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On a quantitative form of Wirsing's mean-value theorem for multiplicative functions

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Dedicated to Professor Zoltán Daróczy on the occasion of his 70th birthday

Abstract. In this paper, the author applies the convolution arithmetic to the investigation of the summatory function of arithmetical functions. Mean value theorems with remainder term estimation are proved for real-valued multiplicative functions of modulus≤ 1.

1. Introduction

In some recent papers we investigated inequalities of the form

$$\left|\frac{1}{x}\sum_{n\leq x}(f(n)-A_x)\right| \ll \left(\frac{1}{\log x}\int_{-\infty}^{\infty}\left|\frac{F(s)-A_x\zeta(s)}{s}\right|^2 dt\right)^{\frac{1}{2}} + O\left(|A_x|\sum_{p\leq x}|f(p)-1|\frac{\log p}{p}\right) \quad (1)$$

$\left(s=1+\frac{1}{\log x}+it\right)$, where A_x is a complex constant, f can be chosen either as von Mangoldt's function Λ or as a completely multiplicative function of modulus ≤ 1,

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and F and ζ denote the generating function of f, defined by

$$F(s)=\sum_{n>1}^{\infty}\frac{f(n)}{n^s},$$

and the Riemann ζ -function, respectively. We used the estimate (1) to prove the prime number theorem and the theorem of Halász in the case of completely multiplicative functions f, $|f|\le 1$. (The assertion for arbitrary multiplicative functions of modulus ≤ 1 follows then by well-known convolution arguments.) The proof of the result can be split into two parts. The first one is contained in

Proposition 1 ([13]). *Assume that f is completely multiplicative with $|f|\le 1$ and $A_x\in\mathbb{C}$. Then if we put $M(x)=\sum_{n\le x}(f(n)-A_x)$,*

$$\frac{|M(x)|}{x}\ll\frac{1}{\log x}\int_1^x\frac{|M(u)|}{u^2}du+O\left(|A_x|\frac{1}{\log x}\sum_{p\le x}|f(p)-1|\,\frac{\log p}{p}\right)+O\left(\frac{1}{\log x}\right)$$

for all $x\ge x_0$.

For the second part we used Schwarz's inequality and Parseval's formula (Re $s=\sigma>1$)

$$\int_{-\infty}^{\infty}\left|\frac{F(s)}{s}\right|^2dt=2\pi\int_0^{\infty}\left|M(e^w)e^{-w\sigma}\right|^2dw \tag{2}$$

to prove

$$\frac{1}{\log x}\int_1^x\frac{|M(u)|}{u^2}du\ll\left(\frac{1}{\log x}\int_{-\infty}^{\infty}\left|\frac{F(s)}{s}\right|^2dt\right)^{\frac12}$$

where $M(x)=\sum_{n\le x}f(n), s=1+\frac{1}{\log x}+it$ and

$$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

is absolutely convergent for $\operatorname{Re}s>1$.

The aim of this paper is to prove Proposition 1 for *arbitrary* multiplicative functions f with $|f|\le 1$, and to apply this result to a *quantitative* proof of

Wirsing's theorem that a real-valued multiplicative function f with values $-1 \leq f(n) \leq 1$ $(n \in \mathbb{N})$ has a mean-value, i.e.

$$\lim_{x\to\infty} x^{-1} \sum_{n\leq x} f(n)$$

exists. For example, if the series

$$\sum_{p} \frac{1-f(p)}{p} \tag{3}$$

diverges we show

$$\frac{1}{x}\left|\sum_{n\leq x} f(n)\right| \ll \exp\left(-c\sum_{p\leq x}\frac{1-f(p)}{p}\right) \tag{4}$$

with $c = \frac{1}{5}$.

Remark 1. The existence of such a constant c is contained in HALÁSZ's result [3] (see P. D. T. A. ELLIOTT [2], chapter 19, where the estimates of Halász are described and refined).

In 1986 A. HILDEBRAND [7] showed

$$\frac{1}{x}\left|\sum_{n\leq x} f(n)\right| \ll \left(1+\sum_{p\leq x}\frac{1-f(p)}{p}\right)^{-\frac{1}{2}}$$

His proof uses the prime number theorem with logarithmic error term.

R. R. HALL and G. TENENBAUM [5] proved (4) with the optimal constant $c = 0,32867\ldots$ but they made use of the prime number theorem in the form $\pi(x) = li\,x + O\left(x\exp\left(-2\sqrt{\log x}\right)\right)$ whereas our proof uses only Selberg's symmetry formula and very elementary properties of the ζ-function on $s = \sigma + it$ for $\sigma > 1$.

Theorem 1. *Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative with $|f| \leq 1$ and let $A_x \in \mathbb{C}$. Put $M(x) = \sum_{n\leq x}(f(n) - A_x)$. Then, for $x \geq 3$,*

$$\begin{aligned}\frac{|M(x)|}{x} \leq{} & \frac{2}{\log x}\int_1^x \frac{|M(u)|}{u^2}du \\ & + O\left(|A_x|\frac{1}{\log x}\sum_{p\leq x}\frac{\log p}{p}\left|1-f(p)\right|\right) + O\left(\frac{1}{\log x}\right).\end{aligned} \tag{5}$$

For $3 \le u \le x$ we define $H_0\big(\frac{1}{\log u}\big)$ and $H_1\big(\frac{1}{\log u}\big)$ by

$$H_0^2\left(\frac{1}{\log u}\right) := \int_{-\infty}^{\infty} \left| \frac{F\big(1+\frac{1}{\log u}+it\big)}{1+\frac{1}{\log u}+it} \right|^2 dt \tag{6}$$

and

$$H_1^2\left(\frac{1}{\log u}\right) := \int_{-\infty}^{\infty} \left| \frac{F'\big(1+\frac{1}{\log u}+it\big)}{1+\frac{1}{\log u}+it} \right|^2 dt \tag{7}$$

respectively. Then we formulate

Theorem 2. *Let $f :\to \mathbb{C}$ such that $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely for $\operatorname{Re} s = \sigma > 1$. Then the following three assertions hold for $x \ge 3$.*

(i) $\displaystyle\int_1^x \frac{|M(u)|}{u^2} du \ll H_0\left(\frac{1}{\log x}\right)(\log x)^{\frac{1}{2}}$

(ii) $\displaystyle\int_1^x \frac{|M(u)|}{u^2} du \ll H_1\left(\frac{1}{\log x}\right)$

(iii) $\displaystyle\int_1^x \frac{|M(u)|}{u^2} du \ll \int_{\frac{1}{2\log x}}^{1} \frac{H_1(y)}{y^{1/2}} dy.$

Remark 2. An appropriate choice of A_x depends on the behaviour of the sum

$$\sum_p \frac{1-\operatorname{Re} f(p)p^{-ia}}{p} \qquad (a \in \mathbb{R}), \tag{8}$$

namely, we define

$$A_x = \begin{cases} \dfrac{x^{ia_0}}{1+ia_0} \prod\limits_{p\le x}\left(1-\dfrac{1}{p}\right)\left(1+\sum\limits_{k=1}^{\infty} f(p)^k\right) p^{-k(1+ia_0)} & \text{if (8) converges for some } a = a_0, \\ 0 & \text{if (8) diverges for all } a \in \mathbb{R}. \end{cases} \tag{9}$$

(see [13]). Here we consider only real-valued functions f and distinguish the cases where $\sum_p (1-f(p))p^{-1}$ converges or diverges, respectively. This ends Remark 2.

As an immediate consequence of Theorem 1 and Theorem 2 we have

Corollary 1. *Let M be given as in Theorem 1 with $A_x = 0$. Then the following assertions are equivalent.*

(i) $\lim\limits_{x\to\infty} x^{-1} M(x) = 0$

(ii) $\lim_{x\to\infty} \frac{1}{\log x} \int_1^x \frac{|M(u)|}{u^2} du = 0$

(iii) $\lim_{x\to\infty} \frac{1}{\log x} \int_1^x \frac{|M(u)|^2}{u^3} du = 0$

(iv) $\lim_{\sigma\to 1^+} (\sigma - 1) \int_{-\infty}^{\infty} \left| \frac{F(\sigma+it)}{\sigma+it} \right|^2 dt = 0$

(v) $\lim_{\sigma\to 1^+} (\sigma - 1)^3 \int_{-\infty}^{\infty} \left| \frac{F'(\sigma+it)}{\sigma+it} \right|^2 dt = 0.$

In the case of real-valued f we obtain a quantitative version of Wirsing's theorem. If the series $\sum_p (1 - f(p))p^{-1}$ converges we can define a function $\delta_0(x)$ tending to zero as x tends to infinity such that if $y(x) = x^{\delta_0(x)}$

$$\sum_{y(x)<p\le x} \frac{1 - f(p)}{p} \le \delta_0(x). \tag{10}$$

For, assume

$$\sum_{p\le x} \frac{1 - f(p)}{p} = c - \varepsilon(x)$$

where $\varepsilon(x) \searrow 0$ as $x \to \infty$. Then, choose $\delta_1(x) > 0$ such that

$$\sum_{p\le x} \frac{1 - f(p)}{p} \log p = \log x \left(-\varepsilon(x) + \frac{1}{\log x} \int_2^x \frac{\varepsilon(u)}{u} du + O\left(\frac{1}{\log x}\right)\right) \le \delta_1(x) \log x$$

and $\delta_1(x) = o(1)$ $(x \to \infty)$. Putting $\delta_0(x) = (\delta_1(x))^{\frac{1}{2}}$ and $y(x) = x^{\delta_0(x)}$ gives

$$\sum_{y(x)<p\le x} \frac{1 - f(p)}{p} \cdot \frac{\log p}{\log p} \le \frac{\delta_1(x)}{\delta_0(x)} = \delta_0(x)$$

which proves (10).

With these notations we formulate the following

Theorem 3. *Let $f : \mathbb{N} \to \mathbb{R}$ be multiplicative and $-1 \le f(n) \le 1$ for every $n \in \mathbb{N}$.*

(i) *If*

$$\sum_{p\le x} \frac{1 - f(p)}{p} \to \infty \quad (x \to \infty)$$

then

$$\frac{1}{x} \sum_{n\le x} f(n) \ll \exp\left(\frac{1}{5} \sum_{p\le x} \frac{f(p) - 1}{p}\right)$$

as $x \to \infty$.

(ii) *If*

$$\sum_{p\le x}\frac{1-f(p)}{p} = c-\varepsilon(x) \quad (x\to\infty)$$

then

$$\frac{1}{x}\sum_{n\le x} f(n) = \prod_{p\le x}\left(1+\sum_{k=1}^{\infty}\frac{f(p^k)}{p^k}\right)\left(1-\frac{1}{p}\right) + O\left((\delta_0(x))^{1/18}\right)$$

as $x\to\infty$.

2. Proof of Theorem 1

Remark 3. In the following we use the convolution arithmetic for functions from

$$\mathcal{S} := \{f:\mathbb{R}\to\mathbb{C},\ f(x)=0 \text{ for } x<1\},$$

which coincides with the Dirichlet convolution for the class

$$\mathcal{A} := \{f\in\mathcal{S} : f(x)=0 \text{ for } x\notin\mathbb{N}\}.$$

of arithmetical functions.

So, for $f,g\in\mathcal{S}$, the convolution $f*g$ in $\mathcal{S}$ is defined by

$$(f*g)(x) = \sum_{1\le n\le x} f\left(\frac{x}{n}\right) g(n).$$

The "action" of this definition on functions of $\mathcal{A}$ is given by the following: if $f\in\mathcal{A},\ g\in\mathcal{S}$ then $f*g\in\mathcal{A}$ and for $n\in\mathbb{N}$,

$$(f*g)(n) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d).$$

In general the binary operation $*$ is not commutative in $\mathcal{S}$, but if $f,g\in\mathcal{A}$ then $f*g=g*f$.

Consider the function ε defined by

$$\varepsilon(x) = \begin{cases} 1 & \text{for } x=1,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $\varepsilon \in \mathcal{A}$, and

$$f * \varepsilon = f \quad \text{for } f \in \mathcal{S}$$

and

$$(\varepsilon * f)(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } f \in . \tag{11}$$

Thus ε serves as a right identity under convolution for all of $\mathcal{S}$, but is a left identity only in $\mathcal{A}$.

The relation (11) suggests that for each $f \in \mathcal{S}$ we define an image $f_0 \in \mathcal{A}$ by

$$f_0 = \varepsilon * f \quad \text{for } f \in \mathcal{S}.$$

The Möbius function μ is defined by

$$\mathbf{1}_0 * \mu = \varepsilon,$$

where $\mathbf{1}_0 = \varepsilon * \mathbf{1}$ and $\mathbf{1} \in \mathcal{S}$ with

$$\mathbf{1}(x) = \begin{cases} 1 & \text{for } x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The well-known Möbius inversion formula says that if $f, g \in \mathcal{S}$ then $f = g * \mathbf{1}_0$ if and only if $g = f * \mu$.

Let $L \in \mathcal{S}$ denote the logarithm function. Then obviously L acts as a derivation on $\mathcal{S}$, that is

$$L \cdot (f * g) = (L \cdot f) * g + f * (L \cdot g) \quad \text{for all } f, g \in \mathcal{S}.$$

Further, we introduce the von Mangoldt function $\Lambda \in \mathcal{A}$ by

$$\varepsilon * L = L_0 = \Lambda * \mathbf{1}_0,$$

i.e.

$$\Lambda = L_0 * \mu.$$

This ends Remark 3.

Now, let f be multiplicative and $|f| \leq 1$. We define a completely multiplicative function $\tilde{f}$ by $\tilde{f}(p) = f(p)$ for all primes p.

Then $f = h * \tilde{f}$ where the multiplicative function h is given by

$$h(p^k) = \begin{cases} 0 & \text{if } k = 1 \\ f(p^k) - f(p)f(p^{k-1}) & \text{if } k \geq 2. \end{cases} \tag{12}$$

This definition corresponds to the equations

$$F(s) = H(s)\tilde{F}(s) = \sum_{n>1}^{\infty} \frac{h(n)}{n^s} \sum_{n>1}^{\infty} \frac{\tilde{f}(n)}{n^s}$$

and

$$F(s) = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}}\right) = \prod_p \left(1 + \sum_{k=2}^{\infty} \frac{h(p^k)}{p^{ks}}\right) \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}. \tag{13}$$

Now, let $M(x) = \sum_{n \le x} (f(n) - A)$ with some constant $A = A_x \in \mathbb{C}$, i.e.

$$M = \mathbf{1} * (f - A\,\mathbf{1}_0)\,.$$

Then

$$LM = L * (f - A\,\mathbf{1}_0) + \mathbf{1} * L_0(f - A\,\mathbf{1}_0).$$

Putting $R_1 = L * (f - A\,\mathbf{1}_0)$ leads to

$$LM = \mathbf{1} * L_0(f - A\,\mathbf{1}_0) + R_1. \tag{14}$$

Since

$$L_0 f = L_0 h * \tilde{f} + h * L_0 \tilde{f} = L_0 h * \tilde{f} + h * \big(\Lambda\tilde{f} * \tilde{f}\big) = L_0 h * \tilde{f} + f * \Lambda\tilde{f}$$

we conclude

$$\mathbf{1} * L_0 f = (\mathbf{1} * f) * \Lambda\tilde{f} + R_2$$

where, because of (12)

$$R_2 = O\bigg(x \sum_{n \le x} n^{-1} |h(n)| \log n\bigg) = O(x).$$

Observing $\Lambda = \Lambda\tilde{f} + \Lambda(\mathbf{1}_0 - \tilde{f})$ gives

$$\begin{aligned} \mathbf{1} * L_0(f - A\mathbf{1}_0) &= \mathbf{1} * f * \Lambda\tilde{f} - A(\mathbf{1} * \mathbf{1}_0 * \Lambda\tilde{f}) - A\big(\mathbf{1} * \mathbf{1}_0 * \Lambda(\mathbf{1}_0 - \tilde{f})\big) + R_2 \\ &= M * \Lambda\tilde{f} - A(\mathbf{1} * \mathbf{1}_0) * \Lambda(\mathbf{1}_0 - \tilde{f}) + R_2. \end{aligned} \tag{15}$$

Collecting (14) and (15) shows

$$LM = M * \Lambda\tilde{f} + R_1 + R_2 + R_3 \tag{16}$$

with

$$R_1(x) = O(x), \quad R_2(x) = O(x)$$

and

$$|R_3| \le |A|\big(\mathbf{1} * \mathbf{1}_0 * \Lambda|\mathbf{1}_0 - \tilde{f}|\big).$$

We multiply (16) with L and obtain

$$L^2 M = \big(LM * \Lambda\tilde{f}\big) + M * L_0\Lambda\tilde{f} + L(R_1 + R_2 + R_3). \tag{17}$$

Then, by substituting (16) in (17) we arrive at

$$\begin{aligned} L^2 M &= \big(M * \Lambda\tilde{f} + R_1 + R_2 + R_3\big) * \Lambda\tilde{f} + M * L_0\Lambda\tilde{f} + L(R_1 + R_2 + R_3) \\ &= M * (\Lambda\tilde{f} * \Lambda\tilde{f} + L_0\,\Lambda\tilde{f}) + R_4 \end{aligned} \tag{18}$$

where

$$R_4(x) = O(x\log x) + O\bigg(x(\log x)|A|\sum_{p\le x}\frac{\log p}{p}\,|1 - f(p)|\bigg)$$

since

$$\big|R_3 * \Lambda\tilde{f}\big| \le |A|(\mathbf{1} * \mathbf{1}_0) * \big(\Lambda * \Lambda|1 - \tilde{f}|\big).$$

Thus by (18), since $|\tilde{f}| \le 1$

$$\begin{aligned} |M(x)|\log^2 x \le &\sum_{n\le x}\left|M\left(\frac{x}{n}\right)\right|\{(\Lambda * \Lambda)(n) + \Lambda(n)\log n\} \\ &+ O(x|A_x)(\log x)\sum_{p\le x}\frac{\log p}{p}\left|1 - \tilde{f}(p)\right| + O(x\log x). \end{aligned}$$

Selberg's formula

$$\sum_{n\le x}(\Lambda * \Lambda)(n) + \sum_{n\le x}\Lambda(n)\log n = 2x\log x + O(x)$$

and partial summation (see, for example, Lemma 3.1 of [12]) yield the assertion of Theorem 1.

3. Proof of Theorem 2

Observing

$$\int_1^x \frac{|M(u)|}{u^2}du \le \left(\int_1^x \frac{|M(u)|^2}{u^3}du\right)^{\frac{1}{2}}\left(\int_1^x \frac{du}{u}\right)^{\frac{1}{2}}$$

shows

$$\frac{|M(x)|}{x} \ll \left(\frac{1}{\log x}\int_1^x \frac{|M(u)|^2}{u^3}du\right)^{\frac{1}{2}}. \tag{19}$$

Since $1 \le u^{2/\log x} \le e^2$ for $1 \le u \le x$ we get

$$\int_1^x \frac{|M(u)|^2}{u^3}du \ll \int_1^x \frac{|M(u)|^2}{u^{3+2\alpha}}du \le \int_1^\infty \frac{|M(u)|^2}{u^{3+2\alpha}}du$$

where $\alpha = \frac{1}{\log x}$. Substituting $u = e^\omega$ and using Parseval's Formula (2) gives

$$\frac{1}{(\log x)^{\frac{1}{2}}}\int_1^x \frac{|M(u)|}{u^2}du \ll \left(\int_0^\infty \frac{|M(e^\omega)|^2}{e^{2\omega(1+\alpha)}}dw\right)^{\frac{1}{2}} = \left(\frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{F(s)}{s}\right|^2 ds\right)^{\frac{1}{2}} \tag{20}$$

where $s = 1 + \frac{1}{\log x} + it$. This gives (i).

Putting $K(u) = \sum_{n \le u} f(n)\log n$ partial summation shows that for $u \ge 3$

$$M(u) = \frac{K(u)}{\log u} + \int_2^u \frac{K(t)}{t(\log t)^2}dt \tag{21}$$

so that

$$\begin{aligned}\int_1^x \frac{|M(u)|}{u^2}du &\le \int_2^x \frac{|K(u)|}{u^2\log u}du + \int_2^x \frac{|K(t)|}{t(\log t)^2}\int_2^x \frac{du}{u^2}dt \\ &\le \left(1 + \frac{1}{\log 2}\right)\int_2^x \frac{|K(u)|}{u^2\log u}du. \end{aligned} \tag{22}$$

Observing

$$\int_2^x \frac{|K(u)|}{u^2\log u}du \le \left(\int_2^x \frac{|K(u)|^2}{u^3}du\right)^{\frac{1}{2}}\left(\int_2^x \frac{du}{u\log^2 u}\right)^{\frac{1}{2}} \tag{23}$$

we arrive in the same way as above at

$$\int_1^x \frac{|M(u)|}{u^2}du \ll \left(\int_0^\infty \frac{|K(e^\omega)|^2}{e^{2\omega(1+\alpha)}}d\omega\right)^{1/2} = \left(\frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{F'(s)}{s}\right|^2 ds\right)^{1/2} \tag{24}$$

where $s = 1 + \frac{1}{\log x} + it$. This proves (ii).

For the proof of inequality (iii) we modify the estimate of the integral on the left hand side in (23) and arrive at

$$\int_2^x \frac{|K(u)|}{u^2\log u}du \ll \int_2^x \frac{|K(u)|}{u^2\log u}\int_u^{u^2} \frac{dv}{v\log v} \ll \int_2^{x^2} \frac{dv}{v\log v}\int_{v^{1/2}}^v \frac{|K(u)|}{u^2\log u}du$$

$$\ll \int_2^{x^2} \frac{dv}{v \log^2 v} \int_2^v \frac{|K(u)|}{u^2} du.$$

The last integral we estimate as in (i) by

$$\int_2^v \frac{|K(u)|}{u^2} du \ll (\log v)^{1/2} H_1 \left(\frac{1}{\log v} \right)$$

which yields

$$\int_2^x \frac{|K(u)|}{u^2 \log u} du \ll \int_2^{x^2} (\log v)^{1/2} H_1 \left(\frac{1}{\log v} \right) \frac{dv}{v \log^2 v}.$$

Substituting $y = \frac{1}{\log v}$ gives assertion (iii).

4. Some lemmas

Lemma 1. *Let f be a nonnegative multiplicative function, $0 \le f \le 1$. Then*

$$x^{-1} \sum_{n \le x} f(n) \ll \exp \left(\sum_{p \le x} \frac{f(p) - 1}{p} \right) \quad \textit{for all } x \ge 2.$$

Proof. Put $M = \mathbf{1} * f$. Then

$$LM = \mathbf{1} * L_0 f + L * f$$

which leads to

$$\begin{aligned}
\log x M(x) &= \sum_{n \le x} f(n) \log n + O\left(\sum_{n \le x} \log \frac{x}{n} \right) = \sum_{n \le x} f(n) \sum_{p^\alpha || n} \log p^\alpha + O(x) \\
&= \sum_{p^\alpha \le x} \log p^\alpha \sum_{\substack{n \le \frac{x}{p^\alpha} \\ p \nmid n}} f(n) f(p^\alpha) + O(x) \\
&\le \sum_{n \le x} f(n) \sum_{p^\alpha \le \frac{x}{n}} \log p^\alpha + O(x) = O\left(x \sum_{n \le x} \frac{f(n)}{n} \right).
\end{aligned}$$

Since

$$\sum_{n \le x} \frac{f(n)}{n} \le \prod_{p \le x} \left(1 + \frac{f(p)}{p} + \frac{f(p^\alpha)}{p^2} + \dots \right) \ll \exp \left(\sum_{p \le x} \frac{f(p)}{p} \right)$$

the assertation of Lemma 2 follows immediately. □

From the representation (13) we deduce

$$\frac{F(s)}{\zeta(s)} = \exp\left(\sum_p \frac{f(p)-1}{p^s} + w(s)\right) \tag{25}$$

where $w(s)$ is holomorphic in a neighbourhood of $s=1$ so that there $w(s)-w(1) = O(|s-1|)$. For general values $F(s)$ we prove

Lemma 2. *For all $t \in \mathbb{R}$ and $\sigma > 1$*

$$|F(\sigma+it)| \ll (\zeta(\sigma))^{\frac{3}{4}} \, |\zeta(\sigma+2it)|^{\frac{1}{4}} . \tag{26}$$

Proof. We have

$$4\left(1 - f(p)\operatorname{Re} p^{it}\right) \geq 4\left(1 - \left|\operatorname{Re}\ p^{it}\right|\right) \geq 1 - \operatorname{Re} p^{2it}$$

since

$$1 - \operatorname{Re}(z_1 z_2) \leq 2\left(1 - \operatorname{Re} z_1\right) + 2\left(1 - \operatorname{Re} z_2\right)$$

holds for all complex numbers z_1, z_2 with $|z_1| \leq 1$, $|z_2| \leq 1$.

Observing

$$|F(s)| \asymp \exp\left(\sum_p \frac{f(p)\operatorname{Re} p^{-it}}{p^\sigma}\right) \quad \text{and} \quad |\zeta(s)| \asymp \exp\left(\sum_p \frac{\operatorname{Re} p^{-it}}{p^\sigma}\right)$$

we obtain the inequality (26). □

Lemma 3. *Let $s = \sigma + it$. Then*

$$|\zeta(s)| \ll \frac{1}{|s-1|} \quad \textit{if } |t| \leq 3$$

and

$$|\zeta(s)| \ll \log|t| \quad \textit{if } |t| > 3$$

uniformly in $\sigma > 1$.

Proof. Partial summation shows

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = s\int_1^\infty \frac{[u]}{u^{s+1}} du = \frac{1}{s-1} + 1 + s\int_1^\infty \frac{[u]-u}{u^{s+1}} du$$

which obviously implies $\zeta(s) = O\left(\frac{1}{|s-1|}\right)$ for $|t| \leq 3$.

In the same manner we conclude for every $\sigma > 1$ and positive integer N

$$\zeta(s) - \sum_{n=1}^{N} n^{-s} = \frac{N^{1-s}}{s-1} + s\int_N^\infty \frac{[u]-u}{u^{s+1}}\,du.$$

Hence

$$\begin{aligned}|\zeta(s)| &\le \sum_{n=1}^{N} n^{-1} + \frac{1}{|s-1|} + |s|\int_N^\infty \frac{du}{u^{1+\sigma}}\\ &\le \log N + \frac{1}{|s-1|} + \frac{|s|}{\sigma}N^{-\sigma} + \text{constant}\end{aligned}$$

and the desired result is obtained by choosing N suitably. □

We may assume that $f(p) = 1$ if $p > x$ since these values do not influence the sum $\sum_{n\le x} f(n)$. Then, choosing

$$A_x = \exp\left(\sum_{p\le x}\frac{f(p)-1}{p} + w(1)\right) \tag{27}$$

we obtain from (25)

$$\frac{F(s)}{A_x\,\zeta(s)} = \exp\left(\sum_{p\le x}(f(p)-1)\left(\frac{1}{p^s}-\frac{1}{p}\right) + O\big(|s-1|\big)\right) \tag{28}$$

if $\sigma > 1$.

Lemma 4. *Let $s = 1 + \frac{1}{\log x} + it$. Then*

$$\sum_{p\le x}|f(p)-1|\left|\frac{1}{p^s}-\frac{1}{p}\right| \ll \log(2+|s-1|\log x) \tag{29}$$

and, if $y(x) = x^{\delta_0(x)}$

$$\sum_{p\le y(x)}|f(p)-1|\left|\frac{1}{p^s}-\frac{1}{p}\right| \ll \log\left(2+\delta_0(x)(s-1)\log x\right). \tag{30}$$

Proof. We put $a = \exp\left(\frac{1}{|s-1|}\right)$ and get

$$\sum_{p\le a}\left|\frac{1}{p^s}-\frac{1}{p}\right| = \sum_{p\le a}\frac{1}{p}|1-\exp((1-s)\log p)| \ll \sum_{p\le a}|1-s|\frac{\log p}{p} \ll 1. \tag{31}$$

For those primes p in the range $a < p \le x$, if there are any, we estimate crudely, and then

$$\sum_{a<p\le x}\left|\frac{1}{p^s}-\frac{1}{p}\right| \le 2\sum_{a<p\le x}\frac{1}{p} = \log\frac{\log x}{\log a} + O(1) \ll \log(2+|s-1|\log x)$$

which proves (29). Then the upper bound (30) is obvious. □

5. Proof of Theorem 3

For the proof of assertion (i) we shall apply the inequality (iii) of Theorem 2. Thus we have to estimate

$$H_1^2(y) = \int_{-\infty}^{\infty} \left| \frac{F'(1+y+it)}{1+y+it} \right|^2 dt$$

for all $\frac{2}{\log x} \le y \le 1$.

For this purpose we divide the range of integration into three parts

$$\begin{aligned} I_1 &:= \{t \in \mathbb{R} : |t| \le K(\sigma - 1)\} \\ I_2 &:= \{t \in \mathbb{R} : K(\sigma - 1) < |t| \le K\} \\ I_3 &:= \{t \in \mathbb{R} : K < |t|\}. \end{aligned}$$

We employ the factorization ($s = 1 + y + it$)

$$F'(s) = \frac{F'(s)}{F(s)} \cdot F(s).$$

Thus

$$\int_{I_1} \left| \frac{F'(s)}{s} \right|^2 dt \le \sup_{t \in I_1} |F(s)|^2 \int_{-\infty}^{\infty} \left| \frac{F'(s)}{sF(s)} \right|^2 dt. \tag{32}$$

Define

$$L(u) := \sum_{n \le u} \Lambda(n) f(n).$$

Then we may apply Parseval's identity to obtain ($\sigma = 1 + y$)

$$\int_{-\infty}^{\infty} \left| \frac{F'(s)}{sF(s)} \right|^2 dt = 2\pi \int_0^{\infty} |L(e^w)|^2 e^{-2w\sigma} dw. \tag{33}$$

Since $L(u) = O(u)$ uniformly for $u \ge 1$ each of the integrals in the equation (33) can be estimated by

$$\ll \int_0^{\infty} e^{2w(1-\sigma)} dw \ll y^{-1}.$$

The last inequality together with (32) shows that

$$\int_{I_1} \left| \frac{F'(s)}{s} \right|^2 dt \ll \sup_{t \in I_1} |F(s)|^2 \cdot y^{-1} \ll \exp\left(-2 \sum_{p \le x} \frac{1 - f(p)}{p} \right) K^2 \cdot y^{-3}$$

since, by Lemma 4 and $t \in I_1$,

$$\begin{aligned}\frac{|F(s)|}{|\zeta(s)|} &\ll \exp\left(-\sum_{p\le x}\frac{1-f(p)}{p}\right)\exp\left(\left|\sum_{p\le x}(f(p)-1)\left(\frac{1}{p^s}-\frac{1}{p}\right)\right|\right)\\ &\ll \exp\left(-\sum_{p\le x}\frac{1-f(p)}{p}\right)K.\end{aligned}$$

Concerning the integral over I_2 we have by Lemma 2 and Lemma 3

$$\sup_{t\in I_2}|F(s)|^2 \ll K^{-1/2}y^{-2}+y^{-3/2}\log K$$

and thus

$$\int_{I_2}\left|\frac{F'(s)}{s}\right|^2 dt \ll K^{-1/2}y^{-3}+(\log K)y^{-5/2}.$$

Let us now handle the integral over I_3.
Again by Parseval's formula we get

$$\begin{aligned}\int_{I_2}\left|\frac{F'(s)}{s}\right|^2 dt &\ll \sum_{m\ge K}\frac{1}{m^2}\int_{|t-m|\le 1}|F'(s)|^2 dt\\ &= \sum_{m\ge K}\frac{1}{m^2}\int_{-1}^{+1}\frac{\left|\sum_{n=1}^{\infty}f(n)n^{imt}\log n n^{-s}\right|^2}{|s|^2}dt \ll K^{-1}y^{-3}.\end{aligned}$$

Choosing $K=\exp\left(\frac{4}{5}\sum_{p\le x}\frac{1-f(p)}{p}\right)$ we obtain assertion (i) of Theorem 3.

Let us now turn to the proof of assertion (ii). For this we make use of inequality (i) of Theorem 2. This means we have to estimate the integral

$$\int_{-\infty}^{\infty}\left|\frac{F(\sigma+it)-A_x\zeta(\sigma+it)}{\sigma+it}\right|^2 dt$$

where $\sigma = 1+\frac{1}{\log x}$ and A_x is defined by (27).

We again split the range of integration into three parts I_1, I_2 and I_3, respectively, as above.

For the integral over I_1 we use (17) together with Lemma 4 and conclude

$$F(s)-A_x\zeta(s) = O(|\zeta(s)|(\delta_0(x)|s-1|\log x+\delta_0(x)))+O(|s-1|)$$

and

$$\int_{I_1}\left|\frac{F(s)-A_x\zeta(s)}{s}\right|^2 dt \ll \left(\delta_0(x)K+\delta_0(x)+\frac{K+1}{\log x}\right)\log x.$$

Concerning the integral over I_2 we have, by Lemma 2 and Lemma 3

$$\int_{I_2} \left|\frac{F(s)}{s}\right|^2 dt = \max_{t\in I_2} |F(\sigma+it)|^{1/2} \int_{I_2} \frac{|F(s)|^{3/2}}{|s|^2} dt$$
$$\ll \frac{1}{K^{1/8}(\sigma-1)^{1/2}} \int_{I_2} \frac{|F(s)|^{3/2}}{|s|^2} dt.$$

It remains to estimate the last integral. We shall proceed as in [1] and [11]. In the halfplane $\sigma > 1$ e have

$$|F(s)|^{3/4} \asymp \left| \exp\left(\sum_p \frac{3}{4} f(p) p^{-s} \right) \right| \asymp \left| \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{\Omega(n)} f(n) n^{-s} \right|$$

and thus, by Lemma 1 and Parseval's equality

$$\int_{I_2} \frac{|F(s)|^{3/2}}{|s|^2} \ll \int_0^{\infty} \exp\left(-2 \sum_{p\le e^{\omega}} \frac{1}{4} p^{-1} \right) e^{2\omega(\sigma-1)} dw \ll \int_0^{\infty} w^{-1/2} e^{-2\omega(\sigma-1)} dw.$$

Collecting the estimates we conclude

$$\int_{I_2} \left|\frac{F(s)}{s}\right|^2 dt \ll K^{-1/8} \log x.$$

Then,

$$\int_{-\infty}^{\infty} \left| \frac{F(s) - A_x \zeta(s)}{s} \right|^2 dt \ll \left(\delta_0(x) K + \frac{1}{K^{1/8}} + \frac{1}{K} \right) \log x \ll H(\delta_0(x))^{1/9}$$

if we choose $K = (\delta_0(x))^{-8/9}$. This ends the proof of Theorem 3.

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