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Linear iterative equations of higher orders and random-valued functions

By KAROL BARON (Katowice)

Dedicated to Professor Zoltán Daróczy on his 70th birthday

Abstract. Given a probability space $(\Omega, \mathcal{A}, P)$, a separable metric space X with the σ-algebra $\mathcal{B}$ of all its Borel subsets and a $\mathcal{B} \otimes \mathcal{A}$-measurable $f : X \times \Omega \to X$ we consider the equation

$$\varphi(x) = \int_\Omega \varphi(f(x,\omega))P(d\omega) \tag{E}$$

and iterates $f^n, n \in \mathbb{N}$, of f defined on $X \times \Omega^{\mathbb{N}}$ by $f^1(x,\omega) = f(x,\omega_1)$ and $f^{n+1}(x,\omega) = f(f^n(x,\omega),\omega_{n+1})$. Assuming that for every $x \in X$ the sequence $(f^n(x,\cdot))_{n\in\mathbb{N}}$ converges in law and $\pi(x,\cdot)$ denotes the limit distribution we show that for every Borel and bounded $u : X \to \mathbb{R}$ the function $x \mapsto \int_X u(y)\pi(x,dy)$, $x \in X$, is a Borel solution of (E) and we study regularity of these solutions.

1. Throughout the paper $(\Omega, \mathcal{A}, P)$ is a probability space and (X, ϱ) is a separable metric space.

Let $\mathcal{B}$ denote the σ-algebra of all Borel subsets of X. We say that $f : X \times \Omega \to X$ is a *random-valued function* (shortly: an *rv-function*) if it is measurable with respect to the product σ-algebra $\mathcal{B} \otimes \mathcal{A}$. The iterates of such an rv-function are

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given by

$$f^1(x,\omega_1,\omega_2,\dots) = f(x,\omega_1),$$
$$f^{n+1}(x,\omega_1,\omega_2,\dots) = f(f^n(x,\omega_1,\omega_2,\dots),\omega_{n+1})$$

for x from X and $(\omega_1,\omega_2,\dots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that $f^n : X\times\Omega^\infty \to X$ is an rv-function on the product probability space $(\Omega^\infty,\mathcal{A}^\infty,P^\infty)$. More exactly, the n-th iterate f^n is $\mathcal{B}\otimes\mathcal{A}_n$-measurable, where $\mathcal{A}_n$ denotes the σ-algebra of all the sets of the form

$$\{(\omega_1,\omega_2,\dots)\in\Omega^\infty : (\omega_1,\dots,\omega_n)\in A\}$$

with A from the product σ-algebra $\mathcal{A}^n$. (See [6; Section 1.4], [3], [4].)

Fix an rv-function $f : X\times\Omega\to X$.

According to R. Kapica [4; Theorem 2] the probability distribution of the limit in measure of $(f^n(x,\cdot))_{n\in\mathbb{N}}$ always produces a bounded solution of the equation

$$\varphi(x) = \int_\Omega \varphi(f(x,\omega))P(d\omega) \tag{1}$$

which in addition is nonconstant provided the limit really depends on x. We generalize this theorem to the weak convergence of distributions and show that a simple additional condition guarantees that both the limit distribution and solutions of (1) generated by this limit are Lipschitzian.

By a distribution (on X) we mean any probability measure defined on $\mathcal{B}$. Recall that a sequence $(\pi_n)_{n\in\mathbb{N}}$ of distributions converges weakly to a distribution π if

$$\lim_{n\to\infty}\int_X u(x)\pi_n(dx) = \int_X u(x)\pi(dx)$$

for any continuous and bounded function $u : X\to\mathbb{R}$. It is well known (see, [1; Theorem 11.3.3]) that this convergence is metrizable by the (Fortet–Mourier, Lévy–Prohorov, Wasserstein) metric:

$$\|\pi_1-\pi_2\|_W = \sup\left\{\left|\int_X u\,d\pi_1 - \int_X u\,d\pi_2\right| :\ u\in \mathrm{Lip}_1(X),\ \|u\|_\infty\le 1\right\},$$

where

$$\mathrm{Lip}_1(X) = \{u : X\to\mathbb{R} \mid |u(x)-u(z)|\le \varrho(x,z) \text{ for } x,z\in X\}$$

and $\|u\|_\infty = \sup\{|u(x)| :\ x\in X\}$ for a bounded $u : X\to\mathbb{R}$.

2. Let $\pi_n(x,\cdot)$ denote the distribution of $f^n(x,\cdot)$, i.e.,

$$\pi_n(x,B) = P^\infty(f^n(x,\cdot)\in B) \tag{2}$$

for $n\in\mathbb{N},\ x\in X$ and $B\in\mathcal{B}$. Clearly, $\pi_1(x,\cdot)$ is the distribution of $f(x,\cdot)$:

$$\pi_1(x,B) = P(f(x,\cdot)\in B) \quad \text{for } x\in X \text{ and } B\in\mathcal{B}. \tag{3}$$

We start with the following lemma.

Lemma 1. *For any $n\in\mathbb{N}$ and $B\in\mathcal{B}$ the function $\pi_n(\cdot,B)$ given by (2) is Borel and*

$$\pi_{n+1}(x,B) = \int_\Omega \pi_n(f(x,\omega),B)P(d\omega) \quad \textit{for } x\in X; \tag{4}$$

moreover, if $u: X\to\mathbb{R}$ is Borel and bounded, then the function

$$x\mapsto \int_X u(y)\pi_n(x,dy), \quad x\in X, \tag{5}$$

is Borel, for every $x\in X$ the function

$$\omega\mapsto \int_X u(y)\pi_n(f(x,\omega),dy), \quad \omega\in\Omega, \tag{6}$$

is $\mathcal{A}$-measurable and

$$\int_X u(y)\pi_{n+1}(x,dy) = \int_\Omega\left(\int_X u(y)\pi_n(f(x,\omega),dy)\right)P(d\omega). \tag{7}$$

Proof. Since

$$C := \{(x,\omega)\in X\times\Omega^\infty : f^n(x,\omega)\in B\}\in\mathcal{B}\otimes\mathcal{A}^\infty,$$

the function

$$x\mapsto P^\infty(C_x), \quad x\in X,$$

i.e. $\pi_n(\cdot,B)$, is (see, e.g., [7; Theorem 6.3.1]) Borel. To get (4) note that (by induction)

$$f^{n+1}(x,\omega_1,\omega_2,\dots) = f^n(f(x,\omega_1),\omega_2,\omega_3,\dots)$$

for $x\in X$ and $(\omega_1,\omega_2,\dots)\in\Omega^\infty$, and observe that

$$\begin{aligned}\pi_{n+1}(x,B) &= P^\infty(\{(\omega_1,\omega_2,\dots)\in\Omega^\infty : f^n(f(x,\omega_1),\omega_2,\omega_3,\dots)\in B\})\\ &= P^\infty(\{(\omega_1,\omega_2,\dots)\in\Omega^\infty : (\omega_2,\omega_3,\dots)\in C_{f(x,\omega_1)}\})\\ &= \int_\Omega P^\infty(C_{f(x,\omega_1)})(P(d\omega_1) = \int_\Omega \pi_n(f(x,\omega_1),B)P(d\omega_1)\end{aligned}$$

for $x\in X$.

If $B \in \mathcal{B}$ and $u = \mathbb{1}_B$, then (5) is the function $\pi_n(\cdot, B)$ – and we already have shown that it is Borel – whereas (6) is the function $\omega \mapsto \pi_n(f(x,\omega), B), \omega \in \Omega$, which is clearly $\mathcal{A}$-measurable, and (7) reduces to (4) for every $x \in X$. A pass to the general case is standard. □

Now we assume the following condition.

(H) For every $x \in X$ the sequence $(\pi_n(x, \cdot))_{n \in \mathbb{N}}$ defined by (2) converges weakly to a distribution $\pi(x, \cdot)$.

The following theorem generalizes [4; Theorem 2].

Theorem 1. *If* (H) *holds, then for every Borel and bounded* $u : X \to \mathbb{R}$ *the function* $\varphi : X \to \mathbb{R}$ *given by*

$$\varphi(x) = \int_X u(y)\pi(x, dy) \tag{8}$$

is a Borel and bounded solution of (1)*; in particular, for any* $B \in \mathcal{B}$ *the function* $\pi(\cdot, B)$ *is a Borel solution of* (1).

Proof. Assume first that $u : X \to \mathbb{R}$ is continuous and bounded. Since for every $n \in \mathbb{N}$ the function (5) is Borel, so is (see [1; Theorem 4.2.2]) the pointwise limit

$$x \mapsto \int_X u(y)\pi(x, dy), \quad x \in X, \tag{9}$$

of the sequence built of these functions. Moreover, making use of (7) and applying the Lebesgue dominated theorem we have also

$$\int_X u(y)\pi(x, dy) = \int_\Omega \left(\int_X u(y)\pi(f(x,\omega), dy) \right) P(d\omega) \quad \text{for } x \in X, \tag{10}$$

which means that $\varphi : X \to \mathbb{R}$ given by (8) solves (1).

Fix now a Borel and bounded function $u_0 : X \to \mathbb{R}$, put

$$M = \|u_0\|_\infty$$

and consider the family $\mathbf{U}$ of all Borel functions $u : X \to [-M, M]$ such that the function (9) is Borel and (10) holds. The previous part of the proof shows that any continuous $u : X \to [-M, M]$ is in $\mathbf{U}$. Moreover, from the Lebesgue dominated convergence theorem it follows that $\mathbf{U}$ contains the limit of any pointwise convergent sequence of functions in $\mathbf{U}$. Consequently (see [7; Theorem 4.5.2]) every Borel function $u : X \to [-M, M]$ belongs to $\mathbf{U}$. In particular, $u_0 \in \mathbf{U}$. This proves the main part of Theorem 1.

To finish the proof observe that if $B \in \mathcal{B}$, then putting $u = \mathbb{1}_B$ in (8) we get $\varphi = \pi(\cdot, B)$. □

Remark 1. Assume (H), let $A \in \mathcal{B}$ and

$$P(f(x,\cdot) \in A) = 1 \quad \text{for } x \in A. \tag{11}$$

If $p \in [0,1]$, $F \subset X$ is closed and

$$P(f(x,\cdot) \in F) \geq p \quad \text{for } x \in A, \tag{12}$$

then

$$\pi(x,F) \geq p \quad \text{for } x \in A. \tag{13}$$

Proof. By induction, making use of (3) and the recurrence (4), we obtain

$$\pi_n(x,F) \geq p \quad \text{for } x \in A.$$

Since F is closed, this jointly with (H) gives (see [1; Theorem 11.1.1])

$$p \leq \limsup_{n\to\infty} \pi_n(x,F) \leq \pi(x,F)$$

for $x \in A$. □

Remark 2. Assume (H) and let a finite $A \subset X$ satisfies (11). If $x_0 \in X$,

$$f(x_0,\cdot) = x_0 \quad \text{a.s.} \tag{14}$$

and

$$P(f(x,\cdot) = x_0) < 1 \quad \text{for } x \in A, \tag{15}$$

then

$$\pi(x,\cdot) \neq \pi(x_0,\cdot) \quad \text{for } x \in A.$$

Proof. Let $B(x_0,r)$ denote the open ball with center at x_0 and radius r. From (15) it follows that

$$0 < P(f(x,\cdot) \in X \setminus \{x_0\}) = \lim_{n\to\infty} P\left(f(x,\cdot) \in X \setminus B\left(x_0, \frac{1}{n}\right)\right)$$

for $x \in A$. Since A is finite it shows that there is a positive integer n such that (12) holds with

$$F = X \setminus B\left(x_0, \frac{1}{n}\right) \quad \text{and} \quad p = \min\{P(f(x,\cdot) \in F) : x \in A\} > 0.$$

By Remark 1 we have (13). On the other hand, from (3) and (14),

$$\pi_1(x_0,B) = P(x_0 \in B) = \mathbb{1}_B(x_0) = \delta_{x_0}(B)$$

for $B \in \mathcal{B}$ which jointly with (4) and (14) shows that $\pi_n(x_0, \cdot) = \delta_{x_0}$ for $n \in \mathbb{N}$. Consequently also

$$\pi(x_0, \cdot) = \delta_{x_0} \tag{16}$$

and since $x_0 \notin F$ we see that

$$\pi(x_0, F) = 0 < p \leq \pi(x, F)$$

for $x \in A$. □

3. To obtain more information about the limit distribution and solutions of (1) generated by this limit we assume that

$$\int_\Omega \varrho(f(x,\omega), f(z,\omega))P(d\omega) \leq \lambda\varrho(x,z) \quad \text{for } x, z \in X. \tag{17}$$

Theorem 2. *Assume* (H). *If* (17) *holds with a* $\lambda \in (0, \infty)$, *then:*

(i) *for every Lipschitzian and bounded* $u : X \to \mathbb{R}$ *the function* $\varphi : X \to \mathbb{R}$ *given by* (8) *is of the first Baire class and a bounded solution of* (1)*;*

(ii) *if* $x, z \in X$ *and* $\pi(x, \cdot) \neq \pi(z, \cdot)$, *then* (1) *has a bounded solution* $\varphi : X \to \mathbb{R}$ *of the first Baire class such that* $\varphi(x) \neq \varphi(z)$.

Proof. From (17) it follows by induction that

$$\int_{\Omega^\infty} \varrho(f^n(x,\omega), f^n(z,\omega))P^\infty(d\omega) \leq \lambda^n \varrho(x,z) \quad \text{for } x, z \in X \text{ and } n \in \mathbb{N}.$$

Hence, if $u : X \to \mathbb{R}$ is bounded and Lipschitzian with a Lipschitz constant L, then

$$\left| \int_{\Omega^\infty} u(f^n(x,\omega))P^\infty(d\omega) - \int_{\Omega^\infty} u(f^n(z,\omega))P^\infty(d\omega) \right| \leq L\lambda^n \varrho(x,z),$$

i.e., by (2),

$$\left| \int_X u(y)\pi_n(x, dy) - \int_X u(y)\pi_n(z, dy) \right| \leq L\lambda^n \varrho(x,z) \quad \text{for } x, z \in X \tag{18}$$

and for $n \in \mathbb{N}$. This shows that the function $\varphi : X \to \mathbb{R}$ given by (8) is the pointwise limit of Lipschitzian functions

$$x \mapsto \int_X u(y)\pi_n(x, dy), \quad x \in X,$$

hence of the first Baire class. This and Theorem 1 give (i).

To get (ii) it is enough to observe that if $\pi(x,\cdot) \neq \pi(z,\cdot)$, then (see [1, Proposition 11.3.2]) there exists a bounded $u \in \mathrm{Lip}_1(X)$ such that

$$\int_X u(y)\pi(x,dy) \neq \int_X u(y)\pi(z,dy)$$

and to apply part (i). □

Corollary 1. *Assume* (H) *and let* X *be compact. If* (17) *holds with a* $\lambda \in (0,\infty)$*, then for every continuous* $u : X \to \mathbb{R}$ *the function* $\varphi : X \to \mathbb{R}$ *given by* (8) *is of the first Baire class and a bounded solution of* (1)*.*

Proof. Fix a continuous function $u : X \to \mathbb{R}$ and (see [1; Theorem 11.2.4]) let $(u_n)_{n\in\mathbb{N}}$ be a sequence of Lipschitzian mappings of X into $\mathbb{R}$ uniformly convergent to u. Defining $\varphi_n : X \to \mathbb{R}$ by

$$\varphi_n(x) = \int_X u_n(y)\pi(x,dy) \tag{19}$$

for $n \in \mathbb{N}$ we see that $(\varphi_n)_{n\in\mathbb{N}}$ uniformly converges to the function $\varphi : X \to \mathbb{R}$ given by (8). It follows from Theorem 2(i) that φ_n is of the first Baire class for every $n \in \mathbb{N}$, and so is (see [7; Theorem 3.5.2]) the uniform limit φ. This and Theorem 1 end the proof. □

Corollary 2. *Assume* (H)*. If* (17) *holds with a* $\lambda \in (0,\infty)$*, then for every closed subset* F *of* X *the function* $\pi(\cdot,F)$ *is of the second Baire class and a solution of* (1)*.*

Proof. Fix a closed $F \subset X$ and for every $n \in \mathbb{N}$ define $u_n, \varphi_n : X \to [0,1]$ by

$$u_n(x) = 1 - \min\{1, n\varrho(x,F)\}$$

and (19). Since u_n is Lipschitzian, by Theorem 2(i) the function φ_n is of the first Baire class for $n \in \mathbb{N}$, and since $(u_n)_{n\in\mathbb{N}}$ pointwise converges to $\mathbb{1}_F$, by the Lebesgue dominated theorem $(\varphi_n)_{n\in\mathbb{N}}$ pointwise converges to $\pi(\cdot,F)$. □

Assuming (17) with $\lambda = 1$ we can obtain much more.

Theorem 3. *Assume* (H)*. If*

$$\int_\Omega \varrho(f(x,\omega), f(z,\omega))P(d\omega) \le \varrho(x,z) \quad \text{for } x,z \in X, \tag{20}$$

then

$$\|\pi(x,\cdot) - \pi(z,\cdot)\|_W \le \varrho(x,z) \quad \text{for } x,z \in X \tag{21}$$

and

(i) *for every Lipschitzian and bounded* $u : X \to \mathbb{R}$ *the function* $\varphi : X \to \mathbb{R}$ *given by* (8) *is a Lipschitzian and bounded solution of* (1)*;*

(i) *if* $x, z \in X$ *and* $\pi(x, \cdot) \neq \pi(z, \cdot)$, *then* (1) *has a Lipschitzian and bounded solution* $\varphi : X \to \mathbb{R}$ *such that* $\varphi(x) \neq \varphi(z)$.

Proof. If $u \in \mathrm{Lip}_1(X)$ is bounded, then we have (18) with $L = 1$ and $\lambda = 1$ for every $n \in \mathbb{N}$ and passing to the limit we see that (21) holds.

If $u : X \to \mathbb{R}$ is bounded and Lipschitzian with a Lipschitz constant L, then we have (18) with $\lambda = 1$ and passing to the limit we see that $\varphi : X \to \mathbb{R}$ given by (8) is Lipschitzian with a Lipschitz constant L.

To get (ii) we argue as in the proof of Theorem 2. □

Remark 3 (cf. [2; Theorem 5.1]). Assume (H). If $\varphi : X \to \mathbb{R}$ is a continuous and bounded solution of (1), then

$$\varphi(x) = \int_X \varphi(y)\pi(x, dy) \quad \text{for } x \in X; \tag{22}$$

in particular, if $x, z \in X$ and $\pi(x, \cdot) = \pi(z, \cdot)$, then $\varphi(x) = \varphi(z)$.

Proof. It follows from (1) and (2) that

$$\varphi(x) = \int_{\Omega^\infty} \varphi(f^n(x, \omega))P^\infty(d\omega) = \int_X \varphi(y)\pi_n(x, dy)$$

for $x \in X$ and $n \in \mathbb{N}$. Passing to the limit we obtain (22). □

Corollary 3. *Assume* (H) *and* (20)*. If* (1) *has a nonconstant continuous and bounded solution* $\varphi : X \to \mathbb{R}$, *then it has also a nonconstant Lipschitzian and bounded solution* $\varphi : X \to \mathbb{R}$.

Proof. It is enough to apply Remark 3 and Theorem 3(ii). □

Corollary 4. *Assume* (H) *and* (20)*. If* (14) *holds for an* $x_0 \in X$ *and any Lipschitzian and bounded solution* $\varphi : X \to \mathbb{R}$ *of* (1) *is a constant function, then for every* $x \in X$ *the sequence* $(f^n(x, \cdot))_{n\in\mathbb{N}}$ *converges to* x_0 *in probability.*

Proof. From (16) and Theorem 3(ii) it follows that

$$\pi(x, \cdot) = \delta_{x_0} \quad \text{for } x \in X.$$

Applying [1; Proposition 11.1.3] we obtain the assertion. □

Remark 4. In case of the convergence in probability [5; Theorem 3.4] brings completely different conditions ensuring that for some continuous and bounded $u : X \to \mathbb{R}$ the function $\varphi : X \to \mathbb{R}$ given by (8) is continuous and nonconstant. In view of Theorem 1, as the proof of [5; Theorem 3.4(iii)] shows it remains valid also in case (H) and for Borel and bounded $u : X \to \mathbb{R}$ satisfying (3.14) of [5].

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KAROL BARON
UNIWERSYTET ŚLĄSKI
INSTYTUT MATEMATYKI
BANKOWA 14
PL–40–007 KATOWICE
POLAND

E-mail: baron@us.edu.pl

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