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On continuous solutions of a class of conditional equations

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This paper is dedicated to Professor Zoltán Daróczy
on the occasion of his 70th birthday

Abstract. Let X be a real linear topological space and $M : \mathbb{R} \to \mathbb{R}$ be continuous and multiplicative. We determine the continuous solutions $f : X \to \mathbb{R}$ of the functional equation

$$f(x + M(f(x))y)f(x)f(y)[f(x + M(f(x))y) - f(x)f(y)] = 0.$$

In this way we generalize in particular a result of Z. Daróczy published in 1966, concerning the continuous solutions of the Gołąb–Schinzel functional equation.

1. Introduction

Let $\mathbb{R}$ denote the set of reals. The functional equation

$$f(x + f(x)y) = f(x)f(y) \tag{GS}$$

has appeared for the first time in the paper [14] by S. Gołąb and A. Schinzel and has been extensively investigated there in the class of functions $f : \mathbb{R} \to \mathbb{R}$. The equation has been obtained by S. Gołąb while looking for subgroups of the centroaffine group of $\mathbb{R}^2$ (see, e.g., [25, p. 12–13]). However, equation (GS) can

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also be derived from the equation

$$f(x+y) = f(x)f\left(\frac{y}{f(x)}\right), \tag{1}$$

obtained a bit earlier by J. Aczél [1] in connection with his research in the theory of geometric objects (it is enough to replace y by $f(x)y$ in (1)). Later equation (GS) has been given the name: *the Gołąb–Schinzel functional equation*. For more details and further references concerning it we refer to [2] and to the survey paper [7]; for some recent results see, e.g., [15], [16], [17], [20], [21].

The first and very elegant description of the continuous solutions of (GS), with more general domains (Hilbert space), has been provided by Z. Daróczy in [12], and it seems that this influenced further research of solutions satisfying some continuity conditions.

Motivated by R. Ger (cf. [13]) and stability results for (GS) proved in [9], [10], [11] the first two authors of this paper have determined in [3] the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(x+f(x)y)f(x)f(y)[f(x+f(x)y) - f(x)f(y)] = 0. \tag{2}$$

Namely the following theorem has been proved.

Theorem 1. *A continuous function $f : \mathbb{R} \to \mathbb{R}$ is a solution of equation* (2) *if and only if one of the following three conditions holds.*

(α) $f \equiv 0$.

(β) *There is $c \in \mathbb{R}$ such that $f(x) = \max\{cx+1, 0\}$ for $x \in \mathbb{R}$ or $f(x) = cx+1$ for $x \in \mathbb{R}$.*

(γ) *There is $a \in (0,\infty)$ such that f is of one of the following two forms:*

$$f(x) \le 1 - \frac{x}{a} \quad \text{for } x \in (a,\infty) \qquad \text{and} \quad f(x) = 0 \quad \text{for } x \in (-\infty, a];$$
$$f(x) \le 1 + \frac{x}{a} \quad \text{for } x \in (-\infty,-a) \quad \text{and} \quad f(x) = 0 \quad \text{for } x \in [-a,\infty).$$

Now, in the present paper we generalize that result by determining the continuous solutions $f : X \to \mathbb{R}$ of the equation

$$f(x+M(f(x))y)f(x)f(y)[f(x+M(f(x))y) - f(x)f(y)] = 0, \tag{3}$$

where X is a real linear topological space and $M : \mathbb{R} \to \mathbb{R}$ is a continuous multiplicative function. Thus we obtain in particular a generalization of the result of Z. Daróczy [12]. We also provide some examples of possible applications.

Note that in the case: $M(x) = 1$ for $x \in \mathbb{R}$, (3) becomes

$$f(x+y)f(x)f(y)[f(x+y) - f(x)f(y)] = 0, \tag{4}$$

which can be regarded as a form of the following conditional version

$$\text{if } f(x)f(y)f(x+y) \neq 0, \quad \text{then } f(x+y) = f(x)f(y)$$

of the well known exponential equation

$$f(x+y) = f(x)f(y) \tag{5}$$

(for details concerning equation (5) see, e.g., [2]); if $M(x) = x$ for $x \in \mathbb{R}$, (3) becomes equation (2), which also can be considered as a conditional form of the Gołąb–Schinzel equation (GS). Therefore solving (3) in some class of functions, we could say that we determine solutions (in that class of functions) of the equations 'lying between' (4) and (2). Moreover, our results correspond to some recent papers (see, e.g., [6], [8], [18], [22], [23], [24]) on conditional versions of (GS), motivated by a problem raised by P. Kahlig and originating in meteorology and fluid mechanics (cf. [18]).

In the sequel X denotes a real linear space, unless explicitly stated otherwise. Next, given $f : X \to \mathbb{R}$, for each $x \in X$ we define $f_x : \mathbb{R} \to \mathbb{R}$ by: $f_x(t) := f(tx)$ for $t \in \mathbb{R}$.

2. The exponential case

We start with the case where $M(x) = 1$ for every $x \in \mathbb{R}$, i.e. we determine the continuous solutions $f : X \to \mathbb{R}$ of (4).

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous solution of* (4) *and $S := \{x \in \mathbb{R} : f(x) \neq 0\}$. Then one of the following two statements is valid.*

(i) $S = \mathbb{R}$.

(ii) $0 \notin S$, *every connected component of S is a finite interval, and $S + S \subset \mathbb{R} \setminus S$.*

Proof. Assume that $S \neq \mathbb{R}$. Let $I = (a, b)$ be a connected component of S. We first show that I is a finite interval. For the proof by contradiction suppose that, for instance, $a = -\infty$ (the case $b = \infty$ is analogous). Clearly we have $b < \infty$ and the continuity of f implies $f(b) = 0$. Take $w \in I$ with $w < 0$. For every $x \in I$, $x < b$ we have $x + w \in I$, whence (4) implies $f(x + w) = f(x)f(w)$. Consequently letting $x \to b$ we obtain $f(b+w) = f(b)f(w) = 0$, which leads to a contradiction, because $b + w \in I$.

Let $w, u \in S$, w belong to a connected component I_1 of S, and u belong to a connected component I_2 of S. As we have just shown, I_1 and I_2 are finite intervals, so we may write that $I_i := (a_i, b_i)$ with some $a_i, b_i \in \mathbb{R}$, for $i = 1, 2$. Suppose $f(b_1 + u) \neq 0$. Since f is continuous, there exists $\varepsilon \in (0, b_1 - a_1)$ such that $f(t + u) \neq 0$ for $t \in (b_1 - \varepsilon, b_1)$. Next, by (4), we get $f(t + u) = f(t)f(u)$ for $t \in (b_1 - \varepsilon, b_1)$, whence, as t goes to b_1, we obtain $f(b_1 + u) = f(b_1)f(u) = 0$ which gives the contradiction. Therefore we have $f(b_1 + u) = 0$. Similarly, we can prove that $f(a_2 + w) = 0$.

Thus we have shown that $f((b_1 + I_2) \cup (I_1 + a_2)) = \{0\}$. Since $I_1 + I_2 \subset (b_1 + I_2) \cup (I_1 + a_2) \cup \{b_1 + a_2\}$, by the continuity of f we obtain $f(I_1 + I_2) = \{0\}$. This proves that $S + S \subset \mathbb{R} \setminus S$, which implies that $0 \notin S$. □

Lemma 2. *Let $f : X \to \mathbb{R}$ be a continuous solution of* (4), $0 \in f(X)$, $x \in X$, $f(x) \neq 0$, *and* $a(x) := \sup\{t \in (-\infty, 1) : f(tx) = 0\}$. *Then* $1 > a(x) \geq 0$ *and* $f(a(x)x + z) = 0$ *for every* $z \in X$ *with* $f(z) \neq 0$.

Proof. There is $w \in X$ with $f(w) = 0$. Write $f_w(t) := f(tw)$ for $t \in \mathbb{R}$ and $S_w := \{t \in \mathbb{R} : f_w(t) \neq 0\}$. Then f_w is a continuous solution of (4) and, by Lemma 1, $0 \notin S_w$, whence $f(0) = f(0\,w) = f_w(0) = 0$. This means that $0 \leq a(x) < 1$. Clearly $f(tx) \neq 0$ for $t \in (a(x), 1)$.

Take $z \in X$ with $f(z) \neq 0$ and suppose that $f(a(x)x + z) \neq 0$. Then there is $\varepsilon \in (0, 1 - a(x))$ such that, for every $t \in (a(x), a(x) + \varepsilon)$, we have $f(tx + z) \neq 0$ and consequently $f(tx + z) = f(tx)f(z)$. Hence

$$0 \neq f(a(x)x + z) = \lim_{t \to a(x)+0} f(tx + z) = \lim_{t \to a(x)+0} f(tx)f(z) = f(a(x)x)f(z) = 0.$$

This contradiction completes the proof. □

Now we are in a position to prove the following.

Proposition 1. *Let $f : X \to \mathbb{R}$ be continuous and $S := \{x \in X : f(x) \neq 0\}$. Then f is a solution of equation* (4) *if and only if one of the following two statements is valid.*

(i) *There is a continuous linear functional $g : X \to \mathbb{R}$ such that $f = \exp \circ g$.*

(ii) $S + S \subset X \setminus S$.

Proof. Obviously, any function $f : \mathbb{R} \to \mathbb{R}$ fulfilling either (i) or (ii) is a solution of (4).

Now, let $f : X \to \mathbb{R}$ be a continuous solution of (4). If $0 \notin f(X)$, then clearly f is a solution of (5) and we have statement (i) with $g := \ln \circ f$ (cf., e.g., [2]). So assume that $f(w) = 0$ for some $w \in X$.

Take $x, y \in S$. If $x = sy$ with some $s \in \mathbb{R}$, then, by Lemma 1 applied to f_y, $x + y \in X \setminus S$. It remains to consider the case where x and y are linearly independent. For the proof by contradiction suppose that $f(x + y) \neq 0$.

In view of Lemma 2, $a(x) \geq 0$. Next, since f is continuous, there is $\varepsilon \in (0, 1 - a(x))$ with $f(y + \varepsilon x) \neq 0$. Write $z := y + \varepsilon x$. Then, on account of Lemma 2, $f((a(x)+\varepsilon)x+y) = f(a(x)x+z) = 0$. Let $\sigma := \sup\{t \in (a(x)+\varepsilon, 1) : f(tx+y) = 0\}$. Clearly $1 > \sigma \geq a(x)+\varepsilon$, $f(\sigma x) \neq 0$, $f(tx) \neq 0$ and $f(tx+y) \neq 0$ for $t \in (\sigma, 1)$. Hence $f(tx + y) = f(tx)f(y)$ for $t \in (\sigma, 1)$ and consequently

$$0 = f(\sigma x + y) = \lim_{t\to\sigma+0} f(tx + y) = \lim_{t\to\sigma+0} f(tx)f(y) = f(\sigma x)f(y) \neq 0.$$

This contradiction proves that $x + y \notin S$. □

The following example shows that without the assumption of continuity of f the statement of Proposition 1 is not valid.

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ for $x < 1$ and $f(x) = e^x$ for $x \geq 1$. Then f is discontinuous at 1 and either of conditions (i), (ii) does not hold.

Example 2. Let $a, b \in (0, \infty)$, $b < 2a$. Then clearly $S := (a, b)$ satisfies the condition: $S + S \subset \mathbb{R} \setminus S$.

3. The Gołąb–Schinzel case

Next, we consider the case $M(x) = x$ for $x \in \mathbb{R}$. We start with a lemma, which follows immediately from Theorem 1 and [7, Remark 1].

Lemma 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $f(\mathbb{R}) \neq \{0\}$. Then the following three statements are valid.*

(i) *f is a solution of equation* (2) *if, and only if, f satisfies equation* (GS) *or it is a solution of the functional equation*

$$f(x + f(x)y)f(x)f(y) = 0. \tag{6}$$

(ii) *f is a solution of* (GS) *if, and only if, statement* (β) *of Theorem 1 holds.*

(iii) *f is a solution of* (6) *if, and only if, statement* (γ) *of Theorem 1 is valid.*

Proposition 2. *Let $f : X \to \mathbb{R}$ be continuous and $f(X) \neq \{0\}$. Then, the following two statements are valid.*

(i) *f is a solution of* (2) *if, and only if, f satisfies equation* (GS) *or it is a solution of the functional equation* (6).

(ii) *f is a solution of equation* (GS) *if, and only if, there exists a continuous linear functional $L : X \to \mathbb{R}$ such that f has one of the two following forms:*

$$f(x) = L(x) + 1 \qquad \text{for} \quad x \in X; \tag{7}$$

$$f(x) = \max\{L(x) + 1, 0\} \quad \text{for} \quad x \in X. \tag{8}$$

Proof. Statement (ii) is already well known (see [7] or [5]) and the sufficient condition of statement (i) is trivial. So, it remains to prove the necessary condition of statement (i).

Assume that f is a solution of (2). With $x = y = 0$ in (2) we have $f(0) \in \{0, 1\}$. Observe also that f_x is a continuous solution of (2) for every $x \in X$.

If $f(0) = 0$, then, for each $x \in X$, $f_x(0) = 0$, whence, by Lemma 3, $f_x : \mathbb{R} \to \mathbb{R}$ is a solution of equation (6) and therefore $f(X) \subset (-\infty, 0]$. Take $z, w \in X$ with $f(z)f(w) \neq 0$. Since $f(z), f(w), f(z + f(z)w) \in (-\infty, 0]$, we have $f(z)f(w) > 0$ and consequently $f(z + f(z)w) \neq f(z)f(w)$. This means that $f(z + f(z)w) = 0$, because f is a solution of (2). Thus we have proved that f satisfies equation (6).

If $f(0) = 1$, then $f_x(0) = f(0) = 1$ for each $x \in X$. In view of Lemma 3, this means that $f_x \not\equiv 0$ satisfies equation (GS) for each $x \in X$.

Let $H = f^{-1}(\{1\})$. If $H = X$, then $f \equiv 1$ and consequently (7) holds with $L \equiv 0$. So, we suppose now that $H \neq X$. The continuity of f implies that H is a closed subset of X. We will prove that H is a closed hyperplane of X.

Observe that $H = \{x \in X : f_x \equiv 1\}$. In fact, if $x \in H$, then there exists $c(x) \in \mathbb{R}$ such that either $f_x(t) = \max\{c(x)t + 1, 0\}$ for $t \in \mathbb{R}$ or $f_x(t) = c(x)t + 1$ for $t \in \mathbb{R}$. Since $f(x) = f_x(1) = 1$, we have $c(x) = 0$, which implies $f_x \equiv 1$.

First we show that H is a linear subspace of X. Obviously, if $x \in H$, then $\lambda x \in X$ for all $\lambda \in \mathbb{R}$. Next, take $x, y \in H$. Then $f_x = f_y \equiv 1$, whence in view of (2), for every $t \in \mathbb{R}$, we get

$$\begin{aligned} 0 &= f(tx + f(tx)ty)f(tx)f(ty)[f(tx + f(tx)ty) - f(tx)f(ty)] \\ &= f(tx + ty)[f(tx + ty) - 1]. \end{aligned}$$

Thus $f_{x+y}(\mathbb{R}) \subset \{0, 1\}$. As f is continuous and $f_{x+y}(0) = f(0) = 1$, this yields $f_{x+y} \equiv 1$ and so $x + y \in H$. We prove now that H is of codimension 1. For the proof by contradiction suppose that there exist two linearly independent vectors

x_1 and x_2 in X such that the linear subspace K of X, generated by x_1 and x_2, satisfies

$$K \cap H = \{0\}. \tag{9}$$

Since $f_{x_i}(\mathbb{R})$ is either $\mathbb{R}$ or $[0, \infty)$ for $i = 1, 2$, there exist $t_1, t_2 \in \mathbb{R}$ such that $f(t_1 x_1) < 1$ and $f(t_2 x_2) > 1$. Next, by the continuity of f, $1 \in f([t_1 x_1, t_2 x_2])$, where $[t_1 x_1, t_2 x_2] = \{\lambda t_1 x_1 + (1 - \lambda) t_2 x_2;\ \lambda \in [0, 1]\}$. So there exists $x_0 \in H \cap [t_1 x_1, t_2 x_2]$. Since x_1 and x_2 are linearly independent, 0 does not belong to $[t_1 x_1, t_2 x_2]$ and therefore $x_0 \neq 0$. This contradicts (9).

Thus we have shown that H is a closed hyperplane of X and consequently there exists a nontrivial continuous linear functional $l : X \to \mathbb{R}$ such that $H = \operatorname{Ker} l$. Furthermore, for every $x_0 \in X \setminus H$, we have $X = H \oplus \mathbb{R} x_0$, whence, for each $x \in X$, there exist unique $z(x) \in H$, $t(x) \in \mathbb{R}$ such that

$$x = z(x) + t(x) x_0 \tag{10}$$

and

$$t(x) = \frac{l(x)}{l(x_0)}. \tag{11}$$

Now, observe that one of the following two possibilities holds:

(a) there are $x_0 \in X \setminus H$ and $c_0 \in \mathbb{R} \setminus \{0\}$ such that $f_{x_0}(t) = 1 + c_0 t$ for $t \in \mathbb{R}$;

(b) for each $x_0 \in X \setminus H$ there is $c_0 \in \mathbb{R} \setminus \{0\}$ with $f_{x_0}(t) = \max\{1 + c_0 t, 0\}$ for $t \in \mathbb{R}$ (which yields $f_{x_0}(\mathbb{R}) \subset [0, \infty)$).

Consider first the case (a). In view of (10), replacing x by $z(x)$ and y by $t(x) x_0$ in (2), we get

$$f(x) = f(z(x) + t(x)x_0) = f(z(x) + f(z(x))t(x)x_0) = f(t(x)x_0) = 1 + c_0 t(x)$$

for $x \in X$ with $f(x) \neq 0$ and $t(x) \neq -\frac{1}{c_0}$. Hence, for every $t \in \mathbb{R} \setminus \{-\frac{1}{c_0}\}$, $f(tx_0 + H) \subset \{1 + c_0 t, 0\}$, which yields $f(tx_0 + H) = \{1 + c_0 t\}$, because f is continuous, H is connected, and $0 \neq 1 + c_0 t = f(tx_0 + 0) \in f(tx_0 + H)$. So $f(z + tx_0) = 1 + c_0 t$ for all $z \in H$, $t \in \mathbb{R} \setminus \{-\frac{1}{c_0}\}$. Next, the continuity of f implies $f(z + tx_0) = 1 + c_0 t$ for all $z \in H$, $t \in \mathbb{R}$. Consequently (10) and (11) imply that, for every $x \in X$, we have $f(x) = 1 + c_0 \frac{l(x)}{l(x_0)}$, which gives (7) with $L = \frac{c_0}{l(x_0)}\, l$.

In the case (b), we argue analogously as in the previous one (note that in this case $f(x) \neq 0$ or $f(-x) \neq 0$ for all $x \in X$). Namely, take $x_0 \in X \setminus H$. Then $f(tx_0 + H) = \{1 + c_0 t\}$ for each $t \in \mathbb{R}$ with $c_0 t > -1$. Without loss of generality we can assume that $c_0 > 0$ (clearly, we can always replace x_0 by $-x_0$). Further, since $f(X) = f_{x_0}(\mathbb{R}) = [0,\infty)$, for each $x \in X$ with $f(x) \neq 0$, there is $d \in (-\frac{1}{c_0},\infty)$ with $f(x) = f(dx_0)$ and therefore, since either $f(f(x)^{-1}(dx_0 - x)) \neq 0$ or $f(f(dx_0)^{-1}(x - dx_0)) \neq 0$, we have

$$f(dx_0) = f\left(x + f(x)\frac{dx_0 - x}{f(x)}\right) = f(x)f\left(\frac{dx_0 - x}{f(x)}\right)$$

or

$$f(x) = f\left(dx_0 + f(dx_0)\frac{x - dx_0}{f(dx_0)}\right) = f(dx_0)f\left(\frac{x - dx_0}{f(dx_0)}\right).$$

This shows that $x-dx_0 \in H$ and consequently $x \in (-\frac{1}{c_0},\infty)x_0+H$ for each $x \in X$ with $f(x) \neq 0$. Hence, from (10) and (11), we obtain (8) with $L = \frac{c_0}{l(x_0)}\, l$. $\square$

The following example shows that statement (i) of Proposition 2 does not hold without the assumption of continuity of f.

Example 3. The function $f : \mathbb{R} \to \mathbb{R}$, given by: $f(x) = x + 1$ for $x < 0$ and $f(x) = 0$ for $x \geq 0$, satisfies equation (2), but it is neither a solution of (6) (take $x = -2$ and $y = -\frac{1}{2}$) nor of (GS) (take $x = -2$ and $y = 0$).

4. The general case

Finally we have the tools to present the main result of this paper. In this part, as before, $M : \mathbb{R} \to \mathbb{R}$ is a continuous multiplicative function with $M(\mathbb{R}) \neq \{0\}$.

Remark 1. According to [19, p. 311, Theorem 6], either $M(\mathbb{R}) = \{1\}$, or there is $a > 0$ such that $M(x) = |x|^a$ for $x \in \mathbb{R}$ or $M(x) = |x|^a sgn\,(x)$ for $x \in \mathbb{R}$. Thus, if $M(\mathbb{R}) \neq \{1\}$, then either M is odd and bijective, or M is even and the function $M_0 := M\big|_{[0,\infty)}$ is a bijection on $[0,\infty)$.

Theorem 2. *Let X be a real linear topological space, $f : X \to \mathbb{R}$ be continuous, and $f(u)f(v)f(u+M(f(u))v) \neq 0$ for some $u, v \in X$. Then f is a solution of equation* (3) *if, and only if, there exists a continuous linear functional $L : X \to \mathbb{R}$ such that,*

1° *in the case where $M(\mathbb{R}) = \{1\}$, $f = \exp \circ L$;*

2° *in the case where M is odd, $f(x) = M^{-1}(L(x) + 1)$ for $x \in X$ or $f(x) = M^{-1}(\max\{L(x) + 1, 0\})$ for $x \in X$.*

3° *in the case where M is even and $M(\mathbb{R}) \neq \{1\}$, $f(x) = M_0^{-1}(\max\{L(x)+1, 0\})$ for $x \in X$, where $M_0 := M\big|_{[0,\infty)}$.*

Proof. The case $M(\mathbb{R}) = \{1\}$ results from Proposition 1. So assume that $M(\mathbb{R}) \neq \{1\}$ and f is a solution of equation (3). Then clearly $g := M \circ f$ is a continuous solution of (2), whence, according to Proposition 2, there exists a continuous linear functional $L : X \to \mathbb{R}$ such that $g(x) = L(x) + 1$ for $x \in X$ or $g(x) = \max\{L(x) + 1, 0\}$ for $x \in X$. In view of Remark 1, this yields the form of f when M is odd. If M is even, then $g(X) \subset [0, \infty)$ and consequently $g(x) = \max\{L(x) + 1, 0\}$ for $x \in X$. Clearly, it is enough to show now that $f(X) \subset [0, \infty)$.

So, let $x \in X$ and $f(x) \neq 0$. Then $M(f_x(t)) = M(f(tx)) = g(tx) = \max\{L(tx) + 1, 0\} = \max\{ct + 1, 0\}$ for $t \in \mathbb{R}$, where $c := L(x)$. Hence, for every $s, t \in \mathbb{R}$ with $ct, cs \in (-1, \infty)$,

$$f_x(s + (cs + 1)t) = f_x(s)f_x(t), \tag{12}$$

because $c(s + M(f_x(s))t) = c[s + (cs + 1)t] = (cs + 1)(ct + 1) - 1 > -1$. Next $M(f_x(1)) = M(f(x)) \neq 0$, whence $0 < M(f_x(1)) = c + 1$ and consequently $c > -1$. Clearly $f_x(s) \neq 0$ for $s \in \mathbb{R}$ with $cs + 1 > 0$.

Note that $c = 0$ yields $f(x) = f_x\left(\frac{1}{2} + \frac{1}{2}\right) = f_x\left(\frac{1}{2}\right)^2 > 0$ (in view of (12)). So it remains to consider the case $c \neq 0$. Let $s := \frac{\sqrt{c+1}-1}{c}$. Then $cs+1 = \sqrt{c+1} > 0$, $f_x(s) \neq 0$, and, by (12), $f(x) = f_x(1) = f_x(s + (cs + 1)s) = f_x(s)f_x(s) > 0$.

The converse is easy to verify (in view of Proposition 2(ii)). $\square$

Theorem 2 yields the following corollary, which is a generalization of statement (i) of Proposition 2.

Corollary 1. *Let X be a real linear topological space and $f : X \to \mathbb{R}$ be continuous. Then f is a solution of equation* (3) *if and only if f satisfies one of the following two functional equations:*

$$f(x)f(y)f(x + M(f(x)y) = 0, \tag{13}$$

$$f(x + M(f(x)y) = f(x)f(y). \tag{14}$$

Remark 2. In view of Corollary 1, the investigation of the solutions $f : X \to \mathbb{R}$ of equation (13) could be of some interest. The form of continuous $f : \mathbb{R} \to \mathbb{R}$, satisfying (13), can be easily deduced from Lemma 3. However, in a more general situation it needs some longer arguments and therefore will be considered in a separate paper. Also solutions $f : X \to \mathbb{R}$ of equation (3), under assumptions weaker than continuity, will be investigated in a next publication.

We complete the paper with three examples of applications of Theorem 2.

Corollary 2. *Let X be a real linear topological space, $z \in X\backslash\{0\}$, $f : X \to \mathbb{R}$ be continuous, $S := \{x \in X : f(x) \neq 0\}$, and $\star : X \times X \to X$ be given by: $x \star y := x + f(x)y$. Then*

$$x \star (y \star z) = (x \star y) \star z \quad \text{for } x, y \in S \quad \text{with } x \star y \in S \tag{15}$$

if, and only if, f satisfies (GS) *(and consequently $\star$ is associative) or f is a solution of equation* (6) *(which means that $x \star y \notin S$ for every $x, y \in S$).*

Proof. It is easy to check that (15) holds if and only if f is a solution of equation (2).

Assume that f is a solution of (2) and does not satisfy equation (6). Then $f(u)f(v)f(u + f(u)v) \neq 0$ for some $u, v \in X$. Consequently, f has the form described in Proposition 2(ii), which means that it is a solution to (GS). A simple calculation shows that then $\star$ is associative. The converse is trivial. □

In what follows, given a real linear space X, we define a binary operation $* : (\mathbb{R}_0 \times X)^2 \to \mathbb{R}_0 \times X$ by: $(a, x) * (b, y) := (ab, x + M(a)y)$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. It is easy to check that $(\mathbb{R}_0 \times X, \star)$ is a group. For more details concerning some special cases of this group see [5, p. 60].

Corollary 3. *Let $M(\mathbb{R}) \neq \{1\}$, X be a real linear topological space, $f : X \to \mathbb{R}$ be continuous, and $D := \{(f(x), x) : x \in X, f(x) \neq 0\}$. Then D is a subsemigroup of $(\mathbb{R} \times X, *)$ if, and only if, f satisfies* (14) *(which implies that D is a subgroup of $(\mathbb{R}_0 \times X, *)$).*

Proof. Let $S := \{x \in X : f(x) \neq 0\}$. Clearly D is a subsemigroup of $(\mathbb{R} \times X, *)$ if, and only if,

$$(f(x), x) * (f(y), y) = (f(x)f(y), x + M(f(x))y) \in D \quad \text{for } x, y \in S. \tag{16}$$

Next, it is easily seen that (16) is equivalent to the following condition:

$$f(x + M(f(x))y) = f(x)f(y) \quad \text{for } x, y \in S. \tag{17}$$

Hence, if D is a subsemigroup of $(\mathbb{R}_0 \times X, *)$, then f satisfies equation (3) and consequently, by Corollary 1, f is a solution of (14). Using similar arguments as in the proof of [5, Theorem 1(ii)], one can prove that this implies that D is a subgroup of $(\mathbb{R}_0 \times X, *)$. □

Corollary 4. *Let X be a real linear topological space and $f : X \to \mathbb{R}$ be continuous. Then f satisfies the conditional functional equation*

$$\text{if } \ f(x) \neq 0, \ \text{then } f(x + y) = f(x) f\left(\frac{y}{f(x)}\right) \tag{18}$$

if, and only if, f is a solution of equation (GS).

Proof. The case $f \equiv 0$ is trivial. So, suppose that $f(z) \neq 0$ for some $z \in X$.

Let f be a solution of (18). Then f is a solution of equation (2) (it is enough to replace y by $f(x)y$ in (18)); whence, by Corollary 1, f satisfies (GS), because (18) (with $x = z$ and $y = zf(z)$) implies that $f(z)f(z + f(z)z) \neq 0$.

Now assume that f satisfies (GS). Take $x, y \in X$ with $f(x) \neq 0$. Then replacing y by $f(x)^{-1}y$ in (GS) we get $f(x + y) = f(x)f\left(\frac{y}{f(x)}\right)$. □

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