0 + h + \beta + m\,d\varphi = A_0 + h_1 + \beta = A_0 + \alpha,\end{aligned} \tag{81}$$

where we put $\alpha := h_1 + \beta$. Then, since $\delta\beta = 0$ and h_1 is harmonic,

$$\delta\beta = \delta h_1 + \delta\beta = 0, \tag{82}$$

and by (80) and $\|h_1\|_{L^p_\ell} \le K_2$,

$$\begin{aligned}\|\alpha\|_{L^p_\ell} = \|h_1 + \beta\|_{L^p_\ell} &\le \|h_1\|_{L^p_\ell} + \|\beta\|_{L^p_\ell} \le K_2 + C\|F_A^+ + \frac{1}{n}\Lambda(F_A)\Phi\|_{L^p_{\ell-1}} + K \\ &= C\|F_A^+ + \frac{1}{n}\Lambda(F_A)\,\Phi\|_{L^p_{\ell-1}} + K',\end{aligned} \tag{83}$$

where $K' = K_2 + K$. Therefore, due to (81), (82) and (83), we have the desired. It is the same for $\varphi^* A$. We have Theorem 7.3.

8. L_1^p-boundedness of F_A or F_A^+

In this section, we show L_1^p-boundedness of F_A or F_A^+ for a solution (A,ψ) of the Seiberg–Witten equation. We show

Theorem 8.1. *Let (M,g) be a compact Kähler manifold of complex dimension $n\geq 2$, and $\widetilde{P}$ the principal $\mathrm{Spin}^c(2n)$ bundle over (M,g), and $\mathcal{L}$ its determinal bundle, respectively. Let the Levi–Civita connection ∇ of (M,g), which acts also $\mathrm{End}(\mathcal{L})$-valued forms on M. Assume that $p\geq 2$. Then, there exists a positive constant C depending only on (M,g) and p such that, for every solution (A,ψ) of the Seiberg–Witten equation, if $n=\dim_{\mathbb{C}} M\geq 3$, then*

$$\|\nabla F_A\|_{L^p}{}^p+\|\nabla(\Lambda F_A)\|_{L^p}{}^p\leq C\sup_{x\in M}|\psi(x)|^{\,p}\|\widetilde{\nabla}^A\psi\|_{L^p}{}^p. \tag{84}$$

If $n=\dim_{\mathbb{C}} M=2$, then

$$\|\nabla F_A^+\|_{L^p}{}^p+\|\nabla(\Lambda F_A)\|_{L^p}{}^p\leq C\sup_{x\in M}|\psi(x)|^{\,p}\|\widetilde{\nabla}^A\psi\|_{L^p}{}^p, \tag{85}$$

where, $\|\nabla F_A\|_{L^p}$, $\|\nabla(\Lambda F_A)\|_{L^p}$, and $\|\widetilde{\nabla}^A\psi\|_{L^p}$ are the L^p-norms, respectively.

Corollary 8.2. *Let (M,g) be a compact Kähler manifold of complex dimension $n\geq 2$, and $\widetilde{P}$ the principal $\mathrm{Spin}^c(2n)$-bundle over (M,g), and $\mathcal{L}$ its determinal bundle, respectively. Assume that $p>2n=\dim_{\mathbb{R}} M$. Then, there exists a positive constant C depending only on (M,g) and p such that, for every solution (A,ψ) of the Seiberg–Witten equation, if $n=\dim_{\mathbb{C}} M\geq 3$, then*

$$\|F_A\|_{L_1^p}{}^p+\|\Lambda(F_A)\|_{L_1^p}{}^p\leq C. \tag{86}$$

If $n=\dim_{\mathbb{C}} M=2$, then

$$\|F_A^+\|_{L_1^p}{}^p+\|\Lambda(F_A)\|_{L_1^p}{}^p\leq C. \tag{87}$$

Proof. For the proof of Theorem 8.1, assume that (A,ψ) is a solution of the Seiberg–Witten equation. Then, it holds that

$$c^+(F_A)=(\psi\otimes\psi^*)_0=\psi\otimes\psi-\frac{1}{2^{n-1}}|\psi|^2\ \mathrm{Id},$$

i.e, for all $\varphi\in\Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$,

$$F_A\,\cdot\,\varphi=\langle\varphi,\psi\rangle\,\psi-\frac{1}{2^{n-1}}\,|\psi|^2\,\varphi. \tag{88}$$

By (88), for every $X \in \mathfrak{X}(M)$,

$$\begin{aligned}\widetilde{\nabla}^A_X(F_A \cdot \varphi) &= (X\langle\varphi,\psi\rangle)\,\psi + \langle\varphi,\psi\rangle\,\widetilde{\nabla}^A_X\psi - \frac{1}{2^{n-1}}(X\,|\psi|^2)\,\varphi - \frac{1}{2^{n-1}}\,|\psi|^2\,\widetilde{\nabla}^A_X\varphi \\ &= \Big(\langle\widetilde{\nabla}^A_X\varphi,\psi\rangle + \langle\varphi,\widetilde{\nabla}^A_X\psi\rangle\Big)\,\psi + \langle\varphi,\psi\rangle\,\widetilde{\nabla}^A_X\psi \\ &\quad - \frac{1}{2^{n-1}}\,\Big(\langle\widetilde{\nabla}^A_X\psi,\psi\rangle + \langle\psi,\widetilde{\nabla}^A_X\psi\rangle\Big)\,\varphi - \frac{1}{2^{n-1}}\,|\psi|^2\,\widetilde{\nabla}^A_X\varphi. \end{aligned} \tag{89}$$

Notice here that $\mathrm{End}(\mathcal{L}) = M \times \sqrt{-1}\mathbb{R}$ since $\mathcal{L}$ is a line bundle, and for every $\mathrm{End}(\mathcal{L})$-valued forms F on M,

$$\nabla^A F = \nabla F, \tag{90}$$

for every connection ∇^A on $\mathcal{L}$, where ∇ is the Levi–Civita connection of (M, g). Because, for every $\eta \in \Gamma(\mathrm{End}(\mathcal{L}))$, it holds that $\nabla^A\eta = d\eta$, where d is the exterior differentiationon M, and if F is an $\mathrm{End}(\mathcal{L})$-valued r-form on M, $(\nabla^A_X F)$ $(X_1,\ldots,X_r)$ coincides with

$$\begin{aligned}&\nabla^A_X(F(X_1,\ldots,X_r)) - \sum_{i=1}^r F(X_1,\ldots,\nabla_X X_i,\ldots,X_r) \\ &= X(F(X_1,\ldots,X_r)) - \sum_{i=1}^r F(X_1,\ldots,\nabla_X X_i,\ldots,X_r) = (\nabla_X F)(X_1,\ldots,X_r),\end{aligned}$$

for all $X, X_1,\ldots,X_r \in \mathfrak{X}(M)$. We have (90).

Due to (90), we have

$$\widetilde{\nabla}^A(F \cdot \varphi) = (\nabla F) \cdot \varphi + F \cdot \widetilde{\nabla}^A\varphi, \tag{91}$$

for every $\mathrm{End}(\mathcal{L})$-valued form F, and $\varphi \in \Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$. In particular, we have

$$\widetilde{\nabla}^A(F^A \cdot \varphi) = (\nabla F^A) \cdot \varphi + F^A \cdot \widetilde{\nabla}^A\varphi. \tag{92}$$

Since for the second term in the right hand side of (92)

$$F_A \cdot \widetilde{\nabla}^A_X\varphi = \langle\widetilde{\nabla}^A_X\varphi,\psi\rangle\psi - \frac{1}{2^{n-1}}\,|\psi|^2\,\widetilde{\nabla}^A_X\varphi,$$

by (88), for $X \in \mathfrak{X}(M)$, we have, by using together with (89) and (92),

$$\begin{aligned}(\nabla_X F^A) \cdot \varphi &= \widetilde{\nabla}^A_X\,(F_A \cdot \varphi) - F_A \cdot (\widetilde{\nabla}^A_X\varphi) = \Big(\langle\widetilde{\nabla}^A_X\varphi,\psi\rangle + \langle\varphi,\widetilde{\nabla}^A_X\psi\rangle\Big)\,\psi \\ &\quad + \langle\varphi,\psi\rangle\,\widetilde{\nabla}^A_X\psi - \frac{1}{2^{n-1}}\,\Big(\langle\widetilde{\nabla}^A_X\psi,\psi\rangle + \langle\psi,\widetilde{\nabla}^A_X\psi\rangle\Big)\,\varphi\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{2^{n-1}} \, |\psi|^2 \, \widetilde{\nabla}^A_X \varphi - \Big\{ \langle \widetilde{\nabla}^A_X \varphi, \psi \rangle - \frac{1}{2^{n-1}} \, |\psi|^2 \, \widetilde{\nabla}^A_X \varphi \Big\} \\
&= \langle \varphi, \widetilde{\nabla}^A_X \psi \rangle \, \psi + \langle \varphi, \psi \rangle \widetilde{\nabla}^A_X \psi - \frac{1}{2^{n-1}} 2 \, \Re \langle \widetilde{\nabla}^A_X \psi, \psi \rangle \, \varphi \\
&= \psi \otimes (\widetilde{\nabla}^A_X \psi)^* \, (\varphi) + \widetilde{\nabla}^A_X \psi \otimes \psi^* \, (\varphi) - \frac{1}{2^{n-1}} 2 \, \Re \langle \widetilde{\nabla}^A_X \psi, \psi \rangle \; \mathrm{Id}(\varphi). \qquad (93)
\end{aligned}
$$

Thus, we obtain

Lemma 8.3. *It holds that, for all $X \in \mathfrak{X}(M)$,*

$$
c^+(\nabla_X F_A) = \psi \otimes (\widetilde{\nabla}^A_X \psi)^* + \widetilde{\nabla}^A_X \psi \otimes \psi^* - \frac{1}{2^{n-1}} 2 \, \Re \langle \widetilde{\nabla}^A_X \psi, \psi \rangle \; \mathrm{Id} \, . \qquad (94)
$$

If $n \geq 3$ and $p \geq 2$, due to Theorem 2.3,

$$
\begin{aligned}
\| c^+(\nabla_X F_A) \|_{L^p}{}^p &= \int_M |c^+(\nabla_X F_A)|^p \, v_g = \int_M \left(|c^+(\nabla_X F_A)|^2 \right)^{p/2} \, v_g \\
&= \int_M \left(2^{n-3} |\nabla_X F_A|^2 + 2^{n-3} |\Lambda(\nabla_X F_A)|^2 \right)^{p/2} \, v_g \\
&\geq \int_M \Big\{ (2^{n-3})^{p/2} \, |\nabla_X F_A|^p + (2^{n-3})^{p/2} \, |\nabla_X (\Lambda F_A)|^p \Big\} \, v_g \\
&= (2^{n-3})^{p/2} \| \nabla_X F_A \|_{L^p}{}^p + (2^{n-3})^{p/2} \| \nabla_X (\Lambda F_A) \|_{L^p}{}^p. \qquad (95)
\end{aligned}
$$

Here, we used

$$
\Lambda(\nabla_X F_A) = \nabla_X (\Lambda F_A), \qquad (96)
$$

which follows from that, by definition, $\Lambda(F_A) = \langle \Phi, F_A \rangle$,

$$
\nabla_X \Lambda(F_A) = X \langle \Phi, F_A \rangle = \langle \nabla_X \Phi, F_A \rangle + \langle \Phi, \nabla_X F_A \rangle = \langle \Phi, \nabla_X F_A \rangle = \Lambda(\nabla_X F_A).
$$

On the other hand, by Lemma 8.3,

$$
\begin{aligned}
\| c^+(\nabla F_A) \|_{L^p}{}^p &:= \int_M \sum_{i=1}^{2n} |c^+(\nabla_{e_i} F_A)|^p \, v_g = \int_M \sum_{i=1}^{2n} | \, \psi \otimes (\widetilde{\nabla}^A_{e_i} \psi)^* + \widetilde{\nabla}^A_{e_i} \psi \otimes \psi^* \\
&\quad - \frac{1}{2^{n-1}} \, 2 \, \Re \, \langle \langle \widetilde{\nabla}^A_{e_i} \psi, \psi \rangle \; \mathrm{Id} \; |^p \, v_g \leq C_1 \sup_{x \in M} |\psi(x)|^p \, \| \widetilde{\nabla}^A \psi \|_{L^p}{}^p, \qquad (97)
\end{aligned}
$$

where C_1 is a constant only on (M, g), n and p. Thus, together with (95) and (97), we have (84).

In the case $n = 2$, due to Theorem 2.3, we have for $F \in \Lambda^2$,

$$
|c^+(F)|^2 = |F^+|^2 + |\Lambda(F)|^2.
$$

Then, we have (85) by the similar way. The detail is omitted. □

Proof. For the proof of Corollary 8.2, assume that $p > 2n = \dim_{\mathbb{R}} M$. Then, the right hand sides of (84) and (85) are estimated by the constant depending only on (M, g) and p due to Theorem 6.3, (2) and (4). Furthermore, we have also that

$$\|F_A\|_{L^p}{}^p + \|\Lambda F_A\|_{L^p}{}^p \quad (n \geq 3); \quad \|F_A^+\|_{L^p}{}^p + \|\Lambda F_A\|_{L^p}{}^p \quad (n = 2)$$

are estimated from above by a constant depending only on (M, g) and p due to Theorem 6.3, we have Corollary 8.2. □

Corollary 8.4. *Let A_0 be any fixed $\sqrt{-1}\mathbb{R}$-valued C^∞ 1-form on M, i.e., ∇^{A_0} be a C^∞ connection on $\mathcal{L}$. Assume that $p > 2n = \dim_{\mathbb{R}} M$. Then, there exists a positive constant K_1 depending only on (M, g), A_0 and p such that, for every solution (A, ψ) of the Seiberg–Witten equation, there exists $A' = A_0 + \alpha$ which is L^p_3-gauge equivalent to A and satisfies that*

$$\delta\alpha = 0 \quad \text{and} \quad \|\alpha\|_{L^p_2} \leq K_1. \tag{98}$$

Proof. Due to Corollary 8.2, there exists a constant C such that, for any solution (A, ψ) of the Seiberg–Witten equation,

$$\begin{cases} \|F_A\|_{L^p_1}{}^p + \|\Lambda(F_A)\|_{L^p_1}{}^p \leq C, & (n \geq 3), \\ \|F_A^+\|_{L^p_1}{}^p + \|\Lambda(F_A)\|_{L^p_1}{}^p \leq C, & (n = 2), \end{cases} \tag{99}$$

On the other hand, due to Theorem 7.3 (L^p_ℓ-gauge fixing lemma), there exist constants C and K such that there exists $A' = A_0 + \alpha$ which is L^p_3-gauge equivalent to A, and satisfies that $\delta\alpha = 0$ and

$$\|\alpha\|_{L^p_2} \leq C\|F_A^+ + \frac{\Lambda(F_A)}{n}\Phi\|_{L^p_1} + K \leq C\|F_A^+\|_{L^p_1} + C_1\|\Lambda(F_A)\|_{L^p_1} + K \leq K_1 < \infty.$$

We have Corollary 8.4. □

9. L^p_ℓ-boundedness of solutions of the Seiberg–Witten equation

In this section, we show the L^p_ℓ-boundedness theorem for any solution (A, ψ) of the Seiberg–Witten equation We first show

Theorem 9.1. *Let (M,g) be a compact Kähler manifold of complex dimension $n \geq 2$, and $\widetilde{P}$ the principal $\mathrm{Spin}^c(2n)$-bundle over (M,g), and $\mathcal{L}$ its determinant bundle. Let A_0 be an arbitrary fixed C^∞ connection on $\mathcal{L}$, and let $p > 2n = \dim_{\mathbb{R}} M$. For every solution (A,ψ) of the Seiberg–Witten equation, we take α to be a $\sqrt{-1}\mathbb{R}$-valued 1-form on M satisfies that A is $L^p_{\ell+1}$-gauge equivalent to $A_0+\alpha$, $\delta\alpha = 0$, and the harmonic projection h of α is decomposed into $h = h_1 + mh_2$, where h_1 is a harmonic 1-form on (M,g) with $\|h_1\|_{L^p_2} \leq K$, and h_2 is harmonic 1-form on (M,g) with periods in $2\pi\sqrt{-1}\mathbb{Z}$. Here $m = 2^{n-1}$.*

Then, for every $\ell \geq 2$, there exists a positive constant $C(\ell)$ depending only on (M,g), A_0, ℓ and p such that, for every solution (A,ψ) of the Seiberg–Witten equation,

$$\|\alpha\|_{L^p_{\ell+1}}{}^p + \|\psi\|_{L^p_\ell}{}^p \leq C(\ell), \tag{100}$$

where the L^p_ℓ-norm for ψ is is taken with respect to $\widetilde{\nabla}^{A_0}$.

Proof. Due to Theorem 6.3 (2), $\|\psi\|_\infty \leq C_1$. By Corollary 8.4, we have $\|\alpha\|_{L^p_2} \leq C_2$. Then, as in 7.3, the corresponding gauge transform belongs to $L^p_3(M,U(1))$. Due to Theorem 6.3 (4), $\|\widetilde{\nabla}^A\psi\|_{L^p} \leq C_3$. Then, we have

$$\|\psi\|_{L^p_1} \leq C_4. \tag{101}$$

Because, since $A = A_0+\alpha$, $\widetilde{\nabla}^A\psi = \alpha\cdot\psi + \widetilde{\nabla}^{A_0}\psi$. Then,

$$\begin{aligned}\|\psi\|_{L^p_1}{}^p &:= \|\psi\|_{L^p}{}^p + \int_M |\widetilde{\nabla}^{A_0}\psi|^p v_g \leq \mathrm{Vol}(M,g)\|\psi\|_\infty{}^p + \int_M |\widetilde{\nabla}^A\psi - \alpha\cdot\psi|^p v_g \\ &\leq \mathrm{Vol}(M,g)\|\psi\|_\infty{}^p + 2^{p-1}\int_M |\widetilde{\nabla}^A\psi|^p v_g + 2^{p-1}\|\psi\|_\infty{}^p\int_M |\alpha|^p v_g \leq C_4' < \infty, \end{aligned}\tag{102}$$

where we used the inequality: $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ $(a>0,\ b>0)$ for every $p>1$.

Furthermore, we have

$$\|\psi\|_{L^p_3} \leq C_5. \tag{103}$$

Indeed, since $A = A^0+\alpha$, we have

$$0 = \mathfrak{D}_A\psi = \mathfrak{D}_{A_0}\psi + \alpha\cdot\psi,$$

so that

$$\mathfrak{D}_{A_0}\psi = -\alpha\cdot\psi. \tag{104}$$

Here, let us recall the Sobolev Multiplication Theorem (cf. [7], pp. 95–96):

$$L^p_\ell \otimes L^p_\ell \ni (\alpha,\psi) \mapsto \alpha\cdot\psi \in L^p_\ell \qquad (\ell\, p > 2n = \dim_{\mathbb{R}} M)$$

is continuous. Therefore, if $\|\alpha\|_{L_1^p} \leq C_2$ and $\|\psi\|_{L_1^p} \leq C_4$, then we have $\|\alpha \cdot \psi\|_{L_1^p} \leq C_6$. Together with (104), we have $\|\psi\|_{L_2^p} \leq C_7$. Since $\|\alpha\|_{L_2^p} \leq C_2$, again by the Sobolev Multiplication Theorem, together with $\|\psi\|_{L_2^p} \leq C_7$, we have $\|\alpha \cdot \psi\|_{L_2^p} \leq C_8$. Thus, by (104), $\|\psi\|_{L_3^p} \leq C_9$. We have (103).

Then, we have

$$\|\alpha\|_{L_4^p} \leq C_{10}. \tag{105}$$

Because, in the Seiberg–Witten equation, $c^+(F_A) = \psi \otimes \psi^* - \frac{1}{2^{n-1}} |\psi|^2 \, \mathrm{Id}$, we know $\|\psi\|_{L_3^p} \leq C_9$. Then, by using the calculation in Lemma 8.3, we have $\|F_A\|_{L_3^p} \leq C_{11}$. By Theorem 7.3 and (96), we have $\|\alpha\|_{L_4^p} \leq C_{12}$. We have (105).

Now we use induction in the bootstrapping argument. Assume that there exists $\ell \geq 3$ such that

$$\|\alpha\|_{L_{\ell+1}^p} \leq C(\ell) \quad \text{and} \quad \|\psi\|_{L_\ell^p} \leq C(\ell).$$

By Sobolev Multiplication Theorem, $\|\alpha \cdot \psi\|_{L_\ell^p} \leq C(\ell)'$. Since

$$0 = \mathfrak{D}_A \psi = \mathfrak{D}_{A_0} \psi + \alpha \cdot \psi,$$

$\|\mathfrak{D}_{A_0}\psi\|_{L_\ell^p} = \| - \alpha \cdot \psi\|_{L_\ell^p} \leq C(\ell)'$. Thus, we have $\|\psi\|_{L_{\ell+1}^p} \leq C(\ell)''$. Here, in the Seiberg–Witten equation, $c^+(F_A) = \psi \otimes \psi^* - \frac{1}{2^{n-1}} |\psi|^2 \, \mathrm{Id}$, by using the calculation in Lemma 8.3, we have $\|F_A\|_{L_{\ell+1}^p} \leq C(\ell)'''$. Due to Theorem 7.3, (71), we have $\|\alpha\|_{L_{\ell+2}^p} \leq C(\ell)^{(4)}$. Now by induction, we obtain the desired for all ℓ. □

10. The Seiberg–Witten moduli space

Let us recall the situation in 7.1. In this section, we want to extend the situation in Chapter 4 in [16] to L_ℓ^p-theory over compact Kähler manifolds (M, g) of complex dimension $n \geq 2$.

10.1. Space of configulations. Fix $p > 2n = \dim_{\mathbb{R}} M$. For every $\ell \geq 1$, we define the space of configulations as follows.

Definition 10.1. The space of configurlations is defined to be

$$\mathcal{C}_\ell^p(\widetilde{P}) := \mathfrak{A}_\ell^p(\mathcal{L}) \times L_\ell^p(S_{\mathbb{C}}^+(\widetilde{P})), \tag{106}$$

where $\mathfrak{A}_\ell^p(\mathcal{L})$ is the space of L_ℓ^p $U(1)$-connections ∇^A on $\mathcal{L}$, i.e., the space of L_ℓ^p $\sqrt{-1}\mathbb{R}$-valued 1-forms on M (cf. Lemmas 3.7, 3.8), where we denote by $L_\ell^p(F)$, the space of all L_ℓ^p sections for a vector bundle F over M.

For each $(A,\psi) \in \mathcal{C}^p_\ell(\widetilde{P})$, the tangent space of $\mathcal{C}^p_\ell(\widetilde{P})$ at (A,ψ) is naturally identified with

$$L^p_\ell\left(\left(T^*M \otimes \sqrt{-1}\mathbb{R}\right) \oplus S^+_{\mathbb{C}}(\widetilde{P})\right).$$

We also define (cf. [16] p. 58) the Seiberg–Witten function

$$F : \mathcal{C}^p_\ell(\widetilde{P}) \to L^p_{\ell-1}(\mathrm{End}(S^+_{\mathbb{C}}(\widetilde{P})) \oplus S^-_{\mathbb{C}}(\widetilde{P})),$$

by

$$F(A,\psi) := (c^+(F_A) - q(\psi), \mathfrak{D}_A\psi), \tag{107}$$

where

$$q(\psi) := \psi \otimes \psi^* - \frac{1}{2^{n-1}}\,\mathrm{Id},$$

i.e., for $\varphi \in \Gamma(S^+_{\mathbb{C}}(\widetilde{P}))$,

$$q(\psi)(\varphi) := \langle \varphi, \psi\rangle\psi - \frac{1}{2^{n-1}}\,|\psi|^2\,\varphi, \qquad c^+(F_A)(\varphi) := F_A \cdot \varphi,$$

respectively. Both $q(\psi)$, $c^+(F_A) \in \mathrm{End}(S^+_{\mathbb{C}}(\widetilde{P}))$. Notice here that the set $F^{-1}(0,0) \subset \mathcal{C}^p_\ell(\widetilde{P})$ is the space of solutions of the Seiberg–Witten equations by definition.

By a direct computation, we have

Lemma 10.2. *The mapping F is smooth, and the differentiation at (A,ψ) is given by*

$$DF_{(A,\psi)} = (c^+ \circ d - Dq_\psi,\ \cdot\psi + \mathfrak{D}_A), \tag{108}$$

i.e, for every $(\alpha,\xi) \in L^p_\ell((T^*M \otimes \sqrt{-1}\mathbb{R}) \oplus S^+_{\mathbb{C}}(\widetilde{P}))$,

$$DF_{(A,\psi)}(\alpha,\xi) = (c^+(d\alpha) - Dq_\psi\xi, \alpha \cdot \psi + \mathfrak{D}_A\xi), \tag{109}$$

where $Dq_\psi(\xi)$ is given by

$$Dq_\psi(\xi) = \xi \otimes \psi^* + \psi \otimes \xi^* - \frac{1}{2^{n-1}}(\langle \xi,\psi\rangle + \langle \psi,\xi\rangle)\,\mathrm{Id}\,. \tag{110}$$

Remark 10.3. In (109), due to the Sobolev Multiplication Theorem: $L^p_k \otimes L^p_k \to L^p_k$ if $kp > \dim_{\mathbb{R}} M$, we have that $\alpha \cdot \psi \in L^p_\ell(S^+_{\mathbb{C}}(\widetilde{P}))$ for every $\alpha \in L^p_\ell(T^*M \otimes \sqrt{-1}\mathbb{R})$ and $\psi \in L^p_\ell(S^+_{\mathbb{C}}(\widetilde{P}))$, where $p > 2n = \dim_{\mathbb{R}} M$ and $\ell \geq 1$.

10.2. Action of gauge transformations. Let us recall Definition 7.2, i.e., $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ acts on $\mathcal{C}^p_\ell(\widetilde{P})$ by

$$(A,\psi) \cdot \sigma = (\sigma^*A, S^+(\sigma^{-1})\psi), \quad (\sigma \in \mathcal{G}^p_{\ell+1}(\widetilde{P}),\ (A,\psi) \in \mathcal{C}^p_\ell(\widetilde{P})). \tag{111}$$

By the same way as Lemma 4.4.1 in [16], we have

Lemma 10.4. *The action (111) of $\mathcal{G}^p_{\ell+1}$ on $\mathcal{C}^p_\ell(\widetilde{P})$ defines a smooth right action. For the Seiberg–Witten function*

$$F : \mathcal{C}^p_\ell(\widetilde{P}) \to L^p_{\ell-1}(\mathrm{End}(S^+_{\mathbb{C}}(\widetilde{P})) \oplus S^-_{\mathbb{C}}(\widetilde{P})),$$

we have

$$F((A,\psi)\,\cdot\,\sigma) = F(A,\psi)\,\cdot\,\sigma, \tag{112}$$

where the action of σ on $L^p_{\ell-1}(\mathrm{End}(S^+_{\mathbb{C}}(\widetilde{P})) \oplus S^-_{\mathbb{C}}(\widetilde{P}))$ is the trivial on the first factor and is given by $S^-(\sigma^{-1})$ on the second factor.

Proof. Denoting simply $\sigma\,\psi := S^+(\sigma)\psi$, we see $q(\sigma\,\psi) = q(\psi)$. In fact, $\sigma\,\psi$ is defined by $\sigma\,\psi(p) := \sigma(\pi(p))\psi(p)$, $(p \in \widetilde{P})$. Then, we have $(\sigma\,\psi)\otimes(\sigma\,\psi)^*$ and $|\sigma\,\psi|^2 = |\psi|^2$, which yield $q(\sigma\,\psi) = q(\psi)$. And we have also $F_{\sigma^* A} = \sigma^{-1}\,F_A\,\sigma = F_A$.

Since $\widetilde{\nabla}^{\sigma^* A}_X\psi = \sigma^{-1}\,\widetilde{\nabla}^A_X(\sigma\,\psi)$ for ψ of $S^+_{\mathbb{C}}(\widetilde{P})$, and the Clifford multiplication commutes with the $S^{\pm}(\sigma)$, we have $\mathfrak{D}_{\sigma^* A}(\sigma^{-1}\,\psi) = \sigma^{-1}\mathfrak{D}_A\psi$. Smoothness of the action follows from the Sobolev Multiplication Theorem for $L^p_{\ell+1}\otimes L^p_\ell \to L^p_\ell$ if $\ell\,p > \dim_{\mathbb{R}} M$. □

10.3. Basic convergence theorems. In this subsection, we assume that $p > 2n = \dim_{\mathbb{R}} M$ and $\ell \geq 1$. Then, we have

Lemma 10.5. *Suppose that (A_s,ψ_s), (B_s,μ_s) $(s = 1,2,\dots)$ are two sequences in $\mathcal{C}^p_\ell(\widetilde{P})$ converging to (A,ψ) and (B,μ) as $s \to \infty$, respectively. Suppose that for each s, we have $\sigma_s \in \mathcal{G}^p_{\ell+1}(\widetilde{P})$ such that*

$$(A_s,\psi_s)\,\cdot\,\sigma_s = (B_s,\mu_s).$$

Then, there exists a subsequence $\{\sigma_{s_k}\}_{k=1}^\infty$ of $\{\sigma_s\}_{s=1}^\infty$ converging to an element $\sigma \in \mathcal{G}^p_{\ell+1}(\widetilde{P})$ as $k \to \infty$.

Furthermore, we have

$$(A,\psi)\,\cdot\,\sigma = (B,\mu).$$

Proof. The proof goes by a similar way as in [16], but is different from its proof at several steps how to use the Sobolev Multiplication Theorems. Since $\sigma_s \in \mathcal{G}^p_{\ell+1} = L^p_{\ell+1}(M,U(1)) \hookrightarrow C^0(M,U(1))$ for $(\ell+1)p > 2n = \dim_{\mathbb{R}} M$. Thus, σ_s are $U(1)$-valued continuous functions on M, so that $\sup_s \|\sigma_s\|_{L^{(\ell+1)p}} < \infty$. Let us take $\tau_s = \det\,\sigma_s = {\sigma_s}^m$ $(m = 2^{n-1})$ if we consider $\det{\sigma_s}^* A_s$. We also have $\sup_s \|\tau_s\|_{L^{(\ell+1)p}} < \infty$. Since

$$(B_s,\mu_s) = (A_s,\psi_s)\,\cdot\,\sigma_s = ({\tau_s}^* A_s, S^+({\sigma_s}^{-1})\psi_s),$$

we have $B_s = {\tau_s}^* A_s = A_s + {\tau_s}^{-1}\, d\tau_s$, i.e., $d\tau_s = \tau_s(B_s - A_s)$. Since the sequences A_s and B_s converge to A and B in L^p_ℓ as $s \to \infty$, respectively, we have $\sup_s \|A_s\|_{L^p_\ell} < \infty$ and $\sup_s \|B_s\|_{L^p_\ell} < \infty$. Using the Sobolev Multiplication Theorem: $L^{(\ell+1)p} \otimes L^p_\ell \to L^p_\ell$ is defined and continuous if $\ell p > \dim_{\mathbb{R}} M$, we have $\sup_s \|d\tau_s\|_{L^p_\ell} < \infty$, i.e., $\sup_s \|\tau_s\|_{L^p_{\ell+1}} < \infty$. By the Sobolev Embedding Theorem: $L^p_{\ell+1} \hookrightarrow L^p_{\ell+\epsilon}$ $(0 < \epsilon < 1)$ is compact, there exists a subsequence $\{\tau_{s_t}\}$ of $\{\tau_s\}$ such that τ_{s_t} converges in $L^p_{\ell+\epsilon}$ to some $\tau \in L^p_{\ell+\epsilon}$ as $t \to \infty$. Then, it holds that $d\tau = \tau(B - A)$. Applyng this to the Sobolev Multiplication Theorem: $L^p_{\ell+\epsilon} \otimes L^p_\ell \to L^p_\ell$ is defined and continuous if $\ell p > \dim_{\mathbb{R}} M$, we have $d\tau \in L^p_\ell$, and it holds that

$$d\tau_{s_t} = \tau_{s_t} \cdot (B_{s_t} - A_{s_t}) \to \tau \cdot (B - A) = d\tau \quad (\text{in } L^p_\ell)$$

as $t \to \infty$. It holds that $\tau \in L^p_{\ell+1}$ and that $\tau_{s_t} = \det \sigma_{s_t} = {\sigma_{s_t}}^m$ converges to τ in $L^p_{\ell+1}$ as $t \to \infty$. Then, we can choose a subsequence $\{\sigma_{s_{t_u}}\}$ of $\{\sigma_s\}$ such that $\sigma_{s_{t_u}}$ converges in $L^p_{\ell+1}$ to some $\sigma \in \mathcal{G}^p_{\ell+1}(\widetilde{P})$ as $u \to \infty$. It holds that $\tau = \det \sigma$ and $(A, \psi) \cdot \sigma = (B, \mu)$. □

10.4. The quotient space. In this subsection, we consider the quotient space of the action of $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ on $\mathcal{C}^p_\ell(\widetilde{P})$. We assume $p > \dim_{\mathbb{R}} M$, and $\ell \geq 1$. By the same way as [16], we have immediately

Lemma 10.6. *The isotropy subgroup* $\mathrm{Stab}(A, \psi)$ *of* $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ *at* (A, ψ) *is* $\{\mathrm{id}\}$ *if* $\psi \not\equiv 0$, *and is the set of constant maps of* M *into* $U(1)$ *which is identified with* S^1 *if* $\psi \equiv 0$.

Proof. Recall that the action of $\sigma \in \mathcal{G}^p_{\ell+1}(\widetilde{P})$ at (A, ψ) is

$$(A, \psi) \cdot \sigma = (\det \sigma^* A, S^+(\sigma^{-1})\psi),$$

$(A, \psi) \cdot \sigma$ is equivalent to

$$\det \sigma^* A = A, \quad \text{and} \quad S^+(\sigma^{-1})\psi = \psi.$$

Since $\det \sigma^* A = A + m\sigma^{-1} d\sigma$, $(m = 2^{n-1})$ and $\sigma^* A = A + \sigma^{-1} d\sigma$, we have $d\sigma = 0$, i.e., σ is a constant map of M into $U(1)$. Since $\sigma \in L^p_\ell(S^+_{\mathbb{C}}(\widetilde{P})) \hookrightarrow C^0(S^+_{\mathbb{C}}(\widetilde{P}))$ $(\ell p > \dim_{\mathbb{R}} M)$ by the Sobolev Embedding Theorem, ψ is a continuous map from $\widetilde{P}$ into $\Delta^+_{\mathbb{C}}$ satisfying that $\psi(pa) = \rho(a^{-1})\psi(p)$, for $p \in \widetilde{P}$ and $a \in \mathrm{Spin}^c(2n)$, where $\rho : \mathrm{Spin}^c(2n) \to GL(\Delta^+_{\mathbb{C}})$ is the complex half-spin representation. Since σ is a constant, $S^+(\sigma^{-1})\psi = \psi$ is equivalent to $\sigma\, \psi(p) = \psi(p)$ for all $p \in \widetilde{P}$, which is equivalent to $\sigma = \mathrm{id}$ if $\psi \not\equiv 0$. □

Definition 10.7. We say a configulation (A,ψ) is irreducible if $\psi \not\equiv 0$, otherwise it is reducible. We denote by $\mathcal{C}^{*p}_{\ell}(\widetilde{P})$ the open subset of irreducible configulations.

Due to Lemma 10.5, we have by the same way as [16],

Lemma 10.8. *The quotient space $\mathcal{B}^p_\ell(\widetilde{P}) := \mathcal{C}^p_\ell(\widetilde{P})/\mathcal{G}^p_{\ell+1}(\widetilde{P})$ is a Hausdorff space.*

Proof. Assume that $\mathcal{B}^p_\ell(\widetilde{P})$ is not Hausdorff. Then, there exists a sequence $\{(A_s,\psi_s)\}$ in $\mathcal{C}^p_\ell(\widetilde{P})$ and a sequence $\{\sigma_s\}$ in $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ such that $(A_s,\psi_s)\to(A,\psi)$, and $(A_s,\psi_s)\cdot\sigma_s \to (B,\mu)$ in $\mathcal{C}^p_\ell(\widetilde{P})$ as $s\to\infty$, but (A,ψ) and (B,μ) are not in the same orbit of the action of $\mathcal{G}^p_{\ell+1}(\widetilde{P})$. But, by Lemma 10.5, there exists a subsequence $\{\sigma_{s_k}\}$ of $\{\sigma_s\}$ converging to an element $\sigma\in\mathcal{G}^p_{\ell+1}(\widetilde{P})$ as $k\to\infty$ and it holds that $(A,\psi)\cdot\sigma=(B,\mu)$, which is a contradiction. □

10.5. The slice theorem. In this subsection, we show

Lemma 10.9 (the slice theorem)**.** *There exist local slices for the action of $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ on $\mathcal{C}^p_\ell(\widetilde{P})$.*

I.e., for each $(A,\psi)\in\mathcal{C}^p_\ell(\widetilde{P})$, there exist a neighborhood U' of (A,ψ) and a closed submanifold S in U' invariant under the action of the isotropy subgroup Stab(A,ψ) *of $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ at (A,ψ), such that the natural map from the equivalence space $S\times_{\mathrm{Stab}(A,\psi)}\mathcal{G}^p_{\ell+1}(\widetilde{P})$ to $\mathcal{C}^p_\ell(\widetilde{P})$,*

$$S\times_{\mathrm{Stab}(A,\psi)}\mathcal{G}^p_{\ell+1}(\widetilde{P})\ni[((B,\mu),\sigma)]\mapsto(B,\mu)\cdot\sigma\in\mathcal{C}^p_\ell(\widetilde{P}),$$

yields a diffeomorphism onto an open neighborhood of the orbit through (A,ψ) in the quotient space $\mathcal{C}^p_\ell(\widetilde{P})/\mathcal{G}^p_{\ell+1}(\widetilde{P})$.

Proof. • Assume that $(A,\psi)\in\mathcal{C}^p_\ell(\widetilde{P})$. By means of a direct computation, the differentiation of the mapping $\mathcal{G}^p_{\ell+1}(\widetilde{P})\ni\sigma\mapsto(A,\psi)\cdot\sigma\in\mathcal{C}^p_\ell(\widetilde{P})$ at id is given by

$$R : L^p_{\ell+1}(M,\sqrt{-1}\mathbb{R})\ni f\mapsto(m\,df,-f\cdot\psi)\in L^p_\ell((T^*M\otimes\sqrt{-1}\mathbb{R})\oplus S^+_{\mathbb{C}}(\widetilde{P})),$$

where $m=2^{n-1}$, if we take the action of $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ to be $\det\sigma^*A)$, due to the Sobolev Multiplication Theorem: $L^p_{\ell+1}\otimes L^p_\ell\to L^p_\ell$ is defined and continuous.

• Define the linear mapping

$$T : L^p_\ell((T^*M\otimes\sqrt{-1}\mathbb{R})\oplus S^+_{\mathbb{C}}(\widetilde{P}))\to L^p_{\ell-1}(M,\sqrt{-1}\mathbb{R})$$

by

$$T(\omega,\mu) := \delta\omega - \sqrt{-1}\,\mathrm{Im}\langle\mu,\psi\rangle, \tag{113}$$

(or $m\,\delta\omega - \sqrt{-1}\,\mathrm{Im}\langle\mu,\psi\rangle$). Then, we have

$$(f, T(\omega,\mu)) = (Rf, (\omega,\mu)), \tag{114}$$

for all $f \in L^p_{\ell+1}(M, \sqrt{-1}\mathbb{R})$ and $(\omega,\mu) \in L^p_\ell((T^*M \otimes \sqrt{-1}\mathbb{R}) \oplus S^+_{\mathbb{C}}(\widetilde{P}))$. Here the L^2-inner products of the both hand sides are given by

$$(f, f') := \int_M \langle f, f'\rangle\, v_g, \qquad ((\omega,\mu),(\omega',\mu')) := \int_M \langle \omega,\omega'\rangle\, v_g + \int_M \Re\langle\mu,\mu'\rangle\, v_g,$$

where each $\langle\, ,\, \rangle$ are the natural Hermitian inner products, respectively.

• The kernel of T, whcih is given by

$$K := \mathrm{Ker}(T) = \{(\omega,\mu) \in L^p_\ell((T^*M \otimes \sqrt{-1}\mathbb{R}) \oplus S^+_{\mathbb{C}}(\widetilde{P}));\ T(\omega,\mu) = 0\},$$

is invariant under the action $\mathrm{Stab}(A,\psi)$.

• If we take an enough small open neighborhood U' of $(0,0)$ in K which is invariant under $\mathrm{Stab}(A,\psi)$, then we want to see that $S := (A,\psi)+U' \subset (A,\psi)+K$ is the desired slice. It only suffices to see that the mapping of $S \times_{\mathrm{Stab}(A,\psi)} \mathcal{G}^p_{\ell+1}(\widetilde{P})$ to $\mathcal{C}^p_\ell(\widetilde{P})$ given by

$$[(A,\psi)+u, \sigma] \mapsto ((A,\psi)+u)\,\cdot\,\sigma$$

yields a diffeomorphism of $U' \times_{\mathrm{Stab}(A,\psi)} \mathcal{G}^p_{\ell+1}(\widetilde{P})$ onto a neighborhood of the orbit through (A,ψ) in $\mathcal{B}^p_\ell(\widetilde{P}) = \mathcal{C}^p_\ell(\widetilde{P})/\mathcal{G}^p_{\ell+1}(\widetilde{P})$.

The mapping is well defined because $(u\,\sigma', \sigma'^{\,-1}\sigma) \mapsto ((A,\psi)+u)\,\cdot\,\sigma$ for all $\sigma' \in \mathrm{Stab}(A,\psi)$.

• The differentiation of the mapping

$$U' \times_{\mathrm{Stab}(A,\psi)} \mathcal{G}^p_{\ell+1}(\widetilde{P}) \ni [u,\sigma] \mapsto ((A,\psi)+u)\,\cdot\,\sigma \in \mathcal{C}^p_\ell(\widetilde{P})$$

at $[0,\mathrm{id}]$ is given by

$$\begin{aligned} H : K \oplus L^p_{\ell+1}(M,\sqrt{-1}\mathbb{R}) &\ni (v,f) \\ &\mapsto H(v,f) \in L^p_\ell((T^*M \otimes \sqrt{-1}\mathbb{R}) \oplus S^+_{\mathbb{C}}(\widetilde{P})), \end{aligned} \tag{115}$$

where $H(v,f)$ is

$$H(v,f) := v + Rf. \tag{116}$$

• H is a bijection.

To see H is injective, notice that $\mathrm{Ker}(H) = \{(v, f);\ -v = Rf\}$. Here $v := (\omega, \mu) \in K = \mathrm{Ker}(T)$, so that

$$0 = (T(\omega, \mu), f) = ((\omega, \mu), Rf) = -(Rf, Rf),$$

which implies that $Rf = 0$, i.e., $v = 0$. By definition of R, f is a constant map of M to $U(1)$, and $f \cdot \psi = 0$. If $\psi \not\equiv 0$, then $f = 0$. Therefore, $(v, f) = (0, f) \in \mathrm{Stab}(A, \psi)$.

To see H is surjective, notice that

$$\mathrm{Im}(H) = \mathrm{Ker}(T) \oplus \mathrm{Im}(R).$$

By Banach's Closed Range Theorem (cf. [37], p. 205), we have $\mathrm{Im}(R) = \mathrm{Ker}(T)^{\perp}$. Therefore,

$$\mathrm{Im}(H) = \mathrm{Ker}(T) \oplus \mathrm{Ker}(T)^{\perp} = L^p_\ell((T^*M \otimes \sqrt{-1}\mathbb{R}) \oplus S^+_{\mathbb{C}}(\widetilde{P})).$$

• Thus, we can apply the Inverse Mapping Theorem, there exist an enough small $\mathrm{Stab}(A, \psi)$-invariant neighborhood U' of $(0, 0)$ in K and also an enough small $\mathrm{Stab}(A, \psi)$-invariant neighborhood V of id in $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ such that the mapping

$$U' \times_{\mathrm{Stab}(A,\psi)} V \ni [u, \sigma] \mapsto ((A, \psi) + u) \cdot \sigma \in \mathcal{C}^p_\ell(\widetilde{P}) \tag{117}$$

yields a diffeomorphism onto an open neighborhood of the orbit through (A, ψ) in the quotient space $\mathcal{B}^p_\ell(\widetilde{P}) = \mathcal{C}^p_\ell(\widetilde{P})/\mathcal{G}^p_{\ell+1}(\widetilde{P})$.

• Furthermore, if we take an enough small neighborhood U'' of $(0, 0)$ in K, the mapping

$$U'' \times_{\mathrm{Stab}(A,\psi)} \mathcal{G}^p_{\ell+1}(\widetilde{P}) \ni [u, \sigma] \mapsto ((A, \psi) + u) \cdot \sigma \in \mathcal{C}^p_\ell(\widetilde{P})$$

yields a diffeomorphism onto an open neighborhood of the orbit through (A, ψ) in the quotient space $\mathcal{B}^p_\ell(\widetilde{P})$.

Indeed, since this mapping is a local diffeomorphism, if we take U'' to be sufficiently small, we show that this mapping is one-to-one. Assume that there is no such neighborhood U'' of $(0, 0)$ in U'. Then, there exist two sequences $\{a_s\}_{s=1}^\infty$ and $\{b_s\}_{s=1}^\infty$ in U' and a sequence $\{\sigma_s\}_{s=1}^\infty$ in $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ such that $a_s \to (0, 0)$, $b_s \to (0, 0)$ as $s \to \infty$, $((A, \psi) + a_s) \cdot \sigma_s = (A, \psi) + b_s$, and $[a_s, \sigma_s] \neq [b_s, \mathrm{id}]$ for each $s = 1, 2, \ldots$. Then, by Lemma 10.5, there exists a subsequence $\{\sigma_{s_k}\}$ of $\{\sigma_s\}$ in $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ such that σ_{s_k} converges in $\mathcal{G}^p_{\ell+1}(\widetilde{P})$ to some $\sigma \in \mathcal{G}^p_{\ell+1}(\widetilde{P})$ as $k \to \infty$.

Then, since $a_{s_k} \to (0,0)$, $b_{s_k} \to (0,0)$ as $k \to \infty$, and $((A,\psi)+a_{s_k})\cdot\sigma_{s_k} = (A,\psi)+b_{s_k}$, we have $(A,\psi)\cdot\sigma = (A,\psi)$, which implies that $\sigma \in \mathrm{Stab}(A,\psi)$. Therefore, both $[a_{s_k},\sigma_{s_k}]$ and $[b_{s_k},\mathrm{id}]$ belong to $U'\times_{\mathrm{Stab}(A,\psi)}V$ for enough large k. But, the above mapping is diffeomorphism on $U'\times_{\mathrm{Stab}(A,\psi)}V$, which contradicts that $((A,\psi)+a_{s_k})\cdot\sigma_{s_k} = (A,\psi)+b_{s_k}$ and $[a_{s_k},\sigma_{s_k}] \neq [b_{s_k},\mathrm{id}]$. Thus, we have the desired conclusion.

We have Lemma 10.9. □

We can summarize

Corollary 10.10. • *The quotient space $\mathcal{B}^p_\ell(\widetilde{P}) = \mathcal{C}^p_\ell(\widetilde{P})/\mathcal{G}^p_{\ell+1}(\widetilde{P})$ is a Hausdorff space.*

• *The complement of the equivalence classes of reducible configulations $[A,0]$, denoted ${\mathcal{B}^*}^p_\ell(\widetilde{P})$ is an open subset in $\mathcal{B}^p_\ell(\widetilde{P})$, and a Banach manifold. The tangent space of ${\mathcal{B}^*}^p_\ell(\widetilde{P})$ at $[A,\psi]$ is identified with*

$$L^p_\ell((T^*M\otimes\sqrt{-1}\mathbb{R})\oplus S^+_{\mathbb{C}}(\widetilde{P}))/\operatorname{Im} R,$$

where R is given by (113).

• *For a reducible equivalence class $[A,0]$, a neighborhood of this point in $\mathcal{B}^p_\ell(\widetilde{P})$ is homeomorphic to the quotient of*

$$L^p_\ell((T^*M\otimes\sqrt{-1}\mathbb{R})\oplus S^+_{\mathbb{C}}(\widetilde{P}))/\big\{(df,0);\ f\in L^p_{\ell+1}(M,\sqrt{-1}\mathbb{R})\big\}$$

by the action of $\mathrm{Stab}(A,0)\cong U(1)=S^1$.

10.6. The tangent space of the moduli space. In this subsection, we consider the linearization of the Seiberg–Witten equations and the action of gauge transformations, and the moduli space of solutions of the Seiberg–Witten equations.

Definition 10.11. The moduli space of solutions of the Seiberg–Witten equations, denoted by $\mathcal{M}^p_\ell(\widetilde{P})$, is the set of equivalence classes of solutions of the Seiberg–Witten equations, i.e.,

$$\mathcal{M}^p_\ell(\widetilde{P}) := F^{-1}(0,0)/\mathcal{G}^p_{\ell+1}(\widetilde{P}) \subset \mathcal{B}^p_\ell(\widetilde{P}) = \mathcal{C}^p_\ell(\widetilde{P})/\mathcal{G}^p_{\ell+1}(\widetilde{P}), \tag{118}$$

due to Lemma 10.4.

We want to describe the tangent space of $\mathcal{M}^p_\ell(\widetilde{P})$ at each point $[A,\psi]\in\mathcal{M}^p_\ell(\widetilde{P})$.

Assume that $(A,\psi) \in \mathcal{C}^p_\ell(\widetilde{P})$ is a solution of the Seiberg–Witten equation, i.e., $(A,\psi) \in F^{-1}(0,0)$. Let us consider the following sequence, denoted by $\mathcal{E}(A,\psi)$:

$$0 \to L^p_{\ell+1}(M;\sqrt{-1}\mathbb{R}) \overset{R}{\to} L^p_\ell((T^*M \otimes \sqrt{-1}\mathbb{R}) \oplus S^+_{\mathbb{C}}(\widetilde{P})) \\ \overset{DF_{(A,\psi)}}{\to} L^p_{\ell-1}(\mathrm{End}(S^+_{\mathbb{C}}(\widetilde{P})) \oplus S^-_{\mathbb{C}}(\widetilde{P})) \to 0.$$

Lemma 10.12. *Assume that $(A,\psi) \in \mathcal{C}^p_\ell(\widetilde{P})$ is a solution of the Seiberg–Witten equations. Then, $\mathcal{E}(A,\psi)$ is a complex, i.e., $DF_{(A,\psi)} \circ R = 0$.*

Proof. Let $\sigma_t \in \mathcal{G}^p_{\ell+1}(\widetilde{P})$ $(-\epsilon < t < \epsilon)$ be a smooth one-parameter family in t through id at $t=0$. Then, $f := \frac{d}{dt}|_{t=0}\,\sigma_t$ belongs to $L^p_{\ell+1}(M,\sqrt{-1}\mathbb{R})$, and due to Lemma 10.4,

$$F((A,\psi)\cdot\sigma_t) = F(A,\psi)\cdot\sigma_t = (0,0),$$

for every $-\epsilon < t < \epsilon$. Differentiate this at $t=0$, we have

$$DF_{(A,\psi)}(R(f)) = (0,0),$$

by (109), and (113). □

Next, let us consider the symbol sequence of $\mathcal{E}(A,\psi)$ for a solution (A,ψ) of the Seiberg–Witten equations: for each $0 \neq \eta \in T^*_xM$ $(x \in M)$,

$$0 \to \sqrt{-1}\mathbb{R} \overset{\sigma(R)(\eta)}{\to} (T^*_xM \otimes \sqrt{-1}\mathbb{R}) \oplus S^+_{\mathbb{C},x}(\widetilde{P}) \\ \overset{\sigma(DF_{(A,\psi)})(\eta)}{\to} \mathrm{End}(S^+_{\mathbb{C},x}(\widetilde{P})) \oplus S^-_{\mathbb{C},x}(\widetilde{P}) \to 0.$$

Then, the symbols are by a direct calculation given as follows:

$$\sigma(R)(\eta)(a) = (\eta\, a, 0) \in (T^*_xM \otimes \sqrt{-1}\mathbb{R}) \oplus S^+_{\mathbb{C},x}(\widetilde{P}), \quad (\forall\ a \in \sqrt{-1}\mathbb{R}), \tag{119}$$

$$\sigma(DF_{(A,\psi)})(\eta)(\beta,\zeta) = ((\eta\wedge\beta)\cdot\ \ , \sqrt{-1}\eta^{\#}\cdot\zeta) \in \mathrm{End}(S^+_{\mathbb{C},x}(\widetilde{P})) \oplus S^-_{\mathbb{C},x}(\widetilde{P}), \\ (\forall\ (\beta,\zeta) \in ((T^*_xM \otimes \sqrt{-1}\mathbb{R}) \oplus S^+_{\mathbb{C},x}(\widetilde{P})), \tag{120}$$

where $\eta^{\#} \in T_xM$ is defined by $g(\eta^{\#}, X) = \eta(X), \forall X \in T_xM$, for all $0 \neq \eta \in T^*_xM$.

Then, we have

Lemma 10.13. *Assume that $(A,\psi) \in \mathcal{C}^p_\ell(\widetilde{P})$ is a solution of the Seiberg–Witten equation. Then,*

(1) *$\sigma(R)(\eta)$ is injective for all $0 \neq \eta \in T_x^*M$.*

(2) *We have*

$$\mathrm{Ker}(\sigma(DF_{(A,\psi)})(\eta)) = \mathrm{Im}(\sigma(R)(\eta)),$$

for all $0 \neq \eta \in T_x^*M$.

Remark 10.14. It is still unsolved at least for us to describe

$$\mathrm{Im}(\sigma(DF_{(A,\psi)})(\eta)) \subset \mathrm{End}(S^+_{\mathbb{C},x}(\widetilde{P})) \oplus S^-_{\mathbb{C},x}(\widetilde{P}), \quad (\forall\ 0 \neq \eta \in T_x^*M),$$

or to extend $\mathcal{E}(A,\psi)$ to a long sequence which would be elliptic (see also [9], p. iii in its preface).

Proof. For (1), it is clear to see, for $0 \neq \eta \in T_x^*M$, that $\eta\, a = 0$ implies $a = 0$ for all $a \in \sqrt{-1}\mathbb{R}$.

For (2), Assume that $(\beta,\zeta) \in ((T_x^*M \otimes \sqrt{-1}\mathbb{R}) \oplus S^+_{\mathbb{C},x}(\widetilde{P}))$ satisfies that

$$\sigma(DF_{(A,\psi)})(\eta)(\beta,\zeta) = ((\eta \wedge \beta)\cdot\ \ , \sqrt{-1}\eta^{\#}\cdot \zeta) = (0,0).$$

Since $\sqrt{-1}\eta^{\#}\cdot\zeta = 0$, we have

$$0 = \sqrt{-1}\eta^{\#}\cdot(\sqrt{-1}\eta^{\#}\cdot\zeta) = -\,|\eta^{\#}|^2\,\zeta,$$

which implies $\zeta = 0$, because $|\eta^{\#}|^2 > 0$ for $0 \neq \eta \in T_x^*M$.

Furthermore, putting $F := \eta \wedge \beta$, we have $c^+(F) = 0$, as an endomorphism of $S^+_{\mathbb{C},x}(\widetilde{P})$. By Lemma 3.4 and Corollary 2.6, we have

$$0 = |c^+(F)|^2 = \begin{cases} |F^+|^2 + |\Lambda(F)|^2 & (n=2), \\ 2^{n-3}(|F|^2 + |\Lambda(F)|^2) & (n \geq 3). \end{cases}$$

If $n \geq 3$, we have $F = 0$, i.e., $\eta \wedge \beta = 0$, which implies that $\beta = a\,\eta$ for some $a \in \sqrt{-1}\mathbb{R}$. Thus, $(\beta,\zeta) \in \mathrm{Im}(\sigma(R)(\eta))$. If $n = 2$, we have $F^+ = 0$ and $\Lambda(F) = 0$, which implies F is anti-self-dual, i.e., $(1+*)(F) = 0$, where $*$ is the Hodge star operator of (M,g). Then, it is known that $\beta = a\,\eta$ for some $a \in \sqrt{-1}\mathbb{R}$ (cf. [2], [10], p. 247, or [9], p. 150). □

Definition 10.15. Let H^i be the i-th cohomology group of the complex $\mathcal{E}(A,\psi)$ $(i = 0,1,2)$.

• For H^0, we have

$$H^0 := \mathrm{Ker}(R) \cong \begin{cases} \{0\} & (\psi \not\equiv 0), \\ \sqrt{-1}\mathbb{R} & (\psi \equiv 0). \end{cases} \tag{121}$$

• For H^1, due to Lemma 10.13, we may use the elliptic P.D.E. theory (for example, [12], p. 196), and we have $\dim H^1 < \infty$, and

$$H^1 := \mathrm{Ker}(DF_{(A,\psi)}) / \mathrm{Im}(R) \cong T_{[A,\psi]}\mathcal{M}^p_\ell(\widetilde{P}), \tag{122}$$

which is the tangent space of $\mathcal{M}^p_\ell(\widetilde{P})$ at a smooth point $[A,\psi]$, and $\dim H^1$ is the dimension of $\mathcal{M}^p_\ell(\widetilde{P})$ near such a point.

• For H^2,

$$H^2 := L^p_{\ell-1}(\mathrm{End}(S^+_{\mathbb{C}}(\widetilde{P})) \oplus S^-_{\mathbb{C}}(\widetilde{P})) / \mathrm{Im}(DF_{(A,\psi)}) \tag{123}$$

could be of infinite dimension. At this moment, we can not say any more about smoothness and the dimension of $\mathcal{M}^p_\ell(\widetilde{P})$.

11. Compactness of the moduli space

Finally, in this section, we show compactness of the moduli space of solutions of the Seiberg–Witten equations.

Theorem 11.1 (compactness of the moduli space)**.** *Let (M,g) be a compact Kähler manifold of complex dimension $n \geq 2$, and $\widetilde{P}$ the principal* $\mathrm{Spin}^c(2n)$*-bundle over (M,g), $\mathcal{L}$ its determinant bundle, and $p > 2n = \dim_{\mathbb{R}} M$. Let (A_m, ψ_m), $m = 1, 2, \ldots$ be any sequence of solutions of the Seiberg–Witten equations. Then, there exist a subsequence (A_{m_k}, ψ_{m_k}) and L^p_3-gauge transformations σ_{m_k} such that $(A_{m_k}, \psi_{m_k}) \cdot \sigma_{m_k}$ is convergent in the C^∞ topology, as $k \to \infty$, to a limit (A,ψ) which is also a solution of the Seiberg–Witten equations. In particular, the moduli space of solutions of the Seiberg–Witten equations is compact.*

Proof. Let us recall the Sobolev Embedding Theorem (cf. [7]), p. 95): The embedding $L^p_\ell \hookrightarrow C^k$ is defined and compact if $\ell\, p - \dim_{\mathbb{R}} M > kp$.

Let $\{(A_m, \psi_m)\}_{m=1}^\infty$ be a sequence of solutions of the Seiberg–Witten equations. Then, due to Theorem 9.1, up to L^p_3-gauge transforms, for all $\ell \geq 1$,

$$\|(A_m, \psi_m)\|_{L^p_\ell} \leq C(\ell) \quad (m = 1, 2, \ldots).$$

Due to the Sobolev Embedding Theorem, there exist a subsequence $\{(A_{m^\ell_i}, \psi_{m^\ell_i})\}$ such that, up to L^p_3-gauge transforms, for all $\ell \geq 1$, $(A_{m^\ell_i}, \psi_{m^\ell_i})$ is convergent in the $C^{\ell-1}$ topology as $i \to \infty$. Then, a subsequence $\{(A_{m_i^i}, \psi_{m_i^i})\}_{i=1}^{\infty}$ is convergent as $i \to \infty$, in the $C^{\ell-1}$ topology for all $\ell \geq 1$, therefore, in the C^{∞} topology. Since $(A_{m_i^i}, \psi_{m_i^i})$ is a solution of the Seiberg–Witten equations, the C^{∞} limit (A, ψ) is also a solution. $\square$

Remark 11.2. Notice that the Seiberg–Witten equations have a solution $(A, 0)$ where $F_A = 0$ at least, so that the moduli space of the solutions $\mathcal{M}_{\ell}^{p}(\widetilde{P})$ is always a non empty set.

We have immediately

Corollary 11.3. *Let (M, g) be a compact Kähler manifold of complex dimension $n \geq 2$, and $\widetilde{P}$ the principal $\mathrm{Spin}^c(2n)$-bundle over (M, g), $\mathcal{L}$ its determinant bundle. Fix $p > 2n = \dim_{\mathbb{R}} M$ arbitrarily. For each $\ell \geq 2$, let $\mathcal{C}_{\ell}^{p}(\widetilde{P})$, the configulation space of L_{ℓ}^{p} pairs (A, ψ), and let $\mathcal{G}_{\ell+1}^{p}(\widetilde{P})$ be the L_{l+1}^{p}-gauge transformations. Let $\mathcal{B}_{\ell}^{p}(\widetilde{P}) = \mathcal{C}_{\ell}^{p}(\widetilde{P})/\mathcal{G}_{\ell+1}^{p}(\widetilde{P})$ be the quotient space, and let ${\mathcal{B}^*}_{\ell}^{p}(\widetilde{P}) = {\mathcal{C}^*}_{\ell}^{p}(\widetilde{P})/\mathcal{G}_{\ell+1}^{p}(\widetilde{P})$ be the space of equivalence classes of irreducible pairs (A, ψ). Then, ${\mathcal{B}^*}_{\ell}^{p}(\widetilde{P})$ is a Banach manifold. Let $\mathcal{M}_{\ell}^{p}(\widetilde{P}) \subset \mathcal{B}_{\ell}^{p}(\widetilde{P})$ be the moduli space of equivalence classes of solutions of the Seiberg–Witten equations. Then, the natural map $\iota_{\ell}^{p} : \mathcal{B}_{\ell}^{p}(\widetilde{P}) \to \mathcal{B}_{2}^{p}(\widetilde{P})$ is an inclusion, and a smooth embedding on the open subset ${\mathcal{B}^*}_{\ell}^{p}(\widetilde{P})$ of irreducible pairs. Furthermore, ι_{ℓ}^{p} induces a homeomorphism of $\mathcal{M}_{\ell}^{p}(\widetilde{P})$ to $\mathcal{M}_{2}^{p}(\widetilde{P})$. At any irreducible solution $[A, \psi] \in \mathcal{M}_{\ell}^{p}(\widetilde{P})$, the differential of ι_{ℓ}^{p} induces an isomorphism between the tangent spaces of the moduli spaces. The open subset of irreducible, smooth points of $\mathcal{M}_{\ell}^{p}(\widetilde{P})$ maps diffeomorphically onto the open subset of irreducible, smooth points of $\mathcal{M}_{2}^{p}(\widetilde{P})$.*

12. Appendix

In this appendix, we give a proof of the following regularity theorem of solutions of the Seiberg–Witten equations.

Theorem 12.1 (cf. Theorem 6.1)**.** *Assume that $p > \dim_{\mathbb{R}} M = 2n$. We take $p = 2$ in the case of $n = 2$. Then, for every solution (A, ψ) in $\mathcal{C}_{1}^{p}(\widetilde{P})$ of the Seiberg–Witten equations, there exists a gauge transform $\sigma \in \mathcal{G}_{2}^{p}(\widetilde{P})$ such that $(A, \psi) \cdot \sigma$ is C^{∞}.*

Proof. Let $p > \dim_{\mathbb{R}} M = 2n$. Assume that $(A, \psi) \in \mathcal{C}_{1}^{p}(\widetilde{P})$ is a solution of the Seiberg–Witten equations, i.e.,

$$\begin{cases} \mathfrak{D}_A\psi = 0, \\ c^+(F_A) = \psi \otimes \psi^* - \dfrac{1}{2^{n-1}}|\psi|^2 \operatorname{Id} =: q(\psi). \end{cases} \tag{124}$$

• Let A_0 be a C^∞ connection of $\mathcal{L}$ and we write $A = A_0 + \alpha$, where $\alpha \in L_1^p(T^*M \otimes \sqrt{-1}\mathbb{R})$. Then, since

$$0 = \mathfrak{D}_A\psi = \mathfrak{D}_{A_0}\psi + \alpha \cdot \psi,$$

we have

$$\mathfrak{D}_{A_0}\psi = -\alpha \cdot \psi. \tag{125}$$

Due to the Sobolev Multiplication Theorem, $-\alpha \cdot \psi \in L_1^p$. Since $\mathfrak{D}_{A_0}$ is a first order elliptic differential operator with C^∞ coefficients, $\psi \in L_2^p$. Due to the Sobolev Multiplication Theorem, $q(\psi) \in L_2^p$. By the second equation of (125), and Lemma 8.3 and (95) (noticing that we used only Corollary 2.6 in its proof), we have $F_A \in L_2^p$ in case of $n \geq 3$. For the case $n = 2$, we also have $F_A = F_{A_0} + d\alpha_1 \in L_2^p$ because $\alpha_1 \in L_3^p$ as (127). By means of L_ℓ^p-fixing lemma (cf. Theorem 7.3), there exists $\sigma_1 \in \mathcal{G}_4^p(\widetilde{P})$ such that ${\sigma_1}^*A = A_0 + \alpha_1$ with $\alpha_1 \in L_3^p(T^*M \otimes \sqrt{-1}\mathbb{R})$, and

$$\|\alpha_1\|_{L_3^p} \leq C\|F_A^+\|_{L_2^p} + K. \tag{126}$$

• Since $(A, \psi) \cdot \sigma_1 = ({\sigma_1}^*A, {\sigma_1}^{-1}\psi) \in \mathcal{C}_3^p(\widetilde{P})$ is also a solution of the Seiberg–Witten equation, we repeat the above argument to the $({\sigma_1}^*A, {\sigma_1}^{-1}\psi)$, we have $\sigma^{-1}\psi \in L_4^p$, and $q(\sigma_1^{-1}\psi) \in L_4^p$. And we have also $F_A = F_{{\sigma_1}^*A} \in L_4^p$. Due to the L_ℓ^p-gauge fixing lemma (cf. Theorem 7.3), there exists $\sigma_2 \in \mathcal{G}_6^p(\widetilde{P})$ such that ${\sigma_2}^*({\sigma_1}^*A) = A_0 + \alpha_2$ with $\alpha_2 \in L_5^p(T^*M \otimes \sqrt{-1}\mathbb{R})$, and

$$\|\alpha_2\|_{L_5^p} \leq C\|F_A^+\|_{L_4^p} + K.$$

• We continue this process, so that we have $F_A \in L_k^p$ for all $k \geq 2$. This means that $F_A \in C^\infty$ due to the Sobolev Embedding Theorem. Since $A = A_0 + \alpha$, we have $F_A = F_{A_0} + d\alpha$, so that $d\alpha = F_A - F_{A_0} \in C^\infty$.

• Now let us recall the de Rham decomposition (see for example, [10] p. 252 (2.33), (2.35)), for every $\alpha \in L_1^p(T^*M \otimes \sqrt{-1}\mathbb{R})$,

$$\alpha = H\alpha + \Delta G\alpha = H\alpha + d\delta G\alpha + \delta dG\alpha,$$

where H is the projection onto harmonic forms, $\Delta = d\delta + \delta d$ is the Laplacian, and G is the Green operator which sends $L_1^p(T^*M \otimes \sqrt{-1}\mathbb{R})$ to $L_3^p(T^*M \otimes \sqrt{-1}\mathbb{R})$.

Let $s := -\delta G\alpha \in L_2^p(M, \sqrt{-1}\mathbb{R})$ and let $\sigma := e^s \in L_2^p(M, U(1)) = \mathcal{G}_2^p(\widetilde{P})$. Then, $\sigma^{-1}d\sigma = ds = -d\delta G\alpha$. Then, we have

$$\sigma^* A = A + \sigma^{-1}d\sigma = A_0 + \alpha + ds = A_0 + H\alpha + \delta dG\alpha,$$

and

$$d\alpha = d\delta dG\alpha = \Delta(dG\alpha), \tag{127}$$

which is C^∞. Since Δ is an elliptic operator, $dG\alpha$ is C^∞ due to (129), and then $H\alpha + dG\alpha$ is C^∞ since $H\alpha$ is harmonic. By (128), $\sigma^* A$ is C^∞.

• Since $(A, \psi) \cdot \sigma = (\sigma^* A, \sigma^{-1}\psi)$ is also a solution of the Seiberg–Witten equations, it holds that

$$\mathfrak{D}_{\sigma^* A}(\sigma^{-1}\psi) = 0.$$

Since $\sigma^* A \in C^\infty$, the equation $\mathfrak{D}_{\sigma^* A}(\sigma^{-1}\psi) = 0$ is the first order elliptic P.D.E. with C^∞-coefficients, so that $\sigma^{-1}\psi \in C^\infty$. Thus, $(A, \psi) \cdot \sigma$ is C^∞.

• If $n = 2$, we take $p = 2$. Then, one can prove the theorem by proceeding the similar argument in the proof of Theorem 5.3.6 in [16], p. 84. We have Theorem 6.1. □

References

[1] K. Akutagawa, Spin^c geometry, the Seiberg–Witten equations, and Yamabe invariants of Kähler surfaces, *Interdscip. Inform. Sci.* **5** (1999), 55–72.

[2] M. F. Atiyah, N. J. Hitchin and I. M. Singer, Self-duality in four dimensional Riemannian geometry, *Phil. Trans. Roy. Soc. London* A**362** (1978), 524–461.

[3] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, *Springer*, 1998.

[4] S. K. Donaldson, The Seiberg–Witten equations and 4-manifold topology, *Bull. Amer. Math. Soc.* **33** (1996), 45–70.

[5] S. K. Donaldson, The Seiberg–Witten equations and almost-Hermitian geometry, *Contemp. Math.* **288** (2001), 32–38.

[6] D. S. Donaldson and P. Kronheimer, The Geometry of Four-Manifolds, *Oxford Univ. Press*, 1990.

[7] D. S. Freed and K. K. Uhlenbeck, Instantons and Four-Manifolds, MSRI Publ., *Springer*, 1984, 1991.

[8] M. Furuta, Donaldson invariants and Seiberg–Witten theory, *Sugaku* **50** (1998), 181–198 (in *Japanese*).

[9] A. Futaki, Lectures on Differential Geometry – Application to General Theory and Modern Phisics, *Saiensu*, 2003 (in *Japanese*).

[10] S. Kobayashi, Differential Geometry of Complex Vector Bundles, *Iwanami and Princeton Univ. Press*, 1987.

[11] P. Kronheimer and T. Mrowka, The genus of embedded surfaces in the projective planes, *Math. Res. Lett.* **1** (1994), 797–808.

[12] H. B. Lawson and M-L. Michelsohn, Spin Geometry, *Princeton Univ. Press*, 1989.

[13] C. LeBrun, Weyl curvature, Einstein metrics, and Seiberg–Witten theory, *Math. Res. Lett.* **5**, no. 4 (1998), 423–438.

[14] C. LeBrun, Hyperbolic manifolds, harmonic forms, and Seiberg–Witten invariants, *Geom. Dedicata* **91** (2002), 137–154.

[15] J. D. Moore, Lectures on Seiberg–Witten Invariants, Lecture Notes in Math. vol. 1629, *Springer*, 1996.

[16] J. W. Morgan, The Seiberg–Witten Equations and Applications to the Topology of Smooth Four-Manifolds, *Princeton Univ. Press*, 1996.

[17] H. Nakajima, Removable singularities for Yang–Mills connections in higher dimensions, *J. Fac. Sci. Univ. Tokyo.* **34** (1987), 299–307.

[18] H. Nakajima, Compactness of the moduli space of Yang–Mills connections in higher dimensions, *J. Math. Soc. Japan* **40** (1988), 383–392.

[19] T. Nitta and T. Taniguchi, Quartenionic Seiberg–Witten equation, *Intern. J. Math.* **7** (1996), 697–703.

[20] N. Seiberg and E. Witten, Electromagnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang–Mills theory, *Nucl. Phys.* **B426** (1994), 19–52.

[21] N. Seiberg and E. Witten, Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD, *Nucl. Phys.* **B431** (1994), 581–640.

[22] K. Seimiya, Kobayashi–Hitchin correspondence for perturbed Seiberg–Witten equations, *Tokyo J. Math.* **21** (1998), 83–93.

[23] L. M. Sibner, The isolated point singularity problem for the coupled Yang–Mills equations in higher dimensions, *Math. Ann.* **271** (1985), 125–131.

[24] B. Speh, Seiberg–Witten equations on locally symmetrix spaces, *Proc. Sympos. Pure Math., Amer. Math. Soc.* **68** (2000), 517–525.

[25] T. Sunada, Heat operator and its applications, *Proceedings of Global Analysis at Katata* (1972), 13–40 (in *Japanese*).

[26] Y. Tanaka, Half-scaled monopole equations in six dimension and codimension two rectifiable currents (*to appear*).

[27] C. H. Taubes, The Seiberg–Witten invariants ans symplectic forms, *Math. Res. Lett.* **1** (1994), 809–822.

[28] C. H. Taubes, More constraints on symplectic forms from Seiberg–Witten invariants, *Math. Res. Lett.* **2** (1995), 9–13.

[29] C. H. Taubes, The Seiberg–Witten and Gromov invariants, *Math. Res. Lett.* **2** (1995), 221–238.

[30] C. H. Taubes, SW $\Longrightarrow$ Gr: from the Seiberg–Witten equations to pseudo-holomorphic curves, *J. Amer. Math. Soc.* **8** (1996), 845–918.

[31] C. H. Taubes, The geometry of the Seiberg–Witten invariants, *Surveys in Differential Geometry* **3** (1998), 299–339.

[32] K. K. Uhlenbeck, Removable singularities in Yang–Mills fields, *Commun. Math. Phys.* **83** (1982), 11–29.

[33] K. K. Uhlenbeck, Connections with L^p bounds on curvature, *Commun. Math. Phys.* **83** (1982), 31–42.

[34] H. Urakawa, Yang–Mills theory over compact symplectic manifolds, *Ann. Global Anal. Geom.* **25** (2004), 365–402.

[35] K. Wehrheim, Uhlenbeck Compactness, EMS Series of Lectures in Mathematics, *European Mathematical Society (EMS)*, *Zürich*, 2004, viii+212.

[36] E. Witten, Monopoles and 4-manifolds, *Math. Res. Lett.* **1** (1994), 764–796.

[37] K. Yosida, Functional Analysis, Third Ed., *Springer*, 1971.

HAJIME URAKAWA
DIVISION OF MATHEMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCES
TOHOKU UNIVERSITY
AOBA 6-3-09
SENDAI, 980-8579
JAPAN

E-mail: urakawa@math.is.tohoku.ac.jp

(Received November 14, 2008; revised February 28, 2009)