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# D'Alembert's functional equation on topological monoids

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*Dedicated to Professor Zoltán Daróczy on the occasion of his 70th birthday*

**Abstract.** We prove that if $f$ is a continuous complex-valued function on the topological monoid $M$ with neutral element $e$ satisfying the functional equation

$$f(xyz) + f(xzy) = 2f(x)f(yz) + 2f(y)f(zx) + 2f(z)f(xy) - 4f(x)f(y)f(z)$$

and $f(e) = 1$, then there is a continuous homomorphism $h : M \to \mathrm{Mat}_2(\mathbb{C})$, the multiplicative monoid of complex $2 \times 2$ matrices such that $f = \frac{1}{2}\mathrm{tr} \circ h$. As a consequence we prove that if $f$ is a continuous function on the topological group $G$ satisfying $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ and $f(e) = 1$ then there is a continuous homomorphism $h : G \to \mathrm{SL}_2(\mathbb{C})$ such that $f = \frac{1}{2}\mathrm{tr} \circ h$.

## 1. Introduction

In our previous work [2] on d'Alembert's equation we found that our method of proof of the result in the abstract worked without reference to the inverse operation on the group, merely the associativity of multiplication and the existence of a neutral element. Moreover the equation (1.1) below whence all results flowed was already clearly central in our previous paper [2, Proposition 2.1] and had been suggested by an exercise in THURSTON [9] (see Example 1.1 below).

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Thus let $M$ be a topological monoid with neutral element $e$. We say that the function $f : M \to \mathbb{C}$ is a *pre-d'Alembert* function if it is continuous, $f(e) = 1$ and, for all $x$, $y$, $z$ in $M$

$$f(xyz)+f(xzy) = 2f(x)f(yz)+2f(y)f(zx)+2f(z)f(xy)-4f(x)f(y)f(z). \tag{1.1}$$

Clearly any continuous homomorphism from $M$ into the monoid $\langle \mathbb{C}, \cdot \rangle$ satisfies (1.1). A non-trivial example, relevant to our results and applications later is given in

*Example 1.1.* The function $X \to \frac{1}{2}\mathrm{trace}(X)$, denoted $\frac{1}{2}\,\mathrm{tr}$, is a pre-d'Alembert function on the multiplicative monoid of $2 \times 2$ complex matrices, $\mathrm{Mat}_2(\mathbb{C})$. For $\frac{1}{2}\,\mathrm{tr}$ is pre-d'Alembert if and only if, for all $X, Y, Z$ in $\mathrm{Mat}_2(\mathbb{C})$

$$\mathrm{tr}(XYZ + XZY) = \mathrm{tr}\,X\,\mathrm{tr}\,YZ + \mathrm{tr}\,Y\,\mathrm{tr}\,XZ + \mathrm{tr}\,Z\,\mathrm{tr}\,XY - \mathrm{tr}\,X\,\mathrm{tr}\,Y\,\mathrm{tr}\,Z. \tag{1.2}$$

Writing $X = \alpha E + X'$ where $\alpha = \frac{1}{2}\,\mathrm{tr}\,X$, and $\mathrm{tr}\,X' = 0$ we see that (1.2) is true if and only if

$$\mathrm{tr}(X'YZ + X'ZY) = \mathrm{tr}\,Y\,\mathrm{tr}\,X'Z + \mathrm{tr}\,Z\,\mathrm{tr}\,X'Y.$$

Now writing $Y = \beta E + Y'$, $\mathrm{tr}\,Y' = 0$, and $Z = \gamma E + Z'$, $\mathrm{tr}\,Z' = 0$ we see that (1.2) is true if and only if

$$\mathrm{tr}(X'(Y'Z' + Z'Y')) = 0$$

for all $X'$, $Y'$, $Z'$ with trace 0. But for $2 \times 2$ matrices of trace 0, $Y'Z' + Z'Y'$ is a scalar matrix, so the result follows.

We see from this example that if $h : M \mapsto \mathrm{Mat}_2(\mathbb{C})$ is a continuous homomorphism then $f : x \mapsto \frac{1}{2}\,\mathrm{tr}\,h(x)$ is a pre-d'Alembert function on $M$. For example $h : \langle \mathbb{C}, + \rangle \to \mathrm{Mat}_2(\mathbb{C})$ given by

$$z \mapsto \begin{bmatrix} e^{iz} & 0 \\ 0 & e^{-iz} \end{bmatrix}$$

yields the function $\cos(z)$. Here we see a connection with the classical d'Alembert functional equation (see [1, Ch. 2, §4]),

$$f(x + y) + f(x - y) = 2f(x)f(y). \tag{1.3}$$

Our main result on pre-d'Alembert functions is

**Theorem 4.12.** *If $f$ is a pre-d'Alembert function on $M$ then there is a continuous monoid homomorphism $h : M \to \mathrm{Mat}_2(\mathbb{C})$ such that*

$$f = \frac{1}{2}\operatorname{tr} \circ h. \tag{1.4}$$

We use this result to prove a structure theorem for d'Alembert functions (see [1], [6]).

*Definition 1.2.* Let $\tau$ be an involution on $M$ (a continuous anti-automorphism such that $\tau(\tau(x)) = x$ for all $x$ in $M$). The function $f : \langle M, \tau\rangle \to \mathbb{C}$ is a *d'Alembert function* if it is continuous, $f(e) = 1$ and for all $x$, $y$ in $M$

$$f(xy) + f(x\tau(y)) = 2f(x)f(y). \tag{1.5}$$

It is easy to see that

$$\mathrm{ad} : \mathrm{Mat}_2(\mathbb{C}) \to \mathrm{Mat}_2(\mathbb{C}) \qquad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \to \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \tag{1.6}$$

is an involution on $\mathrm{Mat}_2(\mathbb{C})$, and $\frac{1}{2}\operatorname{tr}$ is a d'Alembert function on $\langle \mathrm{Mat}_2(\mathbb{C}), \mathrm{ad}\rangle$

$$\frac{1}{2}\operatorname{tr}(XY) + \frac{1}{2}\operatorname{tr}(X\,\mathrm{ad}\,Y) = 2\cdot\frac{1}{2}X\cdot\frac{1}{2}\operatorname{tr} Y \tag{1.7}$$

since $Y + \mathrm{ad}\,Y$ is a scalar matrix equal to $\operatorname{tr} Y \cdot E$. We show in Section 5 that every d'Alembert function is a pre-d'Alembert function (Proposition 5.2), and prove, as a consequence of Theorem 5.4,

**Corollary 5.5.** *If $f$ is a d'Alembert function on $\langle M, \tau\rangle$ then there is a continuous homomorphism $h : M \to \mathrm{Mat}_2(\mathbb{C})$ such that*

$$h \circ \tau = \mathrm{ad} \circ h \tag{1.8}$$

*and*

$$f = \frac{1}{2}\operatorname{tr} \circ h. \tag{1.9}$$

We give an explicit construction for the function $h$ mentioned in Theorem 4.11 (and Corollary 5.5), so we are able in Section 6 to deduce the structure theorem for the 'classical' d'Alembert functions where $M$ is a topological group $G$ and $\tau$ is the group inverse. In particular

**Theorem 6.2.** *If $f$ is a d'Alembert function on the topological group $G$, so that*

$$f(xy) + f(xy^{-1}) = 2f(x)f(y) \tag{1.10}$$

*for all $x$, $y$ in $G$, then there is a continuous (group) homomorphism $h : G \to \mathrm{SL}_2(\mathbb{C})$ such that*

$$f = \frac{1}{2}\,\mathrm{tr} \circ h.$$

**Corollary 6.2.** *If, moreover, $f$ is bounded then $h$ may be chosen such that $h : G \to \mathrm{SU}_2(\mathbb{C})$.*

## 2. Pre-d'Alembert spaces

We say the pair $\langle M, f\rangle$ is a pre-d'Alembert space if $f$ is a pre-d'Alembert function on the topological monoid $M$. In this section we discuss properties common to all pre-d'Alembert spaces and introduce a classification of them. We also introduce Wilson functions on $\langle M, f\rangle$ and derive some of their properties.

*Definition 2.1.* Let $\langle M, f\rangle$ be a pre-d'Alembert space. For each $x$ in $M$ set,

$$f_x : y \mapsto f(xy) - f(x)f(y), \tag{2.1}$$

for all $y$ in $M$. For all $x$, $y$ in $M$ put

$$\Delta(x, y) = f_x(x)f_y(y) - f_x(y)^2. \tag{2.2}$$

Clearly $f_x$, and $\Delta$ are continuous functions. We now prove

**Proposition 2.2.** *Let $\langle M, f\rangle$ be a pre-d'Alembert space. Then for all $x$, $y$, $z$ in $M$*

(i) $$f(yz) = f(zy), \tag{2.3}$$

(ii) $$f_y(z) = f_z(y), \tag{2.4}$$

(iii) $$f_{yz}(yz) = f_{zy}(zy), \tag{2.5}$$

(iv) $$f_x(yz) - f_x(zy) = f(xyz) - f(xzy), \tag{2.6}$$

*and*

(v) $$f_x(yz) + f_x(zy) = 2f(y)f_x(z) + 2f(z)f_x(y). \tag{2.7}$$

Proof. (i) This says that $f$ is central. To see this put $x = e$ in (1.1).

(ii), (iii) and (iv) are immediate consequences of (i).

(v) $$\begin{aligned} f_x(yz) + f_x(zy) &= f(xyz) + f(xzy) - 2f(x)f(yz) \text{ (from (2.3))} \\ &= 2f(y)f(xz) + 2f(z)f(xy) - 4f(x)f(y)f(z) \text{ (from (1.1))} \\ &= 2f(y)f_x(z) + 2f(z)f_x(y) \text{ (from (2.1))}. \end{aligned}$$ □

We observe that (2.3) and (2.7) together imply (1.1). Equation (2.7) for $f_x$ is so powerful that we isolate and name functions with this property.

*Definition 2.3.* Let $\langle M, f\rangle$ be a pre-d'Alembert space. A function $w : M \to \mathbb{C}$ is a *Wilson function* if it is continuous and for all $y$, $z$ in $M$,

$$w(yz) + w(zy) = 2f(y)w(z) + 2f(z)w(y). \tag{2.8}$$

If $M$ is commutative then (2.8) is equivalent to

$$w(yz) = f(y)w(z) + f(z)w(y) \tag{2.9}$$

and in this form was introduced and studied by Wilson [10, equation 7] in 1919. These functions are at the heart of our study of pre-d'Alembert spaces.

An example of a Wilson function on $\langle \mathrm{Mat}_2(\mathbb{C}), \frac{1}{2}\,\mathrm{tr}\rangle$ is

$$w : \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto \beta,$$

as is very easy to check.

**Proposition 2.4.** *Let $\langle M, f\rangle$ be a pre-d'Alembert space. Let $W$ denote the set of all Wilson functions on $\langle M, f\rangle$. Then*

(i) *$W$ is a subspace of the complex linear space of complex-valued functions on $M$,*

(ii) *for each $x$ in $M$, $f_x \in W$,*

(iii) *$w \in W$ is central if and only if $w$ satisfies equation (2.9) for all $y$, $z$ in $M$,*

(iv) $$w(x^2) = 2f(x)w(x) \tag{2.10}$$

*for all $x$ in $M$, and*

(v) *for all $x$, $y$ in $M$, all $w$ in $W$,*

$$w(x^2y) + w(yx^2) - 2w(xyx) = 4[f_x(x)w(y) - f_x(y)w(x)]. \tag{2.11}$$

PROOF. (i), (ii), (iii) These are clear from (2.8) and (2.7).
(iv) Put $y = z = x$ in (2.8).
(v) Let $w \in W$, and $x, y \in M$. Then, using (2.8), we have

$$w(x \cdot xy) + w(xy \cdot x) = 2f(x)w(xy) + 2f(xy)w(x),$$
$$w(x \cdot yx) + w(yx \cdot x) = 2f(x)w(yx) + 2f(yx)w(x).$$

After adding these equations and using (2.8) we deduce that

$$w(x^2y) + 2w(xyx) + w(yx^2) = 4f(x)^2w(y) + 4[f(xy) + f(x)f(y)]w(x). \quad (2.12)$$

Now putting $A := w(x^2y) - 2w(xyx) + w(yx^2)$ and adding this to each side of equation (2.12) we obtain (using (2.8) and (2.10))

$$4f(x^2)w(y) + 8f(x)f(y)w(x) = A + 4f(x)^2w(y) + 4[f(xy) + f(x)f(y)]w(x)$$

which yields the stated expression for $A$. Hence (2.11) is proved. □

**Corollary 2.5.** *For all $x$, $y$ in $M$*

$$f(x^2y^2) - f((xy)^2) = 2\Delta(x, y). \quad (2.13)$$

PROOF. Take $w = f_y$ in (2.11). From (2.6) we see that

$$f_y(x^2y) - f_y(xyx) = f(x^2y^2) - f((xy)^2)$$

and

$$f_y(yx^2) - f_y(xyx) = f(x^2y^2) - f((xy)^2),$$

and so (2.13) follows from (2.11). □

*Definition 2.6.* Let $\langle M, f\rangle$ be a pre-d'Alembert space. We say $\langle M, f\rangle$ is a *Kannappan space* if for all $x$, $y$, $z$ in $M$

$$f(xyz) = f(xzy). \quad (2.14)$$

This condition on $f$ was introduced and used to great effect by KANNAPPAN [4] in his seminal 1968 paper.

We now relate (2.14) to our function $\Delta$ defined by (2.2).

**Proposition 2.7.** *Let $\langle M, f\rangle$ be a pre-d'Alembert space. Then $\langle M, f\rangle$ is a Kannappan space if and only if $\Delta = 0$.*

PROOF. Assume $\langle M, f\rangle$ is a Kannappan space. It is then immediate from (2.13) that $\Delta = 0$.

Assume conversely that $\Delta = 0$. Put

$$S = \left\{(x, y, z) \in M^3 : f(xyz) \neq f(xzy)\right\}.$$

We will show that $S \neq \emptyset$ leads to a contradiction. Note for use below as a consequence of (2.3) that $(x, y, z) \in S$ implies that $(y, x, z) \in S$ and $(z, y, z) \in S$.

Since for all $x, y, z$ in $M$ $\Delta(x, yz) = 0$ and $\Delta(x, zy) = 0$ using (2.5) we deduce

$$f_x(yz)^2 = f_x(zy)^2.$$

So if $(x, y, z) \in S$ then (using (2.6))

$$f_x(yz) + f_x(zy) = 0, \tag{2.15}$$

and so

$$f(y)f_x(z) + f(z)f_x(y) = 0. \tag{2.16}$$

But $f_y(xz)^2 = f_y(zx)^2$ so, similarly

$$f(x)f_y(z) + f(z)f_y(x) = 0. \tag{2.17}$$

From (2.16) and (2.17) we deduce that if $(x, y, z) \in S$ then

$$f(y)f_x(z) = f(x)f_y(z). \tag{2.18}$$

But $f_z(xy)^2 = f_z(yx)^2$ so

$$f(x)f_z(y) + f(y)f_x(z) = 0,$$

which, with (2.18) yields, for all $(x, y, z) \in S$

$$f(x)f_y(z) = 0. \tag{2.19}$$

If $f_y(z) = 0$ then (since $\Delta(y, z) = 0$) either $f_y(y) = 0$ or $f_z(z) = 0$. If $f_y(y) = 0$ then $f_y(xz) = 0$ for all $x, z$ in $M$ so $0 = f_y(xz) - f_y(zx) = f(xzy) - f(yzx) \neq 0$; a contradiction if $(x, y, z) \in S$. Thus $(x, y, z) \in S$ implies that $f_x(x) \neq 0$, $f_y(y) \neq 0$ and $f_z(z) \neq 0$, and from (2.19) $f(x) = 0$, and by symmetry $f(y) = 0$, $f(z) = 0$.

We have shown that if $(x, y, z) \in S$ then

$$f(x) = 0, \quad f(y) = 0, \quad f(z) = 0, \tag{2.20}$$

$$f(xy) \neq 0, \quad f(yz) \neq 0, \quad f(zx) \neq 0, \tag{2.21}$$

$$f(xyz) + f(xzy) = 0. \qquad \text{(from (1.1))} \tag{2.22}$$

Now assume that $(x,y,z)\in S$ and consider $(x,y,xyz)\in M^3$. If $(x,y,xyz)\in S$ then $f(xyz)=0$ by (2.20) and so $f(xzy)=0$ from (2.22). Thus $(x,y,z)\notin S$; a contradiction. So we must have $(x,y,xyz)\notin S$, that is

$$f(x\cdot y\cdot xyz)=f(x\cdot xyz\cdot y). \tag{2.23}$$

Now using $f(z)=0$ and (2.10) for $w=f_z$,

$$f(xyxyz)=f_z((xy)^2)=2f(xy)f(xyz). \tag{2.24}$$

Whereas using $f(x)=0$ and $f(y)=0$

$$\begin{aligned} f(x^2yzy)&=f_{yzy}(x^2)+f(yzy)f(x^2)=2f(x)f_{yzy}(x)+f(x^2)f_z(y^2)\\ &=0+f(x^2)2f(y)f_z(y)\\ \therefore\quad f(x^2yzy)&=0. \end{aligned} \tag{2.25}$$

Thus if $(x,y,z)\in S$ then, from (2.23), (2.24) and (2.25)

$$f(xy)f(xyz)=0.$$

But from (2.21) $f(xy)\neq 0$ so $f(xyz)=0$. However from (2.22) we see that then $f(xzy)=0$ so $(x,y,z)\notin S$; a contradiction.

Thus $S=\emptyset$ and $f(xyz)=f(xzy)$ for all $x$, $y$, $z$ in $M$. □

We can now introduce our classification of pre-d'Alembert spaces.

*Definition 2.8.* Let $\langle M,f\rangle$ be a pre-d'Alembert space. We say

(i) $\langle M,f\rangle$ is *trivial* if it is a Kannappan space and $f_x(x)=0$ for all $x$ in $M$.

(ii) $\langle M,f\rangle$ is *abelian* if it is a Kannappan space and there is $c\in M$ with $f_c(c)\neq 0$.

(iii) $\langle M,f\rangle$ is *non-Kannappan* if there are $a,b$ in $M$ with $\Delta(a,b)\neq 0$.

We complete this section by introducing a function that will play a key role in the proof that $\dim W=3$ if $\langle M,f\rangle$ is a non-Kannappan space.

*Definition 2.9.* Let $\langle M,f\rangle$ be a pre-d'Alembert space. The function $d:M\to\mathbb{C}$ is given by

$$d(x)=2f(x)^2-f(x^2) \tag{2.26}$$

for each $x$ in $M$.

**Proposition 2.10.** *Let $\langle M,f\rangle$ be a pre-d'Alembert space. Then $d$ is a continuous homomorphism from $M$ to $\langle\mathbb{C},\cdot\rangle$.*

Proof. That $d$ is continuous is obvious. Also $d(e) = 1$. So we have to show that $d(xy) = d(x)d(y)$ for all $x$, $y$ in $M$. We do this by transforming the identity below derived from (1.1) with $z = xy$

$$\begin{aligned} &f(x \cdot y \cdot xy) + f(x \cdot xy \cdot y) = 2f(x)f(yxy) + 2f(y)f(xxy) \\ &\qquad + 2f(xy)^2 - 4f(x)f(y)f(xy) \\ \therefore \quad &f((xy)^2) + f_{x^2}(y^2) + f(x^2)f(y^2) = 2f(x)[f_x(y^2) + f(x)f(y^2)] \\ &\qquad + 2f(y)[f_y(x^2) + f(y)f(x^2)] + 2f(xy)^2 - 4f(x)f(y)[f_x(y) + f(x)f(y)]. \end{aligned}$$

Therefore

$$\begin{gathered} f((xy)^2) + 4f(x)f(y)f_x(y) + f(x^2)f(y^2) = 4f(x)f(y)f_x(y) + 2f(x)^2f(y^2) \\ +4f(x)f(y)f_x(y) + 2f(y)^2f(x^2) + 2f(xy)^2 - 4f(x)f(y)f_x(y) - 4f(x)^2f(y)^2, \end{gathered}$$

using (2.1) and (2.10) where appropriate. Hence

$$f((xy)^2) - 2f(xy)^2 + 4f(x)^2f(y)^2 - 2f(x)^2f(y^2) - 2f(y)^2f(x^2) + f(x^2)f(y)^2 = 0.$$

But this is the same as

$$-d(xy) + d(x)d(y) = 0.$$

The result follows. $\square$

We remark that in $\langle \mathrm{Mat}_2(\mathbb{C}), \frac{1}{2}\operatorname{tr}\rangle$, $d(X) = \det(X)$ since $\operatorname{tr}(X^2) = \operatorname{tr}(X)^2 - 2\det X$ from the Cayley–Hamilton theorem.

We also note the following for future reference.

**Proposition 2.11.** *If $G$ is a group with a continuous multiplication and $\langle G, f\rangle$ is a pre-d'Alembert space then*

$$f(xy) + d(y)f(xy^{-1}) = 2f(x)f(y) \tag{2.27}$$

*for all $x$, $y$ in $G$.*

Proof. In equation (1.1) let $z = y$ and replace $x$ by $xy^{-1}$ to deduce that

$$2f(xy) = 2f(xy^{-1})f(y^2) + 4f(y)f(xy^{-1}y) - 4f(xy^{-1})f(y)^2$$

which is equivalent to (2.27) after noting Definition (2.26). $\square$

## 3. Kannappan spaces

In this section we prove structure theorems for the two types of Kannappan spaces; trivial and abelian. First we give examples of each type that play a significant role in the sequel.

*Example 3.1.* $\langle\langle\mathbb{C},\cdot\rangle,\mathrm{id}\rangle$ is the pre-d'Alembert space where the topological monoid is the set of complex numbers under multiplication (stressed by writing $\langle\mathbb{C},\cdot\rangle$ for this monoid) with the usual topology, and $\mathrm{id} : \alpha\mapsto\alpha$ for all $\alpha\in\mathbb{C}$ is the identity function. For this example $d$ is given by $\alpha \mapsto \alpha^2$. This is clearly a Kannappan space as $\langle\mathbb{C},\cdot\rangle$ is commutative and it is clearly trivial since $\mathrm{id}_\alpha(\alpha) = \alpha^2 - \alpha\alpha = 0$ for all $\alpha \in \mathbb{C}$. Finally the function $\alpha \mapsto \left[\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right]$ of $\mathbb{C}$ into $\mathrm{Mat}_2(\mathbb{C})$ is clearly a continuous homomorphism of $\langle\mathbb{C},\cdot\rangle$ into $\langle\mathrm{Mat}_2(\mathbb{C}),\cdot\rangle$. Note that $\frac{1}{2}\,\mathrm{tr}\left[\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right] = \mathrm{id}(\alpha)$ and $\det\left[\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right] = \alpha^2$.

*Example 3.2.* $\langle\langle\mathbb{C}^2,\cdot\rangle,\mathrm{av}\rangle$. Here the monoid is $\mathbb{C}^2 = \{(\alpha,\beta) : \alpha,\beta \in \mathbb{C}\}$ under multiplication $(\alpha,\beta)(\gamma,\delta) = (\alpha\gamma,\beta\delta)$ with the usual topology. The function $\mathrm{av} : (\alpha,\beta) \to \frac{\alpha+\beta}{2}$ is easily seen to be a pre-d'Alembert function. The space is Kannappan since $\langle\mathbb{C}^2,\cdot\rangle$ is commutative. It is abelian since $\mathrm{av}_{(1,-1)}(1,-1) = 1 - 0\cdot 0 = 1 \neq 0$. For this example $d(\alpha,\beta) = \alpha\beta$ as $2\bigl(\frac{\alpha+\beta}{2}\bigr)^2 - \bigl(\frac{\alpha^2+\beta^2}{2}\bigr) = \alpha\beta$. Finally the function

$$(\alpha,\beta) \mapsto \begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix} \quad \text{of } \mathbb{C}^2 \text{ into } \mathrm{Mat}_2(\mathbb{C})$$

is clearly a continuous homomorphism. Note that

$$\frac{1}{2}\,\mathrm{tr}\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix} = \mathrm{av}(\alpha,\beta) \quad\text{and}\quad \det\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix} = d(\alpha,\beta).$$

As we will see each trivial Kannappan space maps into $\langle\langle\mathbb{C},\cdot\rangle,\mathrm{id}\rangle$ and each abelian Kannappan space maps into $\langle\langle\mathbb{C}^2,\cdot\rangle,\mathrm{av}\rangle$. We begin with trivial spaces.

**Proposition 3.3.** *Let $\langle M, f\rangle$ be a pre-d'Alembert space. If $f_x(x) = 0$ for all $x$ in $M$ then $f$ is a continuous (monoid) homomorphism of $M$ into $\langle\mathbb{C},\cdot\rangle$.*

Proof. Since $f(e) = 1$ we need to show that $f(xy) = f(x)f(y)$ for all $x$, $y$ in $M$; equivalently $f_x$ is the zero function for each $x$ in $M$.

Now

$$f(x)^2 = f(x)^2 - f_x(x) = 2f(x)^2 - f(x)^2 = d(x)$$

since $f_x(x) = 0$. Thus, using Proposition 2.10,

$$f(xy)^2 = d(xy) = d(x)d(y) = f(x)^2 f(y)^2$$

or, equivalently, for all $x$, $y$ in $M$

$$f_x(y)[f_x(y) + 2f(x)f(y)] = 0. \tag{3.1}$$

Let $x \in M$ be fixed and put

$$S = \{y \in M : f_x(y) \neq 0\}. \tag{3.2}$$

We will show that $S = \emptyset$ and hence $f(xy) = f(x)f(y)$ for all $y \in M$ is true for each fixed $x$; hence $f$ is a homomorphism.

If $y \in S$ then, from (3.1)

$$f_x(y) + 2f(x)f(y) = 0. \tag{3.3}$$

Thus $f(y) \neq 0$. So $y^2 \in S$ too since

$$f_x(y^2) = 2f(y)f_x(y) \neq 0.$$

Hence we use (3.3) with $y$ replaced by $y^2$, and the fact that $f(y^2) - f(y)^2 = f_y(y) = 0$ to obtain

$$2f(y)f_x(y) + 2f(x)f(y)^2 = 0$$

and so, for $y \in S$,

$$f_x(y) + f(x)f(y) = 0. \tag{3.4}$$

Now (3.3) and (3.4) together yield that $f_x(y) = 0$, so $y \notin S$: a contradiction. Thus $S = \emptyset$, and $f$ is a homomorphism. $\square$

This result leads to the following characterization of trivial spaces.

**Theorem 3.4.** *Let $f : M \to \mathbb{C}$ be a continuous function. Then $\langle M, f\rangle$ is a trivial pre-d'Alembert space if and only if $f$ is a homomorphism of $M$ into $\langle \mathbb{C}, \cdot\rangle$.*

Proof. Assume that $\langle M, f\rangle$ is a trivial pre-d'Alembert space then by Definition 2.8(i) $f_x(x) = 0$ for all $x$ in $M$. From Proposition 3.3 it follows that $f$ is a homomorphism of $M$ into $\langle \mathbb{C}, \cdot\rangle$.

Assume conversely that $f$ is a homomorphism of $M$ into $\langle \mathbb{C}, \cdot\rangle$. Since $f(e)=1$, we need to check (1.1). But the left-hand side and right-hand side of (1.1) are both equal to $2f(x)f(y)f(z)$. Thus $\langle M, f\rangle$ is a pre-d'Alembert space. Since $f(xyz) = f(x)f(y)f(z) = f(x)f(z)f(y) = f(xzy)$ it is a Kannappan space, and since $f_x(x) = f(x^2) - f(x)^2 = 0$ it is trivial. $\square$

**Corollary 3.5.** *If $\langle M, f\rangle$ is a trivial pre-d'Alembert space there is a continuous homomorphism $h : M \to \langle \mathrm{Mat}_2(\mathbb{C}), \cdot\rangle$ such that $f = \frac{1}{2}\operatorname{tr} \circ h$.*

Proof. Define $h : M \to \mathrm{Mat}_2(\mathbb{C})$ by

$$x \mapsto \begin{bmatrix} f(x) & 0 \\ 0 & f(x) \end{bmatrix}.$$

Since $f$ is a homomorphism $h$ is one too. Clearly $f = \frac{1}{2}\operatorname{tr} \circ h$. □

We turn now to abelian spaces. If $\langle M, f\rangle$ is abelian then $f_x$ is central for all $x$ in $M$ : $f_x(yz) - f_x(zy) = f(xyz) - f(xzy) = 0$. So we investigate central Wilson functions on abelian spaces.

**Proposition 3.6.** *Let $\langle M, f\rangle$ be an abelian pre-d'Alembert space and $c \in M$ chosen so that $f_c(c) \neq 0$. Let $w$ be a central Wilson function on $\langle M, f\rangle$ then*

$$w = \frac{w(c)}{f_c(c)}\, f_c. \tag{3.5}$$

Proof. Immediate from (2.11) with $x = c$. □

**Corollary 3.7.** *With $\langle M, f\rangle$, $c$ as above, for all $x$, $y$ in $M$*

$$f(xy) = f(x)f(y) + \frac{1}{f_c(c)} f_c(x)f_c(y). \tag{3.6}$$

Proof. Apply the proposition with $w = f_x$, to deduce that

$$f_x(y) = \frac{f_x(c)}{f_c(c)}\, f_c(y)$$

which is the result stated. □

With (3.6) in hand we can now prove

**Theorem 3.8.** *If $\langle M, f\rangle$ is an abelian pre-d'Alembert space, then there exist two distinct continuous homomorphisms $k, \ell : M \to \langle \mathbb{C}, \cdot\rangle$ such that*

$$f = \frac{k + \ell}{2}.$$

*Conversely, let $k$, $\ell$ be distinct continuous homomorphisms of $M$ into $\langle \mathbb{C}, \cdot\rangle$ then $f := \frac{k+\ell}{2}$ is a pre-d'Alembert function on $M$ and $\langle M, f\rangle$ is an abelian pre-d'Alembert space.*

Proof. Assume $\langle M, f\rangle$ is an abelian pre-d'Alembert space. Let $c \in M$ be chosen so that $f_c(c) \neq 0$ and choose $\lambda \in \mathbb{C}$ such that $\lambda^2 f_c(c) = 1$. Now define $k, \ell$ from $M$ to $\langle \mathbb{C}, \cdot\rangle$ by

$$k(x) = f(x) + \lambda f_c(x), \tag{3.7}$$

$$\ell(x) = f(x) - \lambda f_c(x). \tag{3.8}$$

Then $f = \frac{k+\ell}{2}$, $k(e) = 1$, $\ell(e) = 1$. We need to show that $k(xy) = k(x)k(y)$ and $\ell(xy) = \ell(x)\ell(y)$ for all $x$, $y$ in $M$. Now

$$\begin{aligned} k(xy) &= f(xy) + \lambda f_c(xy) = f(x)f(y) + \frac{1}{f_c(c)} f_c(x)f_c(y) + \lambda[f(x)f_c(y) + f(y)f_c(x)] \\ &= f(x)f(y) + \lambda^2 f_c(x)f_c(y) + \lambda f(x)f_c(y) + \lambda f_c(x)f(y) = k(x)k(y). \end{aligned}$$

The proof for $\ell$ is the same merely replacing $\lambda$ by $-\lambda$.

Finally $k, \ell$ are distinct since

$$k(c) - \ell(c) = 2\lambda f_c(c) \neq 0.$$

Assume conversely that $k, \ell$ are distinct continuous homomorphisms from $M$ to $\langle \mathbb{C}, \cdot\rangle$. Then $x \mapsto (k(x), \ell(x)) \in \langle \mathbb{C}^2, \cdot\rangle$ is a continuous homomorphism. Moreover $f(x) = \mathrm{av}(k(x), \ell(x))$. Since $\langle\langle \mathbb{C}^2, \cdot\rangle, \mathrm{av}\rangle$ is a pre-d'Alembert space so is $\langle M, f\rangle$. Since $\langle \mathbb{C}^2, \mathrm{av}\rangle$ is Kannappan so is $\langle M, f\rangle$ and $\langle M, f\rangle$ is abelian because $f_x(x) = \left(\frac{k(x)-\ell(x)}{2}\right)^2$ is non-zero for some $x$ as $k, \ell$ are distinct. $\square$

**Corollary 3.9.** *If $\langle M, f\rangle$ is an abelian pre-d'Alembert space then there is a continuous homomorphism $h : M \to \langle \mathrm{Mat}_2(\mathbb{C}), \cdot\rangle$ such that $f = \frac{1}{2}\operatorname{tr} \circ h$.*

Proof. Define $h : x \mapsto \begin{bmatrix} k(x) & 0 \\ 0 & \ell(x) \end{bmatrix}$ where $k, \ell$ are given by (3.7) and (3.8). Then $h$ is a continuous homomorphism since $k, \ell$ are continuous homomorphisms as shown above. Clearly $\frac{1}{2}\operatorname{tr} \circ h = f$. $\square$

We end this section with an example of a non-commutative monoid $M$ and pre-d'Alembert function $f$ where $\langle M, f\rangle$ is abelian, and there exist non central Wilson functions.

*Example 3.10.* Let $V$ be a non-zero locally-convex topological vector space over $\mathbb{C}$ and put

$$M = \left\{ \begin{bmatrix} \alpha & x \\ 0 & \beta \end{bmatrix} : \alpha, \beta \in \mathbb{C},\ x \in V \right\}$$

where the associative operation in $M$ is matrix multiplication ($V^* \neq \{0\}$ by [5, Corollary 3.4, p. 59]). Define $f\left[\begin{smallmatrix}\alpha & x\\ 0 & \beta\end{smallmatrix}\right] = \frac{\alpha+\beta}{2}$. Then $\langle M, f\rangle$ is pre-d'Alembert. Also it is Kannappan because

$$\operatorname{tr}(X(YZ - ZY)) = 0 \quad \text{for all } X,\ Y,\ Z \text{ in } M.$$

It is abelian since $f_C(C) = 1$ where $C = \left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$. Let $\varphi : V \to \mathbb{C}$ be a non-zero continuous linear functional. Then $w\left[\begin{smallmatrix}\alpha & x\\ 0 & \beta\end{smallmatrix}\right] := \varphi(x)$ is easily seen to be a Wilson function on $M$, that is not central.

## 4. Non-Kannappan pre-d'Alembert spaces

Let $\langle M, f\rangle$ be a non-Kannappan pre-d'Alembert space. So there exist elements $a$, $b$ in $M$ so that $\Delta(a, b) \neq 0$. *We fix these elements and the space* $\langle M, f\rangle$ and construct Wilson functions based on them to show the existence of a continuous homomorphism $h : M \to \mathrm{Mat}_2(\mathbb{C})$ such that $f = \frac{1}{2}\operatorname{tr} \circ h$.

We show first, that, unlike the situation in Kannappan spaces the space of Wilson functions is 3 dimensional and furthermore there are no non-zero central Wilson functions.

**Proposition 4.1.** *If $w$ is a Wilson function on $\langle M, f\rangle$ such that $w(a) = 0$, $w(b) = 0$, $w(ab) = 0$ then $w = 0$.*

Proof. Let $N = \{x \in M : w(x) = 0\}$. We will show that $N = M$. By hypothesis $a, b, ab \in N$. If $x$, $y$ and $xy$ are in $N$ then so is $yx$, as $w(xy) + w(yx) = 0$ if $x$, $y$ are in $N$.

Now suppose $x, y \in N$. Then for all $z \in M$,

$$w(x \cdot yz) + w(yz \cdot x) = 2f(x)w(yz), \quad w(y \cdot zx) + w(zx \cdot y) = 2f(y)w(zx)$$

and so adding these and simplifying we obtain

$$f(xy)w(z) + f(z)w(xy) + w(yzx) = f(x)w(yz) + f(y)w(zx). \tag{4.1}$$

Interchanging $x$, $y$ in (4.1) and adding the result gives, for all $x$, $y$ in $N$, all $z$ in $M$,

$$w(yzx) + w(xzy) = 2[2f(x)f(y) - f(xy)]w(z). \tag{4.2}$$

In particular if $x \in N$, $z \in M$ we have

$$w(xzx) = d(x)w(z), \tag{4.3}$$

since $d(x) = 2f(x)^2 - f(x^2)$. Now replace $z$ by $yzy$ where $y \in N$ and $z \in M$. Then (4.3) yields

$$w(xyzyx) = d(x)d(y)w(z) = d(xy)w(z)$$

and interchanging $x$, $y$ yet again;

$$w(yxzxy) = d(yx)w(z),$$

so for all $x, y \in N$, $z \in M$,

$$w(xyzyx) + w(yxzxy) = 2d(xy)w(z). \tag{4.4}$$

If $xy, yx \in N$ then (4.2) yields, for all $z$ in $M$,

$$w(xyzyx) + w(yxzxy) = 2[2f(xy)f(yx) - f(xyyx)]w(z). \tag{4.5}$$

Now

$$f(xyyx) = f((xy)^2) + 2\Delta(x, y) \quad \text{from (2.13)}$$

so

$$w(xyzyx) + w(yxzxy) = [2d(xy) - 4\Delta(x, y)]w(z). \tag{4.6}$$

Putting (4.4) and (4.6) together, we have: if $x$, $y$, $xy$ belong to $N$ and $z \in M$ then

$$4\Delta(x, y)w(z) = 0. \tag{4.7}$$

In particular $\Delta(a, b)w(z) = 0$ for all $z \in M$. So $w = 0$ and the proof is complete. $\square$

We use this result to show

**Proposition 4.2.** *Let $\langle M, f\rangle$ be a non-Kannappan pre-d'Alembert space. Then the zero function is the only central Wilson function on $\langle M, f\rangle$.*

Proof. Suppose $w$ is a central Wilson function on $\langle M, f\rangle$. Then

$$w(a^2b) + w(ba^2) - 2w(aba) = 0 \tag{4.8}$$

and

$$w(b^2a) + w(ab^2) - 2w(bab) = 0. \tag{4.9}$$

But, from Proposition 2.4 equation (2.11), we deduce that

$$f_a(a)w(b) - f_a(b)w(a) = 0, \qquad -f_b(a)w(b) + f_b(b)w(a) = 0.$$

Since

$$f_a(a)f_b(b) - f_a(b)^2 = \Delta(a, b) \neq 0$$

we see that $w(b) = 0$ and $w(a) = 0$. But then

$$2w(ab) = w(ab) + w(ba) = 0$$

so by Proposition 4.1 $w = 0$. $\square$

Next we introduce some notation that will remain in place for the remainder of this section.

*Definition 4.3.* $f_1, f_2, f_3 : M \to \mathbb{C}$ are give by

$$f_1 := f_a, \quad f_2 := f_b, \quad f_3 = \frac{1}{2}(f_{ab} - f_{ba}) \tag{4.10}$$

and $\alpha, \beta, \gamma \in \mathbb{C}$ by

$$\alpha = \frac{1}{\Delta} f_a(a), \quad \beta = \frac{1}{\Delta} f_a(b), \quad \gamma = \frac{1}{\Delta} f_b(b) \tag{4.11}$$

where $\Delta := \Delta(a, b)$.

We see immediately that

$$\alpha\gamma - \beta^2 = \frac{1}{\Delta}. \tag{4.12}$$

Using these it is easy to show

**Proposition 4.4.** *$\langle f_1, f_2, f_3 \rangle$ is a basis for $W$. Indeed if $w \in W$ then*

$$w = [\gamma w(a) - \beta w(b)] f_1 + [-\beta w(a) + \alpha w(b)] f_2 - \frac{1}{\Delta} \frac{w(ab) - w(ba)}{2} f_3. \tag{4.13}$$

Since Corollary 3.7 was the central result in obtaining a homomorphism when $\langle M, f \rangle$ was an abelian space, we now look for similar formulae for $f(xy)$, $f_1(xy)$, $f_2(xy)$ and $f_3(xy)$. The first of these is easily obtained from Proposition 4.4:

**Proposition 4.5.** *For all $x$, $y$ in $M$*

$$\begin{aligned} f(xy) = f(x)f(y) &+ \gamma f_1(x) f_1(y) - \beta[f_1(x) f_2(y) + f_2(x) f_1 y)) \\ &+ \alpha f_2(x) f_2(y) - \frac{1}{\Delta} f_3(x) f_3(y). \end{aligned} \tag{4.14}$$

Proof. Apply (4.13) to the Wilson function $f_x$. □

The others follow from

**Theorem 4.6.** *For all $x$, $y$, $z$ in $M$*

$$\frac{f(xyz) - f(xzy)}{2} = \frac{1}{\Delta} \begin{vmatrix} f_1(x) & f_1(y) & f_1(z) \\ f_2(x) & f_2(y) & f_2(z) \\ f_3(x) & f_3(y) & f_3(z) \end{vmatrix}. \tag{4.15}$$

Proof. We note that if two of $x$, $y$, $z$ are equal then each side of (4.15) is 0. We also note that as a function of $z$ the left-hand side is the Wilson function $\frac{1}{2}f_{xy}(z) - \frac{1}{2}f_{yx}(z)$, and the right-hand side is a linear combination of $f_1$, $f_2$ and $f_3$ and hence a Wilson function of $z$ too. But the same argument applies to $x$, $y$ also. So both sides agree if and only if they agree when $\{x, y, z\} = \{a, b, ab\}$. Thus we need only verify that (4.15) is true when $x = a$, $y = b$ and $z = ab$.

Now
$$\frac{f(ab \cdot ab) - f(a \cdot ab \cdot a)}{2} = -\Delta(a, b) = -\Delta$$
by (2.13) whereas
$$\frac{1}{\Delta}\begin{vmatrix} f_1(a) & f_1(b) & f_1(ab) \\ f_2(a) & f_2(b) & f_2(ab) \\ f_3(a) & f_3(b) & f_3(ab) \end{vmatrix} = \frac{1}{\Delta}\begin{vmatrix} f_1(a) & f_1(b) & f_1(ab) \\ f_2(a) & f_2(b) & f_2(ab) \\ 0 & 0 & -\Delta \end{vmatrix} = -\frac{\Delta}{\Delta}\begin{vmatrix} f_1(a) & f_1(b) \\ f_2(a) & f_2(b) \end{vmatrix} = -\Delta.$$

This completes the proof of (4.15). $\square$

**Corollary 4.7.** *For all $x$, $y$ in $M$*
$$\frac{1}{2}[f_1(xy)s - f_1(yx)] = \alpha[f_2(x)f_3(y) - f_3(x)f_2(y)] + \beta[f_3(x)f_1(y) - f_1(x)f_3(y)] \tag{4.16}$$
$$\frac{1}{2}[f_2(xy) - f_2(yx)] = \beta[f_2(x)f_3(y) - f_3(x)f_2(y)] + \gamma[f_3(x)f_1(y) - f_1(x)f_3(y)] \tag{4.17}$$
$$\frac{1}{2}[f_3(xy) - f_3(yx)] = -[f_1(x)f_2(y) - f_2(x)f_1(y)]. \tag{4.18}$$

Proof. For (4.16) put $z = a$ in (4.15) and for (4.17) put $z = b$. For (4.18) first put $z = ab$ in (4.15) to obtain
$$\frac{1}{2}[f_{ab}(xy) - f_{ab}(yx)] = \frac{1}{\Delta}\begin{vmatrix} f_1(x) & f_1(y) & f_1(ab) \\ f_2(x) & f_2(y) & f_2(ab) \\ f_3(x) & f_3(y) & -\Delta \end{vmatrix} \tag{4.19}$$
and then put $z = ba$ to obtain
$$\frac{1}{2}[f_{ba}(xy) - f_{ba}(yx)] = \frac{1}{\Delta}\begin{vmatrix} f_1(x) & f_1(y) & f_1(ba) \\ f_2(x) & f_2(y) & f_2(ba) \\ f_3(x) & f_3(y) & \Delta \end{vmatrix}. \tag{4.20}$$

Since $f_1(ab) = f_1(ba)$ and $f_2(ab) = f_2(ba)$ subtracting (4.20) from (4.19) yields

$$f_3(xy) - f_3(yx) = \frac{1}{\Delta}\begin{vmatrix} ccc f_1(x) & f_1(y) & 0 \\ f_2(x) & f_2(y) & 0 \\ f_3(x) & f_3(y) & -2\Delta \end{vmatrix}$$

from which (4.18) follows. $\square$

Since for any Wilson function

$$w(xy) = \frac{1}{2}[w(xy) + w(yx)] + \frac{1}{2}[w(xy) - w(yx)],$$

$$w(xy) = f(x)w(y) + f(y)w(x) + \frac{1}{2}[w(xy) - w(yx)], \tag{4.21}$$

$$f_j(xy) = f(x)f_j(y) + f_j(x)f(y) + \frac{1}{2}[f_j(xy) - f_j(yx)], \tag{4.22}$$

for $j = 1, 2, 3$.

We proceed to define our function $h : M \to \mathrm{Mat}_2(\mathbb{C})$ that we will show is a homomorphism with $f = \frac{1}{2}\operatorname{tr} \circ h$.

**Lemma 4.8.** *There are complex numbers $\lambda$, $\mu$, $\nu$, $\rho$, $\sigma$ such that*

$$\text{(i)}\quad \lambda^2 = -\frac{1}{\Delta}, \qquad \text{(ii)}\quad \mu\rho = \gamma, \qquad \text{(iii)}\quad \nu\sigma = \alpha$$

$$\text{(iv)}\quad \mu\sigma = -\beta - \lambda, \qquad \text{(v)}\quad \nu\rho = -\beta + \lambda. \tag{4.23}$$

Proof. Choose $\lambda \neq \beta$ such that $\lambda^2 = -\frac{1}{\Delta}$ $(= \beta^2 - \alpha\gamma)$. Then take $\rho = 1$, $\mu = \gamma$, $\nu = -\beta + \lambda$, $\sigma = \frac{\alpha}{\lambda - \beta}$. $\square$

We use these numbers now.

*Definition 4.9.* $h : M \to \mathrm{Mat}_2(\mathbb{C})$ is given by, for each $x$ in $M$,

$$h(x) = \begin{bmatrix} f(x) + \lambda f_3(x), & \mu f_1(x) + \nu f_2(x) \\ \rho f_1(x) + \sigma f_2(x), & f(x) - \lambda f_3(x) \end{bmatrix} \tag{4.24}$$

where $\lambda$, $\mu$, $\nu$, $\rho$, $\sigma$ are given by (4.23).

**Theorem 4.10.** *Let $\langle M, f\rangle$ be a non-Kannappan pre-d'Alembert space. Then $h$ is a continuous homomorphism of $M$ into $\langle \mathrm{Mat}_2(\mathbb{C}), \cdot\rangle$ satisfying $f = \frac{1}{2}\operatorname{tr} \circ h$.*

Proof. Clearly $h$, defined by (4.24) is continuous and $\operatorname{tr} h(x) = 2f(x)$ for all $x \in M$. Thus $f = \frac{1}{2}\operatorname{tr} \circ h$ as claimed.

Since $h(e) = E = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ we need to show that $h(xy) = h(x)h(y)$ for all $x$, $y$ in $M$. Since $\mathrm{Mat}_2(\mathbb{C})$ is a (non-commutative) ring we can write

$$h(x) = f(x)E + f_3(x)L + f_1(x)M + f_2(x)N \tag{4.25}$$

where

$$L = \begin{bmatrix}\lambda & 0\\ 0 & -\lambda\end{bmatrix}, \quad M = \begin{bmatrix}0 & \mu\\ \rho & 0\end{bmatrix}, \quad N = \begin{bmatrix}0 & \nu\\ \sigma & 0\end{bmatrix}. \tag{4.26}$$

Then from (4.23)

$$L^2 = -\frac{1}{\Delta}E, \quad M^2 = \gamma E, \quad N^2 = \alpha E \tag{4.27}$$

and

$$LM + ML = 0, \quad LN + NL = 0, \quad MN = -\beta E - L, \quad NM = -\beta E + L. \tag{4.28}$$

So using (4.27) and (4.28) and (4.14), we see that

$$\begin{aligned} h(x)h(y) = f(xy)E &- [f_1(x)f_2(y) - f_2(x)f_1(y)]L + [f(x)f_3(y) + f_3(x)f(y)]L \\ &+ [f(x)f_1(y) + f_1(x)f(y)]M + [f(x)f_2(y) + f_2(x)f(y)]N \\ &+ [f_3(x)f_1(y) - f_1(x)f_3(y)]LM + [f_2(x)f_3(y) - f_3(x)f_2(y)]NL. \end{aligned} \tag{4.29}$$

On the other hand using (4.22) we find

$$\begin{aligned} h(xy) = f(xy)E + f_3(xy)L + f_1(xy)M + f_2(xy)N &= f(xy)E + f_3(xy)L \\ + [f(x)f_1(y) + f_1(x)f(y)]M &+ \frac{1}{2}[f_1(xy) - f_1(yx)]M \\ + [f(x)f_2(y) + f_2(x)f(y)]N &+ \frac{1}{2}[f_2(xy) - f_2(yx)]N. \end{aligned} \tag{4.30}$$

So $h(xy) = h(x)h(y)$ if

$$\begin{aligned} &\frac{1}{2}[f_1(xy) - f_1(yx)]M + \frac{1}{2}[f_2(xy) - f_2(yx)]N \\ &\qquad = [f_3(x)f_1(y) - f_1(x)f_3(y)]LM + [f_2(x)f_3(y) - f_3(x)f_2(y)]NL. \end{aligned}$$

In view of (4.16), (4.17), and (4.18) this means that it suffices to show that

$$NL = \alpha M + \beta N, \qquad LM = \beta M + \gamma N. \tag{4.31}$$

However this follows from (iv), (v), (ii), (iii) of (4.23). Thus $h$ is multiplicative. $\square$

We complete this section with our main result about general pre-d'Alembert spaces.

**Theorem 4.11.** *Let $\langle M, f\rangle$ be a pre-d'Alembert space. Then there is a continuous homomorphism $h : M \to \langle \mathrm{Mat}_2(\mathbb{C}), \cdot\rangle$ such that $f = \frac{1}{2}\operatorname{tr}\circ h$. Moreover the function $d$ from Definition 2.9 satisfies $d = \det \circ h$.*

Proof. We consider three cases

(i) If $\langle M, f\rangle$ is trivial then Corollary 3.5 gives our result.

(ii) If $\langle M, f\rangle$ is abelian Corollary 3.9 applies, and

(iii) if $\langle M, f\rangle$ is non-Kannappan then Theorem 4.10 gives $h$.

That $d = \det \circ h$ is immediate from the formula $\operatorname{tr}(X^2) = (\operatorname{tr} X)^2 - 2\det X$. □

**Theorem 4.12.** *If $f$ is a pre-d'Alembert function on $M$ then there is a continuous monoid homomorphism $h : M \to \mathrm{Mat}_2(\mathbb{C})$ such that*

$$f = \frac{1}{2}\operatorname{tr}\circ h. \tag{4.32}$$

## 5. D'Alembert spaces I

In this section we define d'Alembert spaces and show that they are necessarily pre-d'Alembert spaces. So using the results of Sections 3 and 4, we will provide structure theorems for d'Alembert spaces.

First we define d'Alembert spaces and obtain some of their properties.

*Definition 5.1.* $\langle M, \tau, m, f\rangle$ is a *d'Alembert space* if $M$ is a topological monoid (with neutral element $e$), $\tau : M \to M$ is a continuous involution (hence $\tau(xy) = \tau(y)\tau(x)$ and $\tau(\tau(x)) = x$), $m : M \to \langle \mathbb{C}, \cdot\rangle$ is a continuous monoid homomorphism satisfying, for all $x$ in $M$,

$$m(x\tau(x)) = 1 \tag{5.1}$$

and $f : M \to \mathbb{C}$ is a continuous function with $f(e) = 1$ satisfying

$$f(xy) + m(y)f(x\tau(y)) = 2f(x)f(y) \tag{5.2}$$

for all $x$, $y$ in $M$.

It is easy to check that for every continuous monoid homomorphism satisfying (5.1) $\langle M, \tau, m, \frac{1+m}{2}\rangle$ is a d'Alembert space. We also see from § 2, Equation (2.27) that if $G$ is a topological group, and $\langle G, f\rangle$ is a pre-d'Alembert space then $\langle G, \mathrm{inv}, d, f\rangle$ is a d'Alembert space. Here $\mathrm{inv} : G \to G$ is the group inverse

$x \mapsto x^{-1}$. For work on d'Alembert's equation for groups with involutions other than inv see STETKÆR [7, 8].

Here is our first main result.

**Proposition 5.2.** *Let $\langle M, \tau, m, f\rangle$ be a d'Alembert space. Then $\langle M, f\rangle$ is a pre-d'Alembert space and*

$$m(y)\, f_x(\tau(y)) = -f_x(y) \tag{5.3}$$

*for all $x$, $y$ in $M$.*

PROOF. Let $x$, $y$, $z$ be in $M$. Then, from (5.2)

$$f(xyz) + m(z)f(xy\tau(z)) = 2f(xy)f(z) \tag{5.4}$$

and

$$f(xzy) + m(y)f(xz\tau(y)) = 2f(xz)f(y). \tag{5.5}$$

Now $\tau(y\tau(z)) = z\tau(y)$ so from (5.2)

$$f(xy\tau(z)) + m(y\tau(z))f(xz\tau(y)) = 2f(x)f(y\tau(z)). \tag{5.6}$$

Multiplying (5.6) by $m(z)$ using the fact that $m$ is a homomorphism that satisfies (5.1) we deduce that

$$m(z)f(xy\tau(z)) + m(y)f(xz\tau(y)) = 2f(x)m(z)f(y\tau(z))$$

and so using (5.2) on $m(z)f(y\tau(z))$ we see that

$$m(z)f(xy\tau(z)) + m(y)f(xz\tau(y)) = 4f(x)f(y)f(z) - 2f(x)f(yz). \tag{5.7}$$

Hence adding equations (5.4) and (5.5) and using equation (5.7) we see that $f$ satisfies equation (1.1). Since $f(e) = 1$ and $f$ is continuous, $\langle M, f\rangle$ is a pre-d'Alembert space.

We now prove (5.3). Let $x, y \in M$. Then

$$\begin{aligned} m(y)f_x(\tau(y)) &= m(y)f(x\tau(y)) - f(x)m(y)f(\tau(y)) \\ &= [2f(x)f(y) - f(xy)] - f(x)f(y) = -f_x(y) \end{aligned}$$

from (5.2) and the definition (2.1) of $f_x$. □

We now import our tripartite classification of pre-d'Alembert spaces to d'Alembert spaces as follows:

*Definition 5.3.* The d'Alembert space $\langle M, \tau, m, f\rangle$ is

(i) *trivial* if $\langle M, f\rangle$ is trivial

(ii) *abelian* if $\langle M, f\rangle$ is abelian

(iii) *non-Kannappan* if $\langle M, f\rangle$ is non-Kannappan.

Here then are the structure results

**Theorem 5.4.** *Let $M$ be a topological monoid (with neutral element $e$), $\tau : M \to M$ a continuous involution $m$ a continuous monoid homomorphism to $\langle \mathbb{C}, \cdot\rangle$ satisfying equation (5.1) and $f : M \to \mathbb{C}$ a continuous function with $f(e) = 1$. Then*

(i) *$\langle M, \tau, m, f\rangle$ is a trivial d'Alembert space if and only if $f$ is a monoid homomorphism and, for all $x$ in $M$,*

$$m(x)f(\tau(x)) = f(x), \tag{5.8}$$

(ii) *$\langle M, \tau, m, f\rangle$ is abelian d'Alembert space if and only if there is a continuous homomorphism $k : M \to \langle \mathbb{C}, \cdot\rangle$ not satisfying equation (5.8) such that*

$$f(x) = \frac{k(x) + m(x)k(\tau(x))}{2} \tag{5.9}$$

*for all $x$ in $M$,*

(iii) *$\langle M, \tau, m, f\rangle$ is a non-Kannappan d'Alembert space if and only if*

(a) *there are elements $a, b$ in $M$ with $f(abab) \neq f(aabb)$*

(b) *there is a continuous homomorphism $h : M \to \langle \mathrm{Mat}_2(\mathbb{C}), \cdot\rangle$ such that*

$$m(x)h(\tau(x)) = \operatorname{ad} h(x) \quad \text{and} \quad (x) = \frac{1}{2}\operatorname{trace} h(x)$$

*for all $x$ in $M$.*

Proof. (i) Assume that $f$ is a monoid homomorphism satisfying (5.8). Then

$$f(xy) + m(y)f(x\tau(y)) = f(x)f(y) + f(x)m(y)f(\tau(y)) = 2f(x)f(y),$$

and so $\langle M, \tau, m, f\rangle$ is a d'Alembert space. Moreover it is trivial since $\langle M, f\rangle$ is trivial (Theorem 3.4).

Assume conversely that $\langle M, \tau, m, f\rangle$ is a trivial d'Alembert space. Then $\langle M, f\rangle$ is a trivial pre-d'Alembert space, so by Theorem 3.4 $f$ is a homomorphism. Putting $x = e$ in equation (5.2) we see that $f$ satisfies equation (5.8).

(ii) Assume that the continuous homomorphism $k$ has the stated properties. Then, using the fact that $m$ satisfies equation (5.1) it is easy to verify that, setting

$$f(x) = \frac{k(x) + m(x)k(\tau(x))}{2}\ ,$$

$\langle M, \tau, m, f\rangle$ is a d'Alembert space. It is abelian since $k$ and $m \cdot k \circ \tau$ are distinct homomorphisms (Theorem 3.8).

Assume, conversely, that $\langle M, \tau, m, f\rangle$ is an abelian d'Alembert space. Then there is $c \in M$ such that $f_c(c) \neq 0$. Choose $\lambda \in \mathbb{C}$ satisfying $\lambda^2 f_c(c) = 1$. Following the proof of Theorem 3.8 we put

$$k(x) := f(x) + \lambda f_c(x).$$

Then, as shown here, $k$ is a continuous homomorphism from $M$ to $\langle \mathbb{C}, \cdot\rangle$. Furthermore, using equation (5.2) we see that

$$m(x)k(\tau(x)) = f(x) - \lambda f_c(x),$$

and so $k(x) + m(x)k(\tau(x)) = 2f(x)$ for all $x$ in $M$, as claimed.

(iii) Assume $h$ is a continuous homomorphism from $M$ to $\langle \mathrm{Mat}_2(\mathbb{C}), \cdot\rangle$ satisfying

$$m(x)h(\tau(x)) = \operatorname{ad} h(x)$$

for all $x$ in $M$. Put $f(x) = \frac{1}{2} \operatorname{tr} h(x)$. Then, for all $x$, $y$ in $M$

$$f(xy) + m(y)f(x\tau(y)) = \frac{1}{2} \operatorname{tr}[h(xy) + m(y)h(x\tau(y))]$$
$$= \frac{1}{2} \operatorname{tr}[h(x)\{h(y) + \operatorname{ad} h(y)\}] = \frac{1}{2} \operatorname{tr} h(x) \operatorname{tr} h(y) = 2f(x)f(y).$$

So $\langle M, \tau, m, f\rangle$ is a d'Alembert space. It is non-Kannappan since (a) is assumed true.

Assume conversely that $\langle M, \tau, m, f\rangle$ is a non-Kannappan d'Alembert space. Then there are $a, b \in M$ with $f(a^2b^2) \neq f((ab)^2)$. So we can define $h$ as in equation (4.24). We see immediately from equation (5.2) that $m(x)h(\tau(x)) = \operatorname{ad} h(x)$. Moreover $h$ is a continuous homomorphism from $M$ to $\langle \mathrm{Mat}_2(\mathbb{C}), \cdot\rangle$ by Theorem 4.10. Finally $f = \frac{1}{2} \operatorname{tr} \circ h$ as claimed. □

**Corollary 5.5.** *If $f$ is a d'Alembert function on $\langle M, \tau\rangle$ there is a continuous homomorphism $h : M \to \mathrm{Mat}_2(\mathbb{C})$ such that*

$$h \circ \tau = \operatorname{ad} \circ h \tag{5.10}$$

*and*

$$f = \frac{1}{2} \operatorname{tr} \circ h. \tag{5.11}$$

We close this section with a discussion of Wilson functions in the context of d'Alembert spaces. This subject too has a long history: see Aczel [1, Ch. 3, §2] and Stetkær [6] for references.

**Proposition 5.6.** *Let $\langle M, \tau, m, f\rangle$ be a d'Alembert space. If $g : M \to \mathbb{C}$ is a continuous function satisfying*

$$g(xy) + m(y)g(x\tau y) = 2g(x)f(y), \tag{5.12}$$

*for all $x$, $y$ in $M$ then there is $\lambda \in \mathbb{C}$ and $w \in W$ (the space of Wilson functions on $\langle M, f\rangle$, such that*

$$g = \lambda f + w. \tag{5.13}$$

*Conversely,*

(i) *if $\langle M, \tau, m, f\rangle$ is a non-Kannappan d'Alembert space then every $w$ in $W$ is a solution of* (5.12). *So the space of solutions to* (5.12) *is 4 dimensional.*

(ii) *if $\langle M, \tau, m, f\rangle$ is an abelian d'Alembert space then every central Wilson function on $\langle M, f\rangle$ satisfies equation* (5.12). *Consequently the space of central solutions to* (5.12) *is 2 dimensional.*

Proof. Suppose $g$ satisfies (5.12). Then, since $f$ also satisfies (5.12)

$$w := g - g(e)f \tag{5.14}$$

also satisfies (5.12). Moreover $w(e) = 0$ and

$$m(x)w(\tau(x)) = -w(x). \tag{5.15}$$

To show that $w$ is a Wilson function let $x, y \in M$. Then

$$w(xy) + m(y)w(x\tau(y)) = 2w(x)f(y), belowdisplayskip = 0pt \tag{5.16}$$

and

$$w(yx) + m(x)w(y\tau(x)) = 2w(y)f(x). \tag{5.17}$$

So using (5.15) we see that

$$\begin{aligned} m(y)w(x\tau(y)) + m(x)w(y\tau(x)) &= m(y)w(x\tau(y)) + m(x)m(y\tau(y))w(y\tau(x)) \\ &= m(y)[w(x\tau(y)) + m(x\tau(y))w(\tau(x\tau(y)))] = m(y)\cdot 0 = 0. \end{aligned}$$

Thus adding equations (5.16) and (5.17) yields

$$w(xy) + w(yx) = 2w(x)f(y) + 2w(y)f(x) \tag{2.8}$$

which is the determining relation for Wilson functions.

For the converse part (i), note that $f_z$ satisfies(5.12) for all $z \in M$:

$$\begin{aligned} f_z(xy) + m(y)f_z(x\tau(y)) = f(zxy) - f(z)f(xy) + m(y)f(zx\tau(y)) \\ - m(y)f(z)f(x\tau(y)) = 2f_z(x)f(y). \end{aligned}$$

Hence every Wilson function on $\langle M, f\rangle$ satisfies (5.12). Thus by Proposition 4.4, $W$ is 4 dimensional. Part (ii) of the converse is obvious since $f_c$ satisfies (5.12) and Proposition 3.6 tells us if $w \in W$ is central then it is a scalar multiple of $f_c$. □

## 6. D'Alembert spaces II

In this section we specialize to the situation where our monoid $M$ is actually a topological group.

**Theorem 6.1.** *Let $G$ be a topological group. If $\langle G, \mathrm{inv}, f\rangle$ is a d'Alembert space, that is to say*

$$f(xy) + f(xy^{-1}) = 2f(x)f(y) \tag{1.10}$$

*for all $x$, $y$ in $G$ then $\langle G, f\rangle$ is a pre-d'Alembert space with $d = 1$.*

*Conversely, if $\langle G, f\rangle$ is a pre-d'Alembert space with $d = 1$ then $\langle G, \mathrm{inv}, 1, f\rangle$ is a d'Alembert space.*

Proof. Assume $\langle G, \mathrm{inv}, 1, f\rangle$ is a d'Alembert space. They by Proposition 5.2, $\langle G, f\rangle$ is a pre-d'Alembert space. Moreover

$$d(x) = 2f(x)^2 - f(x^2) = f(xx) + f(xx^{-1}) - f(x^2) = 1$$

for all $x$ in $G$. Hence $d = 1$.

Assume conversely that $\langle G, f\rangle$ is a pre-d'Alembert space with $d = 1$. Then from Proposition 2.11 we see that

$$f(xy) + f(xy^{-1}) = 2f(x)f(y)$$

for all $x$, $y$ in $G$. Since $x \mapsto x^{-1}$ is continuous, $\langle G, \mathrm{inv}, 1, f\rangle$ is a d'Alembert space. $\square$

Next, we prove a result conjectured in [2].

**Theorem 6.2.** *Let $G$ be a topological group and $f : G \to \mathbb{C}$ a continuous function with $f(e) = 1$ satisfying*

$$f(xy) + f(xy^{-1}) = 2f(x)f(y) \tag{1.10}$$

*for all $x$, $y$ in $G$. Then there is a continuous homomorphism $h : G \to \mathrm{SL}_2(\mathbb{C})$ such that*

$$f = \frac{1}{2}\,\mathrm{tr} \circ h.$$

**Proof.** By Theorem 6.1 $\langle G, f\rangle$ is a pre-d'Alembert space with $d = 1$. By Theorem 4.14 there is a continuous homomorphism $h : G \to \mathrm{Mat}_2(\mathbb{C})$ such that $f = \frac{1}{2}\,\mathrm{tr} \circ h$. Moreover by Proposition 4.11 $\det h(x) = d(x) = 1$ for all $x \in G$, so $h : G \to \mathrm{SL}_2(\mathbb{C})$ as claimed. $\square$

**Corollary 6.3.** *If furthermore $f$ is assumed to be bounded then $h$ may be taken to have codomain* $\mathrm{SU}_2(\mathbb{C})$.

PROOF. Assume $f$ is bounded. We see from the construction of $h$ (equations (3.7), (3.8) and (4.24)) that $h$ is then bounded.

As a result of Weil (see HEWITT and ROSS [3, p. 353, 22.23(c1)]) says that there is a $T \in \mathrm{GL}_2(\mathbb{C})$ such that $Th(x)T^{-1} \in U_2(\mathbb{C})$ for all $x$ in $G$. Since $\frac{1}{2}\operatorname{tr}(Th(x)T^{-1}) = f(x)$ and $Th(x)T^{-1} \in U_2(\mathbb{C}) \cap \mathrm{SL}_2(\mathbb{C}) = \mathrm{SU}_2(\mathbb{C})$ we see that we may choose $ThT^{-1}$ as our continuous homomorphism into $\mathrm{SU}_2(\mathbb{C})$. $\square$

Corollary 6.3 was proved in the special case when $G$ is assumed to be compact in [2], and for connected compact groups in [11].

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