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On (ψ, γ)-stability of Cauchy equation on some noncommutative groups

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Dedicated to Professor Zoltán Daróczy on the occasion of his seventieth birthday

Abstract. In this paper, the (ψ, γ)-stability of the Cauchy functional equation is investigated on some noncommutative groups. It is shown that if γ is invariant with respect to inner automorphisms of a step-two solvable group G, then the Cauchy equation $f(xy) = f(x) + f(y)$ is (ψ, γ)-stable on G. If ψ satisfies the condition $\lim_{n\to\infty} \frac{\psi(n^2)}{n} = 0$, then the Cauchy equation is (ψ, γ)-stable on step-two solvable groups and also on step-three nilpotent groups.

1. Introduction

In 1940, S. M. Ulam [17] posed the following fundamental problem. Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a number $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$? See S. M. Ulam [17] for a discussion of such problems, as well as D. H. Hyers [8], [9], D. H. Hyers and S. M. Ulam [11], [12], Aoki [2], Th. M. Rassias [15], [16], G. L. Forti [7], and J. Aczél and J. Dhombres [1]. The first affirmative answer was given by D. H. Hyers [8] in 1941.

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Theorem 1.1 (Hyers [8]). *Let E_1 and E_2 be Banach spaces. If the function $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| < \varepsilon \tag{1.1}$$

for some $\varepsilon > 0$ and for all $x, y \in E_1$, then there exists a unique function $T : E_1 \to E_2$ such that

$$T(x+y) - T(x) - T(y) = 0 \quad \text{for all } dx, y \in E_1 \tag{1.2}$$

and

$$\|f(x) - T(x)\| < \varepsilon \quad \text{for all } x \in E_1. \tag{1.3}$$

Aoki [2] proved a generalized version of Hyers' result which permitted the Cauchy difference to become unbounded. That is, he assumed that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \quad \text{for all } x, y \in E_1,$$

where ε and p are constants satisfying $\varepsilon > 0$ and $0 \leq p < 1$. By making use of the direct method of Hyers [8], he proved in this case too, that there is an additive function T from E_1 into E_2 given by the formula

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

such that

$$\|T(x) - f(x)\| \leq k\varepsilon\|x\|^p,$$

where k depends on p as well as ε. Independently, Th. M. Rassias [15] in 1978 rediscovered the above result and proved that the mapping T is not only additive, under certain conditions, it is also linear. Rassias's paper [15] provided an impetus for a lot of activities in the development of what we now call Hyers–Ulam–Rassias stability theory of functional equations. On an arbitrary group G, the Cauchy functional equation $f(x+y) = f(x) + f(y)$ takes the form $f(xy) = f(x)+f(y)$ for all $x, y \in G$. The first paper to extend Rassias's result to a class nonabelian groups and semigroups was [5]. In [5] among other results, it was proven that the Cauchy functional equation $f(xy) = f(x) + f(y)$ is (ψ, γ)-stable on any abelian group as well as any metabelian (step-two nilpotent) group. It was also shown that any group A can be embedded into a group G, where the Cauchy functional equation is (ψ, γ)-stable. This paper is a continuation of the study of (ψ, γ)-stability initiated in [5]. In this paper, we study the (ψ, γ)-stability of the Cauchy functional equation on step-two solvable groups and step-three nilpotent groups.

2. The space of (ψ,γ)-pseudoadditive mappings

In this section, we recall some important notions from [5] that we need for this paper. We will denote the set of real numbers by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. Let $\mathbb{R}_0^+ = [0,\infty)$ be the set of non-negative numbers and $\mathbb{R}^+ = (0,\infty)$ be the set of positive numbers. Let S be an arbitrary semigroup and G be a group. Throughout this paper, the function $\psi : \mathbb{R}_0^+ \to \mathbb{R}^+$ is considered to be an increasing function satisfying the following three additional conditions:

(1) $\psi(t_1 t_2) \le \psi(t_1)\psi(t_2)$ for all $t_1, t_2 \in \mathbb{R}_0^+$,

(2) $\psi(t_1 + t_2) \le \psi(t_1) + \psi(t_2)$ for all $t_1, t_2 \in \mathbb{R}_0^+$, and

(3) $\lim_{n\to\infty} \frac{\psi(n)}{n} = 0$, $n \in \mathbb{N}$.

Throughout this paper, by γ we will mean a function $\gamma : S \to \mathbb{R}_0^+$ satisfying the inequality

(1) $\gamma(xy) \le \gamma(x) + \gamma(y)$ for all $x, y \in S$.

It is obvious that for any $x \in S$ and for any $m \in \mathbb{N}$ the inequality

$$\gamma(x^m) \le m\,\gamma(x) \tag{2.1}$$

holds.

Definition 2.1. Let S be an arbitrary semigroup and E a Banach space. Further, let $\psi : \mathbb{R}_0^+ \to \mathbb{R}^+$ and $\gamma : S \to \mathbb{R}_0^+$ be the functions as described above. The mapping $f : S \to E$ is said to be a (ψ,γ)-*quasiadditive mapping* if there exists a $\theta \in \mathbb{R}^+$ such that

$$\|f(xy) - f(x) - f(y)\| \le \theta[\psi(\gamma(x)) + \psi(\gamma(y))] \quad \forall x, y \in S \tag{2.2}$$

holds.

It is clear that the set of all (ψ,γ)-quasiadditive mappings from S to E is a real linear space relative to the usual operations. Let us denote it by $KAM_{\psi,\gamma}(S;E)$.

Definition 2.2. Let $\varphi : S \to E$ be a mapping from the semigroup S to a Banach space E. The mapping φ is said to be a (ψ,γ)-*pseudoadditive mapping* if it is a (ψ,γ)-quasiadditive mapping satisfying $\varphi(x^n) = n\varphi(x)$ for all $x \in S$ and for each $n \in \mathbb{N}$.

We denote the space of all (ψ,γ)-pseudoadditive mappings from a semigroup S to a Banach space E by $PAM_{\psi,\gamma}(S;E)$. By $HOM(S;E)$ we mean the set of all homomorphisms from S to E. By $B_{\psi,\gamma}(S;E)$ we denote the linear space of functions from S to E over reals satisfying the relation:

$$\|f(x)\| \le c\psi(\gamma(x)) \quad \text{for some } c > 0 \quad \text{and for all } x \in S.$$

3. Stability

In this section, we prove some general results related to the (ψ, γ)-stability of the Cauchy functional equation. In [5] the following theorem was established.

Theorem 3.1. *The linear space $KAM_{\psi,\gamma}(S;E)$ is a direct sum of the subspaces $PAM_{\psi,\gamma}(S;E)$ and $B_{\psi,\gamma}(S;E)$, that is*

$$KAM_{\psi,\gamma}(S;E) = PAM_{\psi,\gamma}(S;E) \oplus B_{\psi,\gamma}(S;E).$$

Definition 3.2. The Cauchy functional equation

$$f(xy) = f(x) + f(y), \quad \forall\, x, y \in S \tag{3.1}$$

is said to be (ψ, γ)-*stable* for the pair $(S;E)$ if for any $f : S \to E$ satisfying the functional inequality

$$\|f(xy) - f(x) - f(y)\| \le \theta[\psi(\gamma(x)) + \psi(\gamma(y))] \quad \forall x, y \in S \tag{3.2}$$

there is a solution $g : S \to E$ of functional equation (3.1) such that the function $f(x) - g(x)$ belongs to the space $B_{\psi,\gamma}(S;E)$.

It was shown in [5] that the equation (3.1) is (ψ, γ)-stable for the pair $(S;E)$ if and only if $PAM_{\psi,\gamma}(S;E) = HOM(S;E)$.

The following theorem and its proof are generalizations of a similar result proved in [6].

Theorem 3.3. *Let E_1 and E_2 be Banach spaces over reals. Then the equation* (3.1) *is (ψ, γ)-stable for the pair (S, E_1) if and only if it is (ψ, γ)-stable for the pair (S, E_2).*

Proof. Let E be a Banach space over reals and $\mathbb{R}$ be the set of reals. Let the equation (3.1) be stable for the pair (S, E). Suppose (3.1) is not stable for the pair $(S, \mathbb{R})$. Then there is a nontrivial (ψ, γ)-pseudocharacter f on S. So, for some $\theta \ge 0$ we have

$$\|f(xy) - f(x) - f(y)\| \le \theta[\psi(\gamma(x)) + \psi(\gamma(y))] \quad \forall x, y \in S.$$

Now let $e \in E$ and $\|e\| = 1$. Consider the function $\varphi : S \to E$ given by the formula $\varphi(x) = f(x) \cdot e$. It is clear that φ is a nontrivial (ψ, γ)-pseudoadditive E-valued function, and we obtain a contradiction.

Now suppose that the equation (3.1) is stable for the pair $(S, \mathbb{R})$, that is, $PAM_{\psi,\gamma}(S;\mathbb{R}) = HOM(S;\mathbb{R})$. Denote by E^* the space of linear bounded functionals on E endowed by functional norm topology. It is clear that for any $\varphi \in PAM_{\psi,\gamma}(S;E)$ and any $\lambda \in E^*$ the function $\lambda \circ \varphi$ belongs to the space $PAM_{\psi,\gamma}(S;\mathbb{R})$. Indeed, for some nonnegative θ and any $x, y \in S$ we have $\|\varphi(xy) - \varphi(x) - \varphi(y)\| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(y))]$. Hence

$$\begin{aligned}|\lambda \circ \varphi(xy) - \lambda \circ \varphi(x) - \lambda \circ \varphi(y)| &= |\lambda(\varphi(xy) - \varphi(x) - \varphi(y))| \\ &\leq \|\lambda\| \left(\theta[\psi(\gamma(x)) + \psi(\gamma(y))]\right) = \|\lambda\|\theta[\psi(\gamma(x)) + \psi(\gamma(y))].\end{aligned}$$

Obviously, $\lambda \circ \varphi(x^n) = n\lambda \circ \varphi(x)$ for any $x \in S$ and for any $n \in \mathbb{N}$. Hence the function $\lambda \circ \varphi$ belongs to the space $PAM_{\psi,\gamma}(S;\mathbb{R})$. Let $f : S \to E$ be a nontrivial (ψ, γ)-pseudoadditive mapping. Then there are $x, y \in S$ such that $f(xy) - f(x) - f(y) \neq 0$. Hahn–Banach Theorem implies that there is a $\ell \in E^*$ such that $\ell(f(xy) - f(x) - f(y)) \neq 0$, and we see that $\ell \circ f$ is a nontrivial (ψ, γ)-pseudoadditive real-valued function on S. This contradiction proves the theorem. $\square$

In view of Theorem 3.3, it is not important which Banach space is used on the range. Thus one may consider the (ψ, γ)-stability of the functional equation (3.1) on the pair $(S, \mathbb{R})$. Let us simplify the following notations: In the case $E = \mathbb{R}$ the spaces $KAM_{\psi,\gamma}(S;\mathbb{R})$, $PAM_{\psi,\gamma}(S;\mathbb{R})$, and $HOM(S;\mathbb{R})$ will be denoted by $KX_{\psi,\gamma}(S)$, $PX_{\psi,\gamma}(S)$, $X(S)$, respectively. Further, we will call a (ψ, γ)-additive map a (ψ, γ)-quasicharacter, and a (ψ, γ)-pseudoadditive map a (ψ, γ)-pseudocharacter. We also will use the following properties of the (ψ, γ)-pseudocharacter

(1) $f(xy) = f(yx)$ for any $x, y \in S$,

(2) $f(ab) = f(a) + f(b)$, if $ab = ba$

established in [5]. From the first property it follows that if S is a group, then for any $x, y \in S$, the relation $f(y^{-1}xy) = f(x)$ holds. This implies that every (ψ, γ)-pseudocharacter f is invariant under inner automorphisms of group S. As usually by pseudocharacter we mean a real-valued function $f : S \to \mathbb{R}$ satisfying conditions:

(2) the set $\{f(xy) - f(x) - f(y) \mid \forall\, x, y \in S\}$ is bounded, and

(2) $f(x^n) = nf(x)$ for any $x \in S$ and any $n \in \mathbb{N}$.

The set of pseudocharacters of a semigroup S will be denoted by $PX(S)$. It is clear that if γ is a constant function then $PX_{\psi,\gamma}(S) = PX(S)$.

Lemma 3.4. *Let the group G be the union of its subgroups, $G = \cup_{\alpha \in I} G_\alpha$, such that for any $x, y \in G$ there is $\alpha \in I$ such that $x, y \in G_\alpha$. Suppose that the* *equation* (3.1) *is* (ψ, γ)*-stable for any* G_α*. Then the equation* (3.1) *is* (ψ, γ)*-stable on* G*.*

Proof. Let $f \in PX_{\psi,\gamma}(G)$. Then for some $\theta > 0$ and for any $x, y \in G$ we have the inequality

$$|f(xy) - f(x) - f(y)| \leq \theta[\psi(\gamma(x)) - \psi(\gamma(x))].$$

For any $x, y \in G$ there is an α such that $x, y \in G_\alpha$. The equation (3.1) is stable on G_α. Therefore $f(xy) = f(x) + f(y)$. It means that (3.1) is stable on G, and the proof of the lemma is now complete. □

In [5], it was shown that if G is a group and $f \in PAM_{\psi,\gamma}(G; E)$, then (i) $f(e) = 0$, and (ii) $f(x^{-1}) = -f(x)$ for any $x \in G$.

Now for any group G we introduce the following function γ. Let G' be commutator subgroup of G and $g \in G'$. Then g can be represented as a product $g = c_1 c_2 \dots c_k$ of commutators c_i. By commutator length $|g|$ of g we mean the minimum number of commutators we need to represent g as a product of commutators. For unit element e we set $|e| = 0$. Suppose $G = G'$. Then we define

$$\gamma(g) = |g|. \tag{3.3}$$

We define $\gamma(G) = \sup\{\gamma(g) \mid g \in G\}$. Therefore, $\gamma(G)$ is a nonnegative integer or $+\infty$.

Theorem 3.5. *Let the group* G *be the union of its subgroups,* $G = \cup_{\alpha \in I} G_\alpha$*, such that for any* $x, y \in G$ *there is an* $\alpha \in I$ *such that* $x, y \in G_\alpha$*. Suppose that* $G = G'$*, and that for any* α *there is* β *such that* $G_\alpha \subset G'_\beta$*. Let the function* γ *be defined by* (3.3)*. Assume that* $\gamma(G'_\alpha) < \infty$ *for any* $\alpha \in I$*. Then the equation* (3.1) *is* (ψ, γ)*-stable on* G*.*

Proof. Since $G = G' = \cup_{\alpha \in I} G_\alpha = \cup_{\alpha \in I} G'_\alpha$, by Lemma 3.4 it is necessary and sufficient to show that (3.1) is (ψ, γ)-stable on G'_α for any $\alpha \in I$.

Let $\gamma(G'_\alpha) = k_\alpha \in \mathbb{N}$. Then for any $x \in G'_\alpha$ we have $\gamma(x) \leq k_\alpha$ and $\psi(\gamma(x)) \leq \psi(k_\alpha)$. Therefore if $f \in PX_{\psi,\gamma}(G'_\alpha)$, then

$$|f(xy) - f(x) - f(y)| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(y))], \quad x, y \in G'_\alpha,$$

which yields

$$|f(xy) - f(x) - f(y)| \leq 2\theta\psi(k_\alpha), \quad x, y \in G'_\alpha.$$

From the last relation it follows that $f \in PX(G'_\alpha)$. Consider f on G'_α. Let $a, b \in G_\alpha$ and $w = a^{-1}b^{-1}ab$ their commutator. Let $G_\alpha \subseteq G'_\beta$ for some $\beta \in I$. Then we have $a, b, w \in G'_\beta$ and

$$|f(a^{-1}b^{-1}ab) - f(a^{-1}b^{-1}) - f(ab)| \leq 2\theta\psi(k_\beta),$$

which is

$$|f(a^{-1}b^{-1}ab) - f((ba)^{-1}) - f(ab)| \leq 2\theta\psi(k_\beta).$$

Since $f(x^{-1}) = -f(x)$ (see [5], Lemma 2.8), we have

$$|f(a^{-1}b^{-1}ab) + f(ba) - f(ab)| \leq 2\theta\psi(k_\beta),$$

which simplifies to

$$|f(a^{-1}b^{-1}ab)| \leq 2\theta\psi(k_\beta).$$

Thus f is uniformly bounded on the set of commutators $\{[a, b] \mid a, b \in G_\alpha\}$. Now let $g = w_1 w_2 \ldots w_{k_\alpha}$, where w_i is a commutator for $i = 1, \ldots, k_\alpha$. Then $|f(w_1 w_2 \ldots w_{k_\alpha})| \leq 2k_\alpha\theta\psi(k_\beta)$. Thus f is a bounded function on G'_α. Now from the relation $f(x^n) = nf(x)$, $\forall x \in G'_\alpha$, $\forall n \in \mathbb{N}$ it follows that $f \equiv 0$ on G'_α. But it is known that if a pseudocharacter is zero on commutator subgroup of a group B then it is an additive character of B (see [4]). Therefore f is a character of G_α and $f(xy) = f(x) + f(y)$. This completes the proof of the theorem. $\square$

4. Stability on step-two solvable groups

Let $[x, y]$ denotes commutator of two group elements x and y, that is $[x, y] = x^{-1}y^{-1}xy$. A group G is said to be step-two solvable group if for any x, y, u, v in G we have the equality $[[x, y], [u, v]] = e$, where e is the unit element of G (see [13]). It is obvious that any abelian group is a step-two solvable group. Any extension of an abelian group by another abelian group is a step-two solvable group.

Let $F = F(X)$ be a free group of an arbitrary rank with the set of free generators X. Then a subgroup of F generated by all elements of the form $[[x, y], [u, v]]$, where $x, y, u, v \in F$ is a normal subgroup of F. Let us denote it by F''. Then quotient group $F^{[2]}(X) = F/F''$ is a free step-two solvable group with the free set of generators X. Then for any step-two solvable group H any mapping $\tau : X \to H$ can be extended as an homomorphism of $F^{[2]}$ onto the subgroup of H generated by the set $\tau(X)$.

Let G be a free step-two solvable group with two generators a and b. It is well known (see [3]) that G' is a free abelian group with the set of free generators: $w_{i,j} = a^{-i}b^{-j}[a,b]b^j a^i$ for $i,j \in \mathbb{Z}$. When there is no confusion, we will write $w_{i,j}$ simply as w_{ij}. Let $w = w_{00}$.

Lemma 4.1. *For any $i,j \in \mathbb{Z}$, we have the following relations:*

(1) $a^{-k}w_{i,j}a^k = w_{i+k,j}$,

(2) $b^{-k}w_{0,j}b^k = w_{0,j+k}$.

Proof. The proof is obvious. □

Lemma 4.2. *For any $k \in \mathbb{N}$, we have*

$$a^{-1}b^{-k}ab^k = w_{00}w_{01}w_{02}\dots w_{0(k-1)}.$$

Proof. We prove this lemma by induction on k. If $k = 1$, then we have $a^{-1}b^{-k}ab^k = w_{00}$. Suppose that for any $k \le n$ lemma has been established and let us establish it for $n+1$. Since

$$\begin{aligned} a^{-1}b^{-n-1}ab^{n+1} &= a^{-1}b^{-1}b^{-n}ab^nb = a^{-1}b^{-1}aa^{-1}b^{-n}ab^nb \\ &= a^{-1}b^{-1}a[a,b^n]b = a^{-1}b^{-1}abb^{-1}[a,b^n]b \\ &= [a,b]b^{-1}[a,b^n]b = w_{00}b^{-1}w_{00}w_{01}\dots w_{0n-1}b \\ &\qquad \text{(by induction hypothesis)} \\ &= w_{00}w_{01}w_{02}\dots w_{0n} \quad \text{(by Lemma 4.1 (2))} \end{aligned}$$

the proof of the lemma is now complete. □

In the last two sections, as usual, for $x, y \in G$, the conjugate of x by y will be denoted by x^y and hence $x^y = y^{-1}xy$.

Theorem 4.3. *Let D be an arbitrary step-two solvable group. Suppose that function γ is invariant with respect to inner automorphism of group D. Then the equation* (3.1) *is (ψ,γ)-stable on D.*

Proof. First let $D = G$ be a step-two solvable free group with two generators a and b. Let $f \in PX_{\psi,\gamma}(G)$. Thus for some $\theta > 0$, the map $f : G \to \mathbb{R}$ satisfies the relation

$$|f(xy) - f(x) - f(y)| \le \theta[\psi(\gamma(x)) + \psi(\gamma(y))], \quad x, y \in G. \tag{4.1}$$

We should show that $f \in X(G)$. Since G is a free step-two solvable group there is an additive character ξ of G such that $\xi(a) = f(a)$ and $\xi(b) = f(b)$. Then function

$\phi = f - \xi$ is an element of $PX_{\psi,\gamma}(G)$ such that $\phi(a) = \phi(b) = 0$. It is clear that $f \in X(G)$ if and only if $\phi \in X(G)$. So, from the beginning we can assume that $f(a) = f(b) = 0$. Then for any $k \in \mathbb{N}$, letting $x = a^{-1}$ and $y = b^{-k}ab^k$ in the last inequality, we obtain

$$|f(a^{-1}b^{-k}ab^k) - f(a^{-1}) - f(b^{-k}ab^k)| \le \theta[\psi(\gamma(a^{-1}) + \psi(\gamma(b^{-k}ab^k))]$$

and using relations $f(a) = 0$ and $\gamma(b^{-k}ab^k) = \gamma(a)$ we get

$$|f(a^{-1}b^{-k}ab^k)| \le \theta[\psi(\gamma(a^{-1}) + \psi(\gamma(a))],\ k \in \mathbb{N}. \tag{4.2}$$

Taking into account that $f\big|_{G'}$ is an additive character (since G' is commutative) invariant with respect inner automorphisms of G we get

$$f(a^{-1}b^{-k}ab^k) = f(w_{00}w_{01}\dots w_{0k-1}) = kf(w_{00}). \tag{4.3}$$

Now from (4.2) and (4.3) we obtain

$$|kf(w_{00})| \le \theta[\psi(\gamma(a^{-1}) + \psi(\gamma(a))],\ k \in \mathbb{N}$$

which implies $f(w_{00}) = 0$. Therefore, $f(w_{ij}) = 0$ for any $i, j \in \mathbb{Z}$ and $f\big|_{G'} \equiv 0$. Let A and B be subgroup of G generated by a and b respectively. Let $\overline{B}$ be the subgroup of G generated by B and G'. Then $\overline{B}$ is the semidirect product of B and G', that is $\overline{B} = B \rtimes G'$. Let us verify that $f\big|_{\overline{B}} \equiv 0$.

For any $n \in \mathbb{N}$, any $c \in B$ and any $v \in G'$ we have

$$(cv)^n = c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v. \tag{4.4}$$

Letting $x = c^n$ and $y = v^{c^{n-1}} v^{c^{n-2}} \dots v^c v$ in (4.1), we have

$$\begin{aligned}|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v) - f(c^n) - f(v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)|& \\ \le \theta\big[\psi(\gamma(c^n) + \psi(\gamma(v^{c^{n-1}} v^{c^{n-2}} \dots v^c v))\big]&\end{aligned}$$

for each $n \in \mathbb{N}$. Hence

$$|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)| \le \theta\big[\psi(\gamma(c^n)) + \psi(\gamma(v^{c^{n-1}} v^{c^{n-2}} \dots v^c v))\big].$$

Using the subadditivity of γ, we have

$$|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)| \le \theta\left[\psi(n\gamma(c)) + \psi\left(\sum_{k=0}^{n-1} \gamma\big(v^{c^k}\big)\right)\right].$$

From the last inequality and the fact that γ is invariant with respect to inner automorphisms, we obtain

$$|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)| \leq \theta\psi(n)[\psi(\gamma(c)) + \psi(\gamma(v))]$$

The last relation and (4.4) imply

$$|f((cv)^n)| \leq \theta\psi(n)[\psi(\gamma(c)) + \psi(\gamma(v))].$$

Since $f(x^n) = nf(x)$, we obtain

$$n|f(cv)| \leq \theta\psi(n)[\psi(\gamma(c)) + \psi(\gamma(v))]$$

and therefore

$$|f(cv)| \leq \theta\frac{\psi(n)}{n}[\psi(\gamma(c)) + \psi(\gamma(v))]$$

for each $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality and using the fact that $\lim_{n\to\infty} \frac{\psi(n)}{n} = 0$, we obtain $f(cv) = 0$ and therefore $f\big|_{\overline{B}} \equiv 0$.

Now consider group G. This group is a semidirect product of A and $\overline{B}$, that is $G = A \rtimes \overline{B}$. Every element g of G can be represented in the form $g = du$, where $d \in A$ and $u \in \overline{B}$. Arguing as above we can show that $f(g) = 0$. Therefore $f \equiv 0$ on the group G. It means that equation (3.1) is (ψ,γ)-stable on G.

Now suppose that H be an arbitrary step-two solvable group with two generators α and β. The group G is a free step-two solvable with two generators a and b. Then there is an epimorphism $\tau : G \to H$ such that $\tau(a) = \alpha$ and $\tau(b) = \beta$. Define γ^* by the rule $\gamma^*(x) = \gamma(\tau(x))$ for any $x \in G$. It is clear that γ^* satisfies conditions:

$$\gamma^*(xy) \leq \gamma^*(x) + \gamma^*(y) \quad \text{and} \quad \gamma^*(x^{-1}yx) = \gamma^*(y)$$

for any $x, y \in G$.

Let $f \in PX_{\psi,\gamma}(H)$. Then for some $\theta > 0$, the map f satisfies

$$|f(xy) - f(x) - f(y)| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(y))], \quad \forall x, y \in H.$$

Let us verify that $f \in X(H)$. Suppose that there are $c, d \in H$ such that $f(cd) - f(c) - f(d) \neq 0$. Then function f^* defined by the rule $f^*(x) = f(\tau(x))$ belongs to the space $PX_{\psi,\gamma^*}(G)$. But for elements u and v such that $\tau(u) = c$ and $\tau(v) = d$ we get $f^*(uv) - f^*(u) - f^*(v) = f(cd) - f(c) - f(d) \neq 0$. This contradicts the relation $PX_{\psi,\gamma^*}(G) = X(G)$. Therefore, $f \in X(H)$. So, every step-two solvable group generated by two generators has the (ψ,γ)-stability property.

Now let D be an arbitrary step-two solvable group. Then $D = \cup_{x,y} D(x,y)$, where $D(x,y)$ is a subgroup generated by elements $x, y \in D$. Equation (3.1) is (ψ,γ)-stable on any $D(x,y)$. Therefore by Lemma 3.4 equation (3.1) is (ψ,γ)-stable on D. This completes the proof of the theorem. $\square$

Theorem 4.4. *Let D be an arbitrary step-two solvable group. Suppose the function ψ satisfies an additional condition:* $\lim_{n\to\infty} \frac{\psi(n^2)}{n} = 0$. *Then the equation* (3.1) *is (ψ,γ)-stable on D.*

Proof. As it was done in the previous theorem it is enough to prove this theorem for the case $D = G$, where G is a free step-two solvable group with two generators a and b. Let $f \in PX_{\psi,\gamma}(G)$. Then for some $\theta > 0$, the function $f : G \to \mathbb{R}$ satisfies the relation

$$|f(xy) - f(x) - f(y)| \le \theta[\psi(\gamma(x)) + \psi(\gamma(x))], \quad x, y \in G.$$

Let us assume that $f(a) = f(b) = 0$. Then for any $k \in \mathbb{N}$ we have

$$\big|f(a^{-1}b^{-k}ab^k) - f(a^{-1}) - f(b^{-k}ab^k)\big| \le \theta\big[\psi(\gamma(a^{-1})) + \psi(\gamma(b^{-k}ab^k))\big].$$

From the last inequality, we see that

$$\big|f(a^{-1}b^{-k}ab^k)\big| \le \theta\big[\psi(\gamma(a^{-1})) + \psi(\gamma(a)) + \psi(\gamma(b^{-k})) + \psi(\gamma(b^k))\big]$$

which is

$$\big|f(a^{-1}b^{-k}ab^k)\big| \le \theta\big[\psi(\gamma(a^{-1})) + \psi(\gamma(a)) + \psi(k\gamma(b^{-1})) + \psi(k\gamma(b))\big].$$

Since $\psi(t_1t_2) \le \psi(t_1)\psi(t_2)$ for all $t_1, t_2 \in \mathbb{R}_0^+$, we have

$$\big|f(a^{-1}b^{-k}ab^k)\big| \le \theta\big[\psi(\gamma(a^{-1})) + \psi(\gamma(a))] + \theta\psi(k)[\psi(\gamma(b^{-1})) + \psi(\gamma(b))\big].$$

Taking into account that $f\big|_{G'}$ is an additive character invariant with respect inner automorphisms of G we get

$$f(a^{-1}b^{-k}ab^k) = f(w_{00}w_{01}\dots w_{0k-1}) = kf(w_{00}).$$

Therefore, for each $k \in \mathbb{N}$, we have

$$\big|kf(w_{00})\big| \le \theta\big[\psi(\gamma(a^{-1})) + \psi(\gamma(a))\big] + \theta\psi(k)\big[\psi(\gamma(b^{-1})) + \psi(\gamma(b))\big]$$

and hence

$$|f(w_{00})| \le \frac{\theta}{k}\big[\psi(\gamma(a^{-1}) + \psi(\gamma(a))\big] + \theta\frac{\psi(k)}{k}\big[\psi(\gamma(b^{-1})) + \psi(\gamma(b))\big].$$

The last inequality implies that $f(w_{00}) = 0$. Therefore, $f(w_{ij}) = 0$ for any $i, j \in \mathbb{Z}$ and $f\big|_{G'} \equiv 0$.

Let $\overline{B}$ be a subgroup of G generated by B and G'. Then $\overline{B}$ is a semidirect product of B and G', that is $\overline{B} = B \rtimes G'$. Let us verify that $f\big|_{\overline{B}} \equiv 0$. For any $c \in B$ and any $v \in G'$ we have

$$(cv)^n = c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v \tag{4.5}$$

and for each $n \in \mathbb{N}$

$$\begin{aligned}\big|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v) - f(c^n) - f(v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)\big| \\ \le \theta\big[\psi(\gamma(c^n) + \psi(\gamma(v^{c^{n-1}} v^{c^{n-2}} \dots v^c v))\big].\end{aligned}$$

Hence

$$\big|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)\big| \le \theta\big[\psi(\gamma(c^n)) + \psi(\gamma(v^{c^{n-1}} v^{c^{n-2}} \dots v^c v))\big].$$

Simplifying the above inequality, we obtain

$$\big|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)\big| \le \theta\left[\psi(n\gamma(c)) + \psi\left(\sum_{k=0}^{n-1} \gamma\big(v^{c^k}\big)\right)\right].$$

Using the fact that $v^{c^k} = c^{-k} v c^k$ and the last inequality, we get

$$\big|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)\big| \le \theta\left[\psi(n\gamma(c)) + \psi\left(\sum_{k=0}^{n-1} \big(\gamma(c^{-k}) + \gamma(v) + \gamma(c^k)\big)\right)\right]$$

which implies

$$|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)| \le \theta\left[\psi(n\gamma(c)) + \psi(n\gamma(v)) + \psi\left(\sum_{k=0}^{n-1} \big(\gamma(c^{-k}) + \gamma(c^k)\big)\right)\right].$$

Further, simplifying the last inequality, we have

$$\begin{aligned}&|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)| \\ &\qquad \le \theta\left[\psi(n\gamma(c)) + \psi(n\gamma(v)) + \psi\left(\sum_{k=0}^{n-1} \big(k\left[\gamma(c^{-1}) + \gamma(c)\right]\big)\right)\right].\end{aligned}$$

$$|f(c^n v^{c^{n-1}} v^{c^{n-2}} \dots v^c v)| \\ \leq \theta \left[\psi(n\gamma(c)) + \psi(n\gamma(v)) + \psi(n(n-1))\psi\left(\frac{\gamma(c^{-1}) + \gamma(c))}{2}\right)\right].$$

Therefore

$$|f((cv)^n)| \leq \theta \left[\psi(n\gamma(c)) + \psi(n\gamma(v)) + \psi(n(n-1))\psi\left(\frac{\gamma(c^{-1}) + \gamma(c))}{2}\right)\right].$$

Using the fact $f(x^n) = nf(x)$ and simplifying the resulting expression, we obtain

$$|f(cv)| \leq \theta \frac{\psi(n)}{n}[\psi(\gamma(c)) + \psi(\gamma(v)] + \theta \frac{\psi(n(n-1))}{n}\psi\left(\frac{\gamma(c^{-1}) + \gamma(c))}{2}\right).$$

Since $\lim_{n\to\infty} \frac{\psi(n)}{n} = 0$ and $\lim_{n\to\infty} \frac{\psi(n^2)}{n} = 0$, the last inequality implies $f(cv) = 0$ and hence $f\big|_{\overline{B}} \equiv 0$. Now consider group G. This group is a semidirect product $G = A \rtimes \overline{B}$. Every element g of G can be represented in the form $g = du$, where $d \in A$ and $u \in \overline{B}$. Arguing as above we can show that $f(g) = 0$. Therefore $f \equiv 0$ on the group G. It means that the equation (3.1) is (ψ, γ)-stable on G.

Now suppose that H is an arbitrary step-two solvable group with two generators α and β. If G is a free step-two solvable group with two generators a and b, then there is an epimorphism $\tau : G \to H$ such that $\tau(a) = \alpha$ and $\tau(b) = \beta$. Define γ^* by the rule $\gamma^*(x) = \gamma(\tau(x))$ for any $x \in G$. It is clear that γ^* satisfies conditions:

$$\gamma^*(xy) \leq \gamma^*(x) + \gamma^*(y) \quad \text{and} \quad \gamma^*(x^{-1}yx) = \gamma^*(y)$$

for any $x, y \in G$.

Let $f \in PX_{\psi,\gamma}(H)$. Then, for some $\theta > 0$, the map f satisfies

$$|f(xy) - f(x) - f(y)| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(y))], \quad x, y \in H.$$

Let us verify that $f \in X(H)$. Suppose that there are $c, d \in H$ such that $f(cd) - f(c) - f(d) \neq 0$. Then function f^* defined by the rule $f^*(x) = f(\tau(x))$ belongs to the space $PX_{\psi,\gamma^*}(G)$. But for elements u and v such that $\tau(u) = c$ and $\tau(v) = d$ we get $f^*(uv) - f^*(u) - f^*(v) = f(cd) - f(c) - f(d) \neq 0$. This is a contradiction to the fact that $PX_{\psi,\gamma^*}(G) = X(G)$. Therefore $f \in X(H)$. So every step-two solvable group generated by two elements has the (ψ, γ)-stability property.

Now let D be an arbitrary step-two solvable group. Then $D = \cup_{x,y} D(x, y)$, where $D(x, y)$ is a subgroup generated by elements $x, y \in D$. The equation (3.1) is (ψ, γ)-stable on any $D(x, y)$. Therefore by Lemma 3.4 the equation (3.1) is (ψ, γ)-stable on D. Now the proof is completed. □

Remark 4.5. The function $\psi(t) = t^q + 1$ with $0 < q < 1/2$ satisfies condition $\lim_{n\to\infty} \frac{\psi(n^2)}{n} = 0$.

5. Stability on step-three nilpotent groups

A group G is said to be a step-two nilpotent (or metabelian) group if for any $x, y, u \in G$ we have equality $[[x, y], u] = e$, where e is the unit element of G. A group G with unit element e is said to be a step-three nilpotent group if for any $x, y, u, v \in G$ the equality $[[[x, y], u], v] = e$ holds (see [13]). It is obvious that any abelian group is a step-two nilpotent group, and any step-two nilpotent group is a step-three nilpotent group.

Let K be a commutative field. The set

$$\left\{ \begin{pmatrix} 1 & x_1 & y_1 & z \\ 0 & 1 & x_2 & y_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, x_i, y_i, z \in K, i = 1, 2 \right\}$$

of all 4×4 upper triangular matrices forms a group under matrix multiplication. This group is denoted by $UT(4, K)$, and any subgroup of this group is a step-three nilpotent group. The group $UT(4, K)$ is also known as Heisenberg group $H_4(K)$.

Let $F = F(X)$ be a free group an arbitrary rank with the set of free generators X. Denote by $[[[F, F], F], F]$ the normal subgroup of F generated by all elements of the form $[[[x, y], u], v]$, where $x, y, u, v \in F$. Then the quotient group $F^{(3)}(X) = F/[[[F, F], F], F]$ is a free step-three nilpotent group with a free set of generators X. It means that for any step-three nilpotent group H any mapping $\tau : X \to H$ can be extended to a homomorphism of $F^{(3)}$ onto the subgroup of H generated by the set $\tau(X)$.

Let G be a free step-three nilpotent group with two free generators a and b. It is well known that G has the following presentation (see [14]):

$$\begin{aligned} G = \big\langle a, b \mid b^{-1}ab = ac,\ b^{-1}cb = cd,\ a^{-1}ca = ch, \\ ad = da,\ bd = db,\ ah = ha,\ bh = hb \big\rangle. \end{aligned} \tag{5.1}$$

From (5.1) it follows that for any integers n, m the following relations

$$a^{-n}c^m a^n = c^m h^{nm}, \tag{5.2}$$

$$b^{-n}c^m b^n = c^m d^{nm}, \tag{5.3}$$

hold. Suppose that $\varphi \in PX_{\psi,\gamma}(G)$.

Theorem 5.1. *Suppose the function ψ satisfies $\lim_{n\to\infty} \frac{\psi(n^2)}{n} = 0$. Then for any step-three nilpotent group G the equation (3.1) is (ψ, γ)-stable.*

Proof. As we know we can consider only the case when G is free step-three nilpotent group with two generators a, b. Let $\varphi \in PX_{\psi,\gamma}(G)$. We must show that $\varphi \in X(G)$. We can assume that $\varphi(a) = \varphi(b) = 0$.

Then from (5.2), we get

$$\varphi(a^{-n}c^m a^n) = \varphi(c^m h^{nm}). \tag{5.4}$$

From Theorem 2.11 from [5] it follows that $\varphi(u^{-1}vu) = \varphi(v)$ for any u and v. Now taking into account this relation, Theorem 2.10 from [5] and (5.4) we get

$$\varphi(c^m) = \varphi(c^m) + \varphi(h^{nm}).$$

So $\varphi(h) = 0$. Similarly, we get $\varphi(d) = 0$. From presentation (5.3) it follows that $b^{-n}ab^n = ac^n d^{\frac{n(n-1)}{2}}$, for any $n \in \mathbb{N}$. Therefore, $\varphi\big(ac^n d^{\frac{n(n-1)}{2}}\big) = 0$ and $\varphi(ac^n) = 0$. Thus from

$$|\varphi(ac^n) - \varphi(a) - \varphi(c^n)| \le \theta[\psi(\gamma(a)) + \psi(\gamma(c^n))]$$

we have

$$|\varphi(c^n)| \le \theta[\psi(\gamma(a)) + \psi(\gamma(c^n))].$$

Since $\varphi \in PX_{\psi,\gamma}(G)$, we have

$$n|\varphi(c)| \le \theta[\psi(\gamma(a)) + \psi(n\gamma(c))]$$

and hence

$$|\varphi(c)| \le \theta\left[\frac{\psi(\gamma(a))}{n} + \frac{\psi(n)}{n}\psi(\gamma(c))\right].$$

The last inequality implies that $\varphi(c) = 0$. So, we have $\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = \varphi(h) = 0$.

Now let us show that $\varphi \equiv 0$ on G. First note that φ is a function on factor group $G/Z(G)$, where $Z(G)$ denotes center of G. Indeed, $Z(G)$ is a free abelian group generated by elements d and h. From relations $\varphi(d) = \varphi(h) = 0$ it follows that $\varphi \equiv 0$ on $Z(G)$ and for any $u \in G$ and any $w \in Z(G)$ we have $\varphi(uw) = \varphi(u)$. Taking into account this note we get the following relations:

$$a^n b^m c^k a^{n_1} b^{m_1} c^{k_1} = a^{n+n_1} b^{m+m_1} c^{n_1 m+k+k_1} \pmod{Z(G)},$$

and

$$(a^n b^m c^k)^p = a^{pn} b^{pm} c^{nm\frac{p(p-1)}{2}+pk} \pmod{Z(G)}. \tag{5.5}$$

For any $x, y, z \in G$, we have

$$|\varphi(xyz) - \varphi(xy) - \varphi(z)| \leq \theta[\psi(\gamma(xy)) + \psi(\gamma(z))]$$

and

$$|\varphi(xy) - \varphi(x) - \varphi(y)| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(y))].$$

Therefore

$$|\varphi(xyz) - \varphi(x) - \varphi(y) - \varphi(z)| \leq \theta[\psi(\gamma(xy)) + \psi(\gamma(z)) + \psi(\gamma(x)) + \psi(\gamma(y))].$$

Since $\psi(\gamma(xy)) \leq \psi(\gamma(x) + \gamma(y)) \leq \psi(\gamma(x)) + \psi(\gamma(y))$, the last inequality yields

$$|\varphi(xyz) - \varphi(x) - \varphi(y) - \varphi(z)| \leq 2\theta[\psi(\gamma(x)) + \psi(\gamma(y)) + \psi(\gamma(z))]. \tag{5.6}$$

Now let $v = a^n b^m c^k d^q h^\ell$ be an arbitrary element of G. From (5.5), it follows that for any $p \in \mathbb{N}$ there is a $w_p \in Z(G)$ such that $v^p = a^{pn} b^{pm} c^{nm\frac{p(p-1)}{2}+pk} w_p$. Hence we have

$$\varphi(v^p) = \varphi\big(a^{pn} b^{pm} c^{nm\frac{p(p-1)}{2}+pk} w_p\big) = \varphi\big(a^{pn} b^{pm} c^{nm\frac{p(p-1)}{2}+pk}\big).$$

From (5.6), we get

$$\begin{aligned} &\big|\varphi(v^p) - \varphi(a^{pn}) - \varphi(b^{pm}) - \varphi(c^{nm\frac{p(p-1)}{2}+pk})\big| \\ &\qquad\qquad \leq 2\theta\big[\psi(\gamma(a^{pn})) + \psi(\gamma(b^{pm})) + \psi(\gamma(c^{nm\frac{p(p-1)}{2}+pk}))\big]. \end{aligned}$$

Hence from the last inequality, we have

$$|p\varphi(v))| \leq 2\theta\big[\psi(\gamma(a^{pn})) + \psi(\gamma(b^{pm})) + \psi(\gamma(c^{nm\frac{p(p-1)}{2}+pk}))\big]$$

which simplifies to

$$|p\varphi(v))| \leq 2\theta\big[\psi(p\gamma(a^n)) + \psi(p\gamma(b^m)) + \psi(\gamma(c^{nm\frac{p(p-1)}{2}})) + \psi(\gamma(c^{pk}))\big].$$

Thus simplifying further, we see that

$$\begin{aligned} |\varphi(v))| \leq\ & 2\theta\frac{\psi(p)}{p}\big[\psi(\gamma(a^n)) + \psi(\gamma(b^m)) + \psi(\gamma(c^k))\big] \\ & + 2\theta\frac{\psi(p(p-1))}{p}\psi(1/2)\psi(\gamma(c^k)). \end{aligned}$$

Since $\lim_{p\to\infty} \frac{\psi(p)}{p} = 0$ and $\lim_{p\to\infty} \frac{\psi(p(p-1))}{p} = 0$ the last inequality implies $\varphi(v) = 0$. Therefore $\varphi \equiv 0$ on G and equation (3.1) is (ψ, γ)-stable on G. The proof of the theorem is now complete. $\square$

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