Publ. Math. Debrecen
75/1-2 (2009), 85–94

On almost everywhere convergence of Fourier series on unbounded Vilenkin groups

By GYÖRGY GÁT (Nyíregyháza)

Dedicated to Professor Zoltán Daróczy on the occasion of his seventieth birthday

Abstract. In 1973 Gosselin [9] proved that if we have a bounded Vilenkin system, then the Vilenkin–Fourier series of a function in the Lebesgue class L^p for $1 < p$ converges a.e. to the function. It is the most celebrated problem in the harmonic analysis on unbounded Vilenkin groups to give function classes for the elements of which the Fourier series converges almost everywhere. No positive answer is known even for continuous functions in the Lipschitz class. In this paper we give a discretized version of the theorem of Carleson and Hunt, and apply it in order to prove the following theorem with respect to unbounded Vilenkin systems. Let $f \in L^2(G_m)$, and $\sum_{A=0}^{\infty} A^2 \sum_{k=M_A}^{M_{A+1}-1} |\hat{f}(k)|^2 < \infty$. Then we have the a.e. relation $S_n f \to f$. This immediately implies the a.e. convergence $S_n f \to f$ for all $f \in \mathrm{Lip}(\alpha, 2)$ $(\alpha > 0)$.

1. The discrete Fourier series

In the first part of this paper we pass from the statement of Carleson and Hunt [6], [10] on Fourier series, as a statement about Fourier series on the real line, to a statement about Fourier series for groups $\mathbb{Z}_m$. Namely, the integers mod m. We note that this kind of observation, a transference result,

---

Mathematics Subject Classification: 42C10.
Key words and phrases: Schwartz functions, unbounded Vilenkin groups, Vilenkin series, maximal partial sums, almost everywhere convergence.
Research supported by the Hungarian National Foundation for Scientific Research (OTKA), grant no. T048780.

has several antecedents in the literature. For instance the paper by Máté [12] transfers the Carleson [6] theorem to the integers. The paper by Billard [3] discusses the Carleson theorem in the context of Walsh systems. The papers by Thiele [19] discuss several different proofs of Carleson theorem in the Walsh setting. Campbell and Petersen [5] transfer the Carleson theorem to the integers (unaware of Máté) and then to dynamical systems. Auscher and Carro [2] discuss general transference between the three Euclidean groups $\mathbb{R}$, $\mathbb{T}$, and $\mathbb{Z}$.

The second part of the paper concerns a positive result on the convergence of Fourier series for unbounded Vilenkin groups and certain square integrable functions on such groups.

Denote by $e : \mathbb{R} \to \mathbb{C}$ and $e_m : \mathbb{R} \to \mathbb{C}$ the following functions:

$$e(x) = \exp(2\pi\imath x), \quad e_m(x) = e(x/m).$$

The jth partial sum of the Vilenkin–Fourier series of the function $g : \mathbb{Z}_m \to \mathbb{C}$ is:

$$S_j g(k) := \sum_{i=0}^{j-1} \hat{g}(i) e_m(ik),$$

where

$$\hat{g}(i) = \frac{1}{m} \sum_{n=0}^{m-1} g(n) \bar{e}_m(in)$$

is the ith Fourier coefficient. The maximal function of the partial sums of the Fourier series of g is

$$S^* g := \sup_{j \in \mathbb{Z}_m} |S_j g|.$$

The aim of this section is to prove that this maximal operator is of type (L^p, L^p) that is to prove:

Theorem 1.1.

$$\frac{1}{m} \sum_{n=0}^{m-1} |S^* g(n)|^p \le C \frac{1}{m} \sum_{k=0}^{m-1} |g(k)|^p,$$

where $1 < p < \infty$, and the constant C depends only on p.

Proof. For the basic idea of the proof of this theorem see 5.3 Lemma 3, 6.4 Theorem 12, and 5.5 Theorem 16 in the book of Schipp, Wade and Simon [16]. Also see Theorem 7.5 in chapter X in the book of Zygmund [22]. For a positive integer n let K_n be the nth classical trigonometric Fejér kernel function and for an integrable function $f \in L^1[0,1]$ define the function $Vf : \mathbb{Z}_m \to \mathbb{C}$ in the following way:

$$Vf(x) := \int_0^1 \left(2K_{2m}\left(\frac{x}{m} - t\right) - K_m\left(\frac{x}{m} - t\right)\right) f(t)dt \quad (x \in \mathbb{Z}_m).$$

It is clear that if $g : \mathbb{Z}_m \to \mathbb{C}$ and $G := \sum_{i=0}^{m-1} \hat{g}(i)e^i$, then for every $j \in \mathbb{Z}_m$ we have

$$S_j g = V\left(\sum_{i=0}^{j-1} \hat{g}(i)e^i\right) = V(s_j G),$$

where $s_j G$ denotes the jth partial sum of the trigonometric Fourier series of G. That is, for every $x \in \mathbb{Z}_m$ we have

$$\begin{aligned}
&\sup_{0\le j<m} |S_j g(x)| \\
&\quad = \sup_{0\le j<m} |V(s_j G)(x)| \\
&\quad = \sup_{0\le j<m} \left|\int_0^1 \left(2K_{2m}\left(\frac{x}{m} - t\right) - K_m\left(\frac{x}{m} - t\right)\right) s_j G(t)dt\right| \\
&\quad \le \sup_{0\le j<m} \left|\int_0^1 2K_{2m}\left(\frac{x}{m} - t\right) s_j G(t)dt\right| + \sup_{0\le j<m} \left|\int_0^1 K_m\left(\frac{x}{m} - t\right) s_j G(t)dt\right| \\
&\quad \le 2\int_0^1 2K_{2m}\left(\frac{x}{m} - t\right) \sup_{0\le j<m} |s_j G(t)|dt + \int_0^1 K_m\left(\frac{x}{m} - t\right) \sup_{0\le j<m} |s_j G(t)|dt \\
&\quad = 2\sigma_{2m}\left(\sup_{0\le j<m} |s_j G|\right)(x) + \sigma_m\left(\sup_{0\le j<m} |s_j G|\right)(x),
\end{aligned}$$

where $\sigma_q h(x) := \int_0^1 K_q\left(\frac{x}{m} - t\right) h(t)dt$ ($h \in L^1[0,1]$, $q = 1, 2, \dots$). Since

$$\|\sigma_q h\|_\infty = \max_{x\in\mathbb{Z}_m} |\sigma_q h(x)| \le \max_{x\in\mathbb{Z}_m} \left|\int_0^1 K_q\left(\frac{x}{m} - t\right) dt\right| \|h\|_\infty = \|h\|_\infty,$$

and besides,

$$\|\sigma_q h\|_1 = m^{-1} \sum_{l=0}^{m-1} |\sigma_q h(l)| \le \int_0^1 m^{-1} \sum_{l=0}^{m-1} K_q\left(\frac{l}{m} - t\right) |h(t)|dt = \|h\|_1,$$

then for every $1 < p < +\infty$ we get by the help of the interpolation theorem of Marczinkiewicz the operators σ_q are uniformly of type (L^p, L^p). Consequently, the theorem of Carleson and Hunt [6], [10] gives

$$\Big\| \sup_{0\le j<m} |S_j g| \Big\|_p \le C_p \Big\| \sup_{0\le j<m} |s_j G| \Big\|_p \le C_p \|G\|_p = C_p \|V^* g\|_p \le C_p \|g\|_p,$$

where V^* denotes the adjoint operator of V. This completes the proof of the theorem. □

2. The Vilenkin systems

First we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by N. Ja. Vilenkin in 1947 (see e.g. [20], [1]) as follows.

Let $m := (m_k, k \in \mathbb{N})$ $(\mathbb{N} := \{0, 1, \dots\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0, 1, \dots, m_k - 1\}$, with the group operation $\mod m_k$ addition. Since the groups is discrete, then every subset is open. The normalized Haar measure on Z_{m_k}, μ_k is defined by $\mu_k(\{j\}) := 1/m_k$ $(j \in \{0, 1, \dots, m_k - 1\})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbb{N})$. The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group.

The Vilenkin group metrizable in the following way:

$$d(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by below are the same. A base for the neighborhoods of G_m can be given by the intervals:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m, n \in \mathbb{P} := \mathbb{N} \setminus \{0\}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the nullelement of $G_m, I_n := I_n(0)\,(n \in \mathbb{N})$.

Furthermore, let $L^p(G_m)\,(1 \le p \le \infty)$ denote the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G_m, $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in G_m)$, and E_n the conditional expectation operator with respect to $\mathcal{A}_n\,(n \in \mathbb{N})$ $(E_{-1}f := 0\,(f \in L^1))$.

Let $M_0 := 1, M_{n+1} := m_n M_n\,(n \in \mathbb{N})$ be the so-called generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, \dots, m_i - 1\},\ i \in \mathbb{N}),$$

where only a finite number of n_i 's differ from zero. We introduce the following notations:

$$n^{(k)} := \sum_{i=k}^{\infty} n_i M_i, \ (n, k \in \mathbb{N}) \quad |n| := \max\{k \in \mathbb{N} : n_k \neq 0\} \ (1 \leq n \in \mathbb{N}).$$

The generalized Rademacher functions are defined as

$$r_n(x) := \exp\left(2\pi\imath \frac{x_n}{m_n}\right) \quad (x \in G_m, n \in \mathbb{N}, \imath := \sqrt{-1}\,).$$

It is known that $\sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0, & \text{if } x_n e 0, \\ m_n, & \text{if } x_n = 0 \end{cases}$ $(x \in G_m,\ n \in \mathbb{N})$.

The n^{th} Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m, and all the characters of G_m are of this form.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, and the Fejér (or $(C,1)$) means with respect to the Vilenkin system ψ as follows

$$\hat{f}(n) := \int_{G_m} f\bar{\psi}_n d\mu, \qquad S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n(y,x) = D_n(y-x) := \sum_{k=0}^{n-1} \psi_k(y)\bar{\psi}_k(x), \qquad \sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f,$$

$(n \in \mathbb{P},\ y, x \in G_m,\ \hat{f}(0) := \int_{G_m} f d\mu,\ S_0 f = D_0 = 0,\ f \in L^1(G_m))$. It is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y-x) d\mu(x)$$

$(n \in \mathbb{P},\ y \in G_m,\ f \in L^1(G_m))$. It is also well-known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n(0) \\ 0 & \text{if } x \notin I_n(0), \end{cases}$$

$$S_{M_n} f(x) = M_n \int_{I_n(x)} f d\mu = E_n f(x) \quad (f \in L^1(G_m), n \in \mathbb{N}).$$

Moreover, [1] for $n \in \mathbb{N}$

$$D_n = \psi_n \sum_{k=0}^{\infty} D_{M_k} \sum_{i=m_k-n_k}^{m_k-1} r_k^i = \sum_{k=0}^{\infty} \psi_{n^{(k+1)}} D_{M_k} \sum_{i=0}^{n_k-1} r_k^i. \tag{1}$$

If $1 \le p < \infty$ and k is any nonnegative integer, then the integrated modulus of continuity of order k for $f \in L^p(G_m)$ is

$$\omega_p(f,k) = \sup\left\{\|\tau_y f - f\|_p : y \in I_k\right\},$$

where $\tau_y f(x) = f(x+y)$. Onneweer [13, p. 680] defined $\mathrm{Lip}(\alpha,p)$ to be the space of all f in $L^p(G_m)$ such that $\omega_p(f,k) = O(M_k^{-\alpha})$ [4], [14].

The almost everywhere convergence of the full partial sums for $L^p(G_m)$, $p > 1$, is known in the bounded case [9] but not in the unbounded case. There is no known result with respect to the a.e. convergence of the partial sums $S_n f$ even for continuous functions in Lipschitz classes.

On the other hand, mean convergence of the full partial sums for $L^p, p > 1$, is known for the unbounded case. Namely, in 1999 the author [7] proved that if $f \in L^p(G_m)$, where $p > 1$, then $\sigma_n f \to f$ almost everywhere. This was the very first "positive" result with respect to the a.e. convergence of the Fejér means of functions on unbounded Vilenkin groups. With respect to norm convergence that is, the fact that the partial sums $S_n f$ converges to f in L^p-norm for all $f \in L^p(G_m)$, and $1 < p < \infty$ one can see the papers (written independently at the same time) of Young [21], Schipp [15] and Simon [18], [17]. On the other hand, much is unknown for unbounded Vilenkin systems. We mention for instance the convergence of (C,α) means with negative α. For the Walsh–Paley system this means is investigated by Goginava [8], but for unbounded Vilenkin systems, nothing can be said.

In this paper we prove the following

Theorem 2.1. *Let G_m be any Vilenkin group (bounded or not), and $f \in L^2(G_m)$ such that*

$$\sum_{A=0}^{\infty} A^2 \sum_{k=M_A}^{M_{A+1}-1} |\hat{f}(k)|^2 < \infty.$$

Then we have the a.e. relation $\lim_{n\to\infty} S_n f = f$.

We could say that with respect to bounded Vilenkin systems since $\log(M_A) \sim A$, then Theorem 2.1 is nothing else but the well known Rademacher–Menshov theorem for general orthonormal systems. But, in this paper we are talking about unbounded ones, and for this reason $\log(M_A)/A$ can converge to $+\infty$ arbitrary fast. Consequently, for these Vilenkin groups Theorem 2.1 is much stronger than the Rademacher–Menshov theorem. The same can be said with respect to the theorem of Kolmogorov, Seliverstov, Plessner and Kaczmarz (see e.g. [11]) which states for bounded Vilenkin groups that the inequality $\sum_{k=0}^{\infty}|\hat{f}(k)|^2\log k<\infty$ implies the almost everywhere convergence of the partial sum of the Fourier series. We emphasize once again that in this paper we are talking about of Vilenkin groups of any kind. That is, the sequence m can grow "very fast".

Proof. Introduce the operators $T_{n,k}: L^1(G_m)\to L^0(G_m)$ $(n,k\in\mathbb{N})$:

$$T_{n,k}f(y)$$
$$:=\sum_{j=0}^{n_k-1}\left(\frac{1}{m_k}\sum_{s=0}^{m_k-1}\left(M_{k+1}\int\limits_{I_{k+1}(y_k(s))}f(t)\bar{\psi}_{n^{(k+1)}}(t)d\mu(t)\right)\psi_{n^{(k+1)}}(y)\bar{r}_k^j(y_k(s))\right)r_k^j(y),$$

where $y, y_k(s):=(y_0,\dots,y_{k-1},s,0,0,\dots)\in G_m$. In other words,

$$T_{n,k}f=\sum_{j=0}^{n_k-1}E_k\left(E_{k+1}(f\bar{\psi}_{n^{(k+1)}})\psi_{n^{(k+1)}}\bar{r}_k^j\right)r_k^j. \tag{2}$$

(1) and (2) imply that

$$S_nf=\sum_{k=0}^{|n|}T_{n,k}f.$$

Fix (for a moment) $n^{(k+1)}$ and y and let the function g be defined as

$$g(y_0,\dots,y_{k-1},s):=M_{k+1}\int_{I_{k+1}(y_k(s))}f(t)\bar{\psi}_{n^{(k+1)}}(t)d\mu(t)\quad(s\in\mathbb{Z}_{m_k}).$$

Then we have

$$|T_{n,k}f(y)|=\left|\sum_{j=0}^{n_k-1}\frac{1}{m_k}\sum_{s=0}^{m_k-1}g(y_0,\dots,y_{k-1},s)\bar{r}_k^j(y_k(s))r_k^j(y)\right|.$$

By Theorem 1.1 we have

$$\frac{1}{m_k}\sum_{y_k=0}^{m_k-1}\sup_{n_k}|T_{n,k}f(y)|^2\le C\frac{1}{m_k}\sum_{s=0}^{m_k-1}|g(y_0,\dots,y_{k-1},s)|^2$$
$$=CE_k\left(\left|E_{k+1}(f\bar{\psi}_{n^{(k+1)}})\right|^2\right)(y) \tag{3}$$

Apply the conditional Hölder inequality for $E_{k+1}(f\bar{\psi}_{n^{(k+1)}})$. Then We get

$$\left|E_{k+1}(f\bar{\psi}_{n^{(k+1)}})\right|^2 \le E_{k+1}(|f|^2). \tag{4}$$

Consequently, by (3) and (4) we have the following inequality

$$\frac{1}{m_k}\sum_{y_k=0}^{m_k-1}\sup_{n_k}|T_{n,k}f|^2 \le CE_k(E_{k+1}(|f|^2)) \le CE_k(|f|^2).$$

This inequality immediately gives

$$\left\|\sup_n |T_{n,k}f|\,\right\|_2^2 \le C\|f\|_2^2 \tag{5}$$

for any $k \in \mathbb{N}$.

For $k < A = |n|$ we have $T_{n,k}f = T_{n,k}(E_{A+1}f)$, and $T_{n,k}(E_Af) = 0$. Consequently in this case we get

$$\left\|\sup_n |T_{n,k}f|\,\right\|_2^2 = \left\|\sup_n |T_{n,k}(E_{A+1}f - E_Af)|\,\right\|_2^2 \le C\|E_{A+1}f - E_Af\|_2^2.$$

Next, we discuss the case $k = A$. More exactly, the question is that: What can be said about $\sup_A \sup_{|n|=A}|T_{n,A}f|$? It is easy to have that $T_{n,A}f = T_{n,A}E_{A+1}f$, and $T_{n,A}E_Af = E_Af$. Thus,

$$\begin{aligned}\left\|\sup_A \sup_{|n|=A}|T_{n,A}f|\,\right\|_2^2 &= \left\|\sup_A \sup_{|n|=A}|T_{n,A}(E_{A+1}f - E_Af) + E_Af|\,\right\|_2^2\\ &\le C\left\|\sup_A \sup_{|n|=A}|T_{n,A}(E_{A+1}f - E_Af)|\,\right\|_2^2 + C\left\|\sup_A |E_Af|\,\right\|_2^2 =: i_1 + i_2.\end{aligned}$$

By (5) for i_1 we have

$$i_1 \le C\sum_{A=0}^{\infty}\left\|\sup_{|n|=A}|T_{n,A}(E_{A+1}f - E_Af)|\,\right\|_2^2 \le C\sum_{A=0}^{\infty}\|E_{A+1}f - E_Af\|_2^2 \le C\|f\|_2^2.$$

On the other hand, for i_2 we get the same bound, that is, $i_2 \le C\|f\|_2^2$. (We recall that the maximal function $\sup_A |E_Af|$ is of type (L^2, L^2)). Finally, the equality $S_nf = \sum_{k=0}^{|n|}T_{n,k}f = T_{n,|n|}f + \sum_{k=0}^{|n|-1}T_{n,k}f$ gives

$$\begin{aligned}\left\|\sup_n |S_nf|\,\right\|_2^2 &\le C\|f\|_2^2 + 2\left\|\sup_A \sup_{|n|=A}\sum_{k=0}^{A-1}|T_{n,k}f|\,\right\|_2^2\\ &\le C\|f\|_2^2 + C\sum_{A=0}^{\infty}A^2\|E_{A+1}f - E_Af\|_2^2.\end{aligned}$$

Since

$$\|E_{A+1}f - E_A f\|_2^2 = \sum_{k=M_A}^{M_{A+1}-1} |\hat{f}(k)|^2$$

and

$$\|f\|_2^2 = \sum_{j=0}^{\infty} |\hat{f}(j)|^2 \le |\hat{f}(0)|^2 + \sum_{A=0}^{\infty} (A+1)^2 \sum_{k=M_A}^{M_{A+1}-1} |\hat{f}(k)|^2,$$

then we have

$$\big\| \sup_n |S_n f| \,\big\|_2^2 \le C\left(|\hat{f}(0)|^2 + \sum_{A=0}^{\infty} (A+1)^2 \sum_{k=M_A}^{M_{A+1}-1} |\hat{f}(k)|^2 \right). \tag{6}$$

Let

$$L^2_\dagger(G_m) := \Bigg\{ f \in L^2(G_m) : \|f\|_\dagger := \sqrt{ |\hat{f}(0)|^2 + \sum_{A=0}^{\infty} (A+1)^2 \sum_{k=M_A}^{M_{A+1}-1} |\hat{f}(k)|^2 } < +\infty \Bigg\}.$$

It is clear that $\|.\|_\dagger$ is a norm and by (6) the maximal operator $L^2_\dagger(G_m) \ni f \mapsto \sup_n |S_n f| \in L^2(G_m)$ is bounded.
Since the set of Vilenkin polynomials is dense in $L^2_\dagger(G_m)$ and $\lim_{n\to\infty} S_n f = f$ holds trivially for all Vilenkin polynomials f, we have by standard argument the a.e. relation $\lim_{n\to\infty} S_n f = f$ for every $L^2_\dagger(G_m)$. The proof of Theorem 2.1 is complete. □

Corollary 2.2. *Let $f \in \mathrm{Lip}(\alpha, 2)$ for some $\alpha > 0$. Then $S_n f \to f$ a.e.*

Proof. $\sum_{k=M_A}^{M_{A+1}-1} |\hat{f}(k)|^2 \le \sum_{k=M_A}^{\infty} |\hat{f}(k)|^2 = \|f - E_A f\|_2^2 \le (\omega_2(f, A))^2 \le C M_A^{-2\alpha} \le C 2^{-2A\alpha}$. □

Acknowledgement. The author is indebted to the referee for the help given to improve the manuscript.

References

[1] G. H. Agaev, N. Ja. Vilenkin, G. M. Dzhafarli and A. I. Rubinstein, Multiplicative systems of functions and harmonic analysis on 0-dimensional groups, *Izd. "ELM", Baku* (1981), 180 (in *Russian*).

[2] P. Auscher and M. J. Carro, On relations between operators on $\mathbb{R}^{\mathbb{N}}$, $\mathbb{T}^{\mathbb{N}}$ and $\mathbb{Z}^{\mathbb{N}}$, *Stud. Math.* **101**, no. 2 (1992), 165–182.

[3] P. Billard, Sur la convergence presque partout des séries de Fourier–Walsh des fonctions de l'espace $L^2(0,1)$, *Stud. Math.* **28** (1967), 363–388.

[4] W. R. Bloom and J. J. F. Fournier, Generalized Lipschitz spaces on Vilenkin groups, *Math. Nachr.* **132** (1987), 67–80.

[5] J. Campbell and K. Petersen, The spectral measure and Hilbert transform of a measure-preserving transformation, *Trans. Am. Math. Soc.* **313**, no. 1 (1989), 121–129.

[6] L. Carleson, On convergence and growth of partial sums of Fourier series, *Acta Math.* **116** (1966), 135–157.

[7] G. Gát, Pointwise convergence of the Fejér means of functions on unbounded Vilenkin groups, *Journal of Approx. Theory* **101**, no. 1 (1999), 1–36.

[8] U. Goginava, On the approximation properties of Cesàro means of negative order of Walsh–Fourier series, *Journal of Approx. Theory* **115**, no. 1 (2002), 9–20.

[9] J. Gosselin, Almost everywhere convergence of Vilenkin–Fourier series, *Trans. Amer. Math. Soc.* **185** (1973), 345–370.

[10] R. A. Hunt, On the convergence of Fourier series, Proc. Conf. Edwardsville, 1967, 235–255.

[11] S. Kaczmarz and H. Steinhaus, Theorie der Orthogonalreihen, *Chelsea Publishing Co., New York*, 1951.

[12] A. Máté, Die Konvergenz der Fourierreihen quadratisch integrierbarer Funktionen, *Mat. Lapok* **18** (1967), 195–242.

[13] C. W. Onneweer, Absolute convergence of Fourier series on certain groups, II, *Duke Math. J.* **41** (1974), 679–688.

[14] T. S. Quek and L. H. Yap, Absolute convergence of Vilenkin–Fourier series, *J. of Math. Anal. and Appl.* **74** (1980), 1–14.

[15] F. Schipp, On Lp-norm convergence of series with respect to product system, *Analysis Math.* **2** (1976), 49–64.

[16] F. Schipp, W. R. Wade and P. Simon, Walsh series: An Introduction to Dyadic Harmonic Analysis. With the Collaboration of J. Pál, *Adam Hilger, Ltd.*, *Bristol and New York*, 1990.

[17] P. Simon, On the concept of a conjugate function, Colloquia Math. Soc. János Bolyai 19, *Fourier Analysis and Approximation Theory*, *Budapest*, 1976, 747–755.

[18] P. Simon, Verallgemeinerte Walsh–Fourierreihen II, *Acta Math. Acad. Sci. Hung.* **27**(3–4) (1976), 329–341.

[19] C. Thiele, The quartile operator and pointwise convergence of Walsh series, *Trans. Am. Math. Soc.* **352**, no. 12 (2000), 5745–5766.

[20] N. Ja. Vilenkin, On a class of complete orthonormal systems, *Izv. Akad. Nauk. SSSR, Ser. Math.* **11** (1947), 363–400 (in *Russian*).

[21] W. S. Young, Mean convergence of generalized Walsh–Fourier series, *Trans. Amer. Math. Soc.* **218** (1976), 311–320.

[22] A. Zygmund, Trigonometric Series, *University Press, Cambridge*, 1959 (in *English*).

GYÖRGY GÁT
COLLEGE OF NYÍREGYHÁZA
INST. OF MATH. AND COMP. SCI.
NYÍREGYHÁZA, P.O.BOX 166., H–4400
HUNGARY

E-mail: gatgy@nyf.hu

(Received September 22, 2008; revised March 14, 2009)