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Lightlike curves in Lorentz manifolds

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To the memory of Professor András Rapcsák

The purpose of the present paper is to initiate a general study of differential geometry of lightlike curves in Lorentz manifolds. First, we construct a complementary vector subbundle to the tangent bundle of a lightlike curve. Then in section 2 we obtain the Frenet equations with respect to a general Frenet field of frames and prove theorems of reduction of the codimension of a lightlike curve (Theorems 3, 4 and 5). According to the Theorem 5, any lightlike curve of a Minkowski space whose the seventh curvature vanishes, lies in a 5-dimensional plane. This is a surprisingly result and it might have applications in multi-dimensional physical theories. Finally, we prove an existence and uniqueness theorem for lightlike curves in Lorentz manifolds.

§1. A complementary vector subbundle to the tangent bundle of a lightlike curve

Let M be a real $(m+2)$-dimensional Lorentz manifold, i.e., in M there exists a semi-Riemannian metric g of index $\nu = 1$, (cf. O'Neill [6]). Suppose C is a differentiable curve in M locally given by

$$x^i = x^i(t), \quad t \in [a, b]\,. \tag{1.1}$$

In case the tangent vector field

$$\frac{d}{dt} = \left(\frac{dx^1}{dt}, \dots, \frac{dx^{m+2}}{dt}\right),$$

has a non-null length with respect to g we have a complete study of the geometry of C (cf. Spivak [7]). Surprisingly, though theory of curves is one of the intensively studied theory of differential geometry, till now we do not have a method of studying curves whose tangent vector field is lightlike, i.e., we have

$$g\left(\frac{d}{dt}, \frac{d}{dt}\right) = 0\,. \tag{1.2}$$

As far as we know, the results obtained on this class of curves refer to the case when the ambient space is one of the Minkowski spaces R_1^3 or R_1^4 (cf. Cartan [3], Castagnino [2], Bonnor [1], Graves [4], Ikawa [5]).

That is why, we consider as a need a general theory of such curves in a Lorentz manifold. The present paper is concerned with such a study and it might give more insights for a general study of lightlike submanifolds of semi-Riemannian manifolds.

We say that C is a *lightlike curve* in M if there exists a lightlike vector field ξ tangent to C, that is, we have

$$g(\xi, \xi) = 0\,. \tag{1.3}$$

Certainly, in this case there exists a diffrentiable function $k_0 \neq 0$ such that

$$\frac{d}{dt} = k_0 \xi\,, \tag{1.4}$$

and henceforth (1.2) and (1.3) are equivalent with each other. Denote by TC the tangent bundle of C and define as in case of nondegenerate curves

$$TC^\perp = \bigcup_{x \in C} TC_x^\perp; \quad TC_x^\perp = \{v \in T_xM,\ g(v, \xi_x) = 0\}\,. \tag{1.5}$$

Then $TC^\perp$ is a vector bundle over C whose fibres are $(m+1)$–dimensional and ξ is a differentiable section of $TC^\perp$. Thus TC is a 1-dimensional vector subbundle of $TC^\perp$. Suppose sC is a complementary vector subbundle to TC in $TC^\perp$, i.e., we have

$$TC^\perp = TC \perp sC\,, \tag{1.6}$$

where $\perp$ means orthogonal direct sum. It follows that sC is a non–degenerate m–dimensional vector subbundle of TM. Then denote by $sC^\perp$ the 2–dimensional complementary orthogonal vector subbundle to sC in TM, i.e., we have

$$TM = sC \perp sC^\perp\,. \tag{1.7}$$

Throughout the paper we denote by $F(C)$ the algebra of differentiable functions on C and by $\Gamma(E)$ the $F(C)$–module of differentiable sections of a vector bundle E over C. We use the same notation for any other vector bundle.

As in any theory of submanifolds appears as a necessity the construction of a complementary vector bundle of the tangent bundle of the submanifolds in the tangent bundle of the ambient space, we state first the following result.

Theorem 1. *Let C be a lightlike curve in a Lorentz manifold M. Then for a given vector subbundle sC as in* (1.6) *there exists a unique 1-dimensional vector subbundle nC of $sC^{\perp}$ such that on each neighbourhood of coordinates $\mathcal{U} \subset C$, for any $\xi \in \Gamma(TC_{|\mathcal{U}})$ there exists $N \in \Gamma(nC_{|\mathcal{U}})$ satisfying*

$$g(\xi, N) = 1\,, \tag{1.8}$$

and

$$g(N, N) = 0\,. \tag{1.9}$$

Proof. Since TC is a vector subbundle of $sC^{\perp}$ we may consider a complementary vector subbundle δ of TC in $sC^{\perp}$. For any $\xi \in \Gamma(TC_{|\mathcal{U}})$ there exists $V \in \Gamma(\delta_{|\mathcal{U}})$ such that $g(\xi, V) \neq 0$, otherwise TM would be degenerate with respect to g. Then it follows that any $N \in \Gamma(sC^{\perp}_{|\mathcal{U}})$ satisfying (1.8) and (1.9) is given by

$$N = \frac{1}{g(\xi, V)}\left\{V - \frac{g(V,V)}{2g(\xi, V)}\xi\right\}\,. \tag{1.10}$$

Moreover, it is easy to check that N depends neither on vector bundle δ nor or local section V. Hence for any $\xi \in \Gamma(TC_{|\mathcal{U}})$ there exists a unique vector field $N \in \Gamma(sC^{\perp}_{|\mathcal{U}})$ satisfying (1.8) and (1.9). Next, consider another neighborhood of cooriditanes $\mathcal{U}^* \subset C$ such that $\mathcal{U} \cap \mathcal{U}^* \neq \emptyset$. Then $\xi^* = f\xi$, where f is nowhere zero differentiable function on $\mathcal{U} \cap \mathcal{U}^*$, and by using (1.10) obtain $N^* = \frac{1}{f}N$. Therefore we obtain a unique 1-dimensional vector subbundle nC of $sC^{\perp}$ whose local sections N satisfy (1.8) and (1.9). The proof is complete.

Next we consider the vector bundle

$$NC = nC \perp sC\,, \tag{1.11}$$

which according to the proof of Theorem 1 is complementary to TC in TM, i.e., we have

$$TM = (TC \oplus nC) \perp sC\,, \tag{1.12}$$

where $\oplus$ means direct sum but not orthogonal. It is important to note that the induced metrics on $TC \oplus nC$ and sC are of index $\nu = 1$ (Lorentz metric) and $\nu = 0$ (Riemannian metric) respectively.

§2. The Frenet equations for a lightlike curve in a Lorentz manifold

Let C be a lightlike curve of an $(m+2)$–dimensional Lorentz manifold M and $\{\xi, N\}$ be the lightlike vector fields from Theorem 1. Suppose ∇

is the Levi–Civita connection on M. Then by using (1.3), (1.8) and (1.9) we obtain the following *general Frenet equations* for the lightlike curve C:

$$
\begin{aligned}
\nabla_\xi \xi &= A\xi + k_1 W_1 \\
\nabla_\xi N &= -AN + k_2 W_1 + k_3 W_2 \\
\nabla_\xi W_1 &= -k_2\xi - k_1 N + k_4 W_2 + k_5 W_3 \\
\nabla_\xi W_2 &= -k_3\xi - k_4 W_1 + k_6 W_3 + k_7 W_4 \\
\nabla_\xi W_3 &= -k_5 W_1 - k_6 W_2 + k_8 W_4 + k_9 W_5 \\
&\ \vdots \\
\nabla_\xi W_{m-2} &= -k_{2m-5} W_{m-4} - k_{2m-4} W_{m-3} + \\
&\quad + k_{2m-2} W_{m-1} + k_{2m-1} W_m \\
\nabla_\xi W_{m-1} &= -k_{2m-3} W_{m-3} - k_{2m-2} W_{m-2} + k_{2m} W_m \\
\nabla_\xi W_m &= -k_{2m-1} W_{m-2} - k_{2m} W_{m-1}\,,
\end{aligned}
\tag{2.1}
$$

where A and $\{k_1, \dots, k_{2m}\}$ are differentiable functions and $\{W_1, \dots, W_m\}$ is an orthonormal basis of $\Gamma(sC)$. We call

$$
\{\xi, N, W_1, \dots, W_m\}
\tag{2.2}
$$

the *lightlike Frenet field of frames* on M along C, and $\{k_1, \dots, k_{2m}\}$ the *curvature functions* of C. Then by using (1.3), (1.8) and (1.9) we obtain

$$
\begin{cases}
\bar{\xi} = f\xi; \quad \bar{N} = -\dfrac{f}{2}\displaystyle\sum_{\alpha=1}^{m}(c_\alpha)^2 + \dfrac{1}{f}N + \sum_{\alpha=1}^{m} c_\alpha W_\alpha \\
\bar{W}_\alpha = \displaystyle\sum_{\beta=1}^{m} a_{\alpha\beta}(W_\beta - a c_\beta \xi)\,,
\end{cases}
\tag{2.3}
$$

where $f \neq 0$, c_α and $a_{\alpha\beta}$ are differentiable functions, and for any point x on C, $[a_{\alpha\beta}(x)]$ is an element of the orthogonal group $O(m)$, and $\{\bar{\xi}, \bar{N}, \bar{W}_1, \dots, \bar{W}_m\}$ is another lightlike Frenet field of frames on M along C.

Next, from the first Frenet equation of (2.1) written for both Frenet fields of frames we obtain

$$
\begin{cases}
\bar{A} = \bar{k}_1 \displaystyle\sum_{\alpha=1}^{m} a_{1\alpha} c_\alpha + \xi(f) + fA; \quad \bar{k}_1 a_{11} = f^2 k_1 \\
\bar{k}_1 a_{12} = \cdots = \bar{k}_1 a_{1m} = 0\,.
\end{cases}
\tag{2.4}
$$

Suppose $k_1 = 0$ on C. Then $\bar{k}_1 = 0$ on C, otherwise from (2.4) obtain $a_{11} = a_{12} = \cdots = a_{1m} = 0$, which is a contradiction because $[a_{\alpha\beta}]$ is an orthogonal matrix. A lightlike curve C on which $k_1 = 0$ is called a *lightlike geodesic* of the Lorentz manifold M. In this case the first Frenet equation becomes

$$\nabla_{\bar{\xi}}\bar{\xi} = \bar{A}\bar{\xi}\,, \tag{2.5}$$

with respect to the lightlike Frenet field of frames $\{\bar{\xi}, \bar{N}, \bar{W}_1, \dots, \bar{W}_m\}$. Now, choose f from the first relation (2.3) as a solution of partial differential equation

$$\xi(f) + fA = 0\,. \tag{2.6}$$

Then $\bar{A} = 0$ and (2.5) becomes

$$\nabla_{\bar{\xi}}\bar{\xi} = 0\,. \tag{2.7}$$

Suppose $\dfrac{d}{dt} = \bar{\varepsilon}\bar{\xi}$ and consider u given by $\dfrac{du}{dt} = \bar{\varepsilon}$ as a new parameter on C, provided $\bar{\varepsilon} > 0$ on C. Then $\dfrac{d}{du} = \bar{\xi}$ and (2.7) becomes

$$\frac{d^2x^i}{du^2} + \Gamma_j{}^i{}_k \frac{dx^j}{du}\frac{dx^k}{du} = 0\,, \tag{2.8}$$

where $\Gamma_j{}^i{}_k$ are the Christoffel symbols induced by ∇. Then by using (2.8) we obtain

Theorem 2. *A lightlike curve C of R_1^{m+2} is a straight line if and only if $k_1 = 0$ on C.*

For the particular case $m = 2$, Theorem 2 is due to Bonnor [1]. We call u the *pseudo-arc* on C (cf. Vessiot [8]).

The above study enables us to suppose, from now on, $k_1 \neq 0$ at every point of the lightlike curve C. Then $\bar{k}_1 \neq 0$ and (2.4) becomes

$$\begin{cases} a_{11} = 1,\ a_{1\alpha} = a_{\alpha 1} = 0, \quad \alpha \in \{2, \dots, m\} \\ \bar{A} = f^2 k_1 c_1 + \xi(f) + fA; \quad \bar{k}_1 = f^2 k_1\,. \end{cases} \tag{2.9}$$

Remark 1. For the particular case $m = 2$, that is, for a lightlike curve C of a 4-dimensional Lorentz manifold the transformation of lightlike Frenet fileds of frames (2.3) becomes

$$\begin{cases} \bar{\xi} = f\xi;\ \bar{N} = -\frac{f}{2}((c_1)^2 + (c_2)^2)\xi + \frac{1}{f}N + c_1 W_1 + c_2 W_2\,, \\ \bar{W}_1 = W_1 - fc_1\xi; \quad \bar{W}_2 = W_2 - fc_2\xi\,. \end{cases} \tag{2.10}$$

According to the above remark we are further concerned with the study of relationships between $\{A, k_1, \ldots, k_{2m}\}$ and $\{\bar{A}, \bar{k}_1, \ldots, \bar{k}_{2m}\}$ with respect to the transformation of lightlike Frenet field of frames:

$$\text{(2.11)} \qquad \begin{cases} \bar{\xi} = f\xi; \quad \bar{N} = -\dfrac{f}{2}\displaystyle\sum_{\alpha=1}^{m}(c_\alpha)^2\xi + \dfrac{1}{f}N + \sum_{\alpha=1}^{m} c_\alpha W_\alpha \\ \bar{W}_\alpha = W_\alpha - f c_\alpha \xi, \quad \alpha \in \{1, \ldots, m\}\,. \end{cases}$$

Thus by direct calculations using (2.11) and the last $m+1$ equations in (2.1) for both lightlike Frenet field of frames we obtain

$$\text{(2.12)} \qquad \begin{cases} \bar{k}_2 = k_2 + fc_1A + \xi(fc_1) + \dfrac{f^2k_1}{2}((c_1)^2 - (c_2)^2 - (c_3)^2) - \\ \qquad - f(c_2k_4 + c_3k_5) \\ \bar{k}_3 = k_3 + fc_2A + \xi(fc_2) + f^2c_1c_2k_1 + fc_1k_4 - fc_3k_6 \\ \bar{k}_4 = f(k_4 + fc_2k_1); \; \bar{k}_5 = f(k_5 + fc_3k_1)\,, \\ \bar{k}_\alpha = fk_\alpha, \quad \alpha \in \{6, \ldots, m\}\,, \end{cases}$$

and

$$\text{(2.13)} \qquad \begin{gathered} fAc_3 + \xi(fc_3) + f^2c_1c_3k_3 + fc_1k_5 + fc_2k_6 = 0 \\ c_2k_7 + c_3k_8 = 0; \; c_3k_9 = 0; \; c_\alpha = 0, \; \alpha \in \{4, \ldots, m\}\,. \end{gathered}$$

Now we choose c_1 given by

$$\text{(2.14)} \qquad c_1 = -\frac{1}{f^2k_1}(\xi(f) + fA)\,.$$

Then from (2.9) it follows $\bar{A} = 0$. Therefore, we always may consider a lightlike Frenet field of frames $\left\{\dfrac{d}{du} = \xi, N, W_1, \ldots, W_m\right\}$ with respect to with the Frenet equations are given by

$$\text{(2.15)} \qquad \begin{aligned} \frac{D\xi}{Du} &= k_1W_1 \\ \frac{DN}{Du} &= k_2W_1 + k_3W_2 \\ \frac{DW_1}{Du} &= -k_2\xi - k_1N + k_4W_2 + k_5W_3 \\ \frac{DW_2}{Du} &= -k_3\xi - k_4W_1 + k_6W_3 + k_7W_4 \end{aligned}$$

$$\frac{DW_3}{Du} = -k_5 W_1 - k_6 W_2 + k_8 W_4 + k_9 W_5$$
$$\vdots$$
$$\frac{DW_{m-2}}{Du} = -k_{2m-5} W_{m-4} - k_{2m-4} W_{m-3} + \\ + k_{2m-2} W_{m-1} + k_{2m-1} W_m$$
$$\frac{DW_{m-1}}{Du} = -k_{2m-3} W_{m-3} - k_{2m-2} W_{m-2} + k_{2m} W_m$$
$$\frac{DW_m}{Du} = -k_{2m-1} W_{m-2} - k_{2m} W_{m-1}\,,$$

where $\frac{D}{Du} = \nabla_{\frac{d}{du}}$.

Theorem 3. *Let C be a lightlike curve of a $(m+2)$–dimensional $(m > 4)$ Lorentz manifold M with $k_1 \neq 0$ on C. Then with respect to the lightlike Frenet field of frames $\{\xi, N, W_1, \ldots, W_m\}$ we have $k_8 = k_9 = 0$ on C.*

Proof. We choose $c_3 = 1,\ c_2 = 0$ and f as a non-null solution of (2.6). Then from (2.14) we obtain $c_1 = 0$ and the assertion of the theorem follows from (2.13).

Theorem 4. *Let C be a lightlike curve of a Minkowski space R_1^{m+2} $(m > 3)$ with k_α, $\alpha \in \{1, \ldots, 2p-4\}$, $p \in \{3, \ldots, m\}$ nowhere zero and k_{2p-3}, k_{2p-2} and k_{2p-1} everywhere zero on C. Then C lies in some p–dimensional Minkowski space of R_1^{m+2}. In case k_{2m-1} and k_{2m} are everywhere zero, C lies in a Minkowski hyperplane of R_1^{m+2}.*

Proof. Let $\{\xi = \frac{d}{du}, N, W_1, \ldots, W_{p-2}\}$ be a part of a lightlike Frenet field of frames along C, and $\Delta(u) \subset T_{x(u)} R_1^{m+2}$ be the p–dimensional subspace spanned by $\{\xi(u), N(u), W_1(u), \ldots, W_{p-2}(u)\}$. All these subspace are parallel as p–dimensional planes of R_1^{m+2}. In order to prove this we first note that $\frac{DX}{Du}$ is just $X'(u)$ in R_1^{m+2} and by using (2.15) obtain

$$W_i'(u) = \sum_{j=1}^{p} A_{ij}(u) W_j(u), \quad i \in \{1, \ldots, p\}\,, \tag{2.16}$$

where $W_{p-1} = N$ and $W_p = \xi$. Suppose now C is given by the equations

$$x^i = x^i(u), \quad u \in [a, b]\,,$$

and V is a constant vector field on C such that

$$g(W_i(a),V)=0,\quad i\in\{1,\dots p\}\,. \tag{2.17}$$

Then by using (2.16) we obtain the system

$$\frac{d}{du}(g(W_i(u),V)=g(W_i'(u),V)=\sum_{j=1}^{p}A_{ij}(u)g(W_j(u),V)\,, \tag{2.18}$$

with initial conditions (2.17). By the uniqueness of solutions of (2.18) we infer $g(W_i(u),V)=0$ for all u. Hence all p–planes $\Delta(u)$ are parallel with $\Delta(a)$. The proof is complete by the following general result.

Proposition 1. (Spivak [7], p.39). *Let $C : x^i = x^i(u)$, $u\in[a,b]$ be an immersed curve of R^{m+2} such that $\dfrac{dx^i}{du}\in\Delta(u)$ for all u, where $\Delta(u)$ are parallel p–dimensional planes of R^{m+2}. Then C is a curve in some p–dimensional plane of R^{m+2}.*

Remark 2. The p–dimensional plane H wherein C lies is a Minkowski space since both linear independent lightlike vector fields ξ and N belong to H. The second assertion of the theorem follows in a similar way as the first one.

From Theorems 3 and 4 we obtain the following surprising result.

Theorem 5. *Let C be a lightlike curve of a Minkowski space R_1^{m+2} $(m>4)$ with $\{k_1,\dots,k_6\}$ nowhere zero and k_7 everywhere zero on C. Then C lies in a 5-dimensional plane of R_1^{m+2}.*

§3. The fundamental existence and uniqueness theorem for lightlike curves

Let M be a $(m+2)$–dimensional Lorentz manifold. In the previous sections we have seen that the lightlike Frenet field of frames $\{\xi,N,W_1,\dots,W_m\}$ constructed along a lightlike curve is *quasi-orthonormal* (cf. Vrănceanu–Roşca [9]), that is, $\{W_1,\dots,W_m\}$ is an orthonormal basis and ξ and N are lightlike vector fields satisfying (1.8).

Consider R_1^{m+2} with the Lorentz metric

$$g(x,y)=\sum_{\alpha=1}^{m+1}x^\alpha y^\alpha - x^{m+2}y^{m+2}\,. \tag{3.1}$$

Then we define the quasi-orthonormal basis

$$(3.2)\qquad \begin{cases} \overset{\circ}{W}_1 = (1,0,\dots,0),\dots,\overset{\circ}{W}_m = (0,\dots,1,0,0)\,, \\ \overset{\circ}{W}_{m+1} = (0,\dots,\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}),\ \overset{\circ}{W}_{m+2} = (0,\dots,\dfrac{1}{\sqrt{2}},-\dfrac{1}{\sqrt{2}}) \end{cases}$$

that is we have

$$(3.3)\qquad \begin{cases} g(\overset{\circ}{W}_\alpha,\overset{\circ}{W}_\beta) = \delta_{\alpha,\beta};\ g(\overset{\circ}{W}_{m+1},\overset{\circ}{W}_{m+1}) = g(\overset{\circ}{W}_{m+2},\overset{\circ}{W}_{m+2}) = 0 \\ g(\overset{\circ}{W}_{m+1},\overset{\circ}{W}_{m+2} = 1,\ \alpha,\beta\in\{1,\dots,m\}\,. \end{cases}$$

It is easy to see that

$$(3.4)\qquad \sum_{\alpha=1}^{m}\overset{\circ}{W}{}^i_\alpha\overset{\circ}{W}{}^j_\alpha + \overset{\circ}{W}{}^i_{m+1}\overset{\circ}{W}{}^j_{m+2} + \overset{\circ}{W}{}^j_{m+1}\overset{\circ}{W}{}^i_{m+2} = g^{ij}\,,$$

where we put

$$g^{ij} = \begin{cases} 1, & i=j\neq m+2 \\ -1, & i=j=m+2 \\ 0, & i\neq j\,. \end{cases}$$

Theorem 6. *Let M be a Lorentz manifold, let $k_1,\dots,k_{2m}\colon[-\varepsilon,\varepsilon]\to\mathbb{R}$ be everywhere continuous functions and let $\{\overset{\circ}{W}_1,\dots,\overset{\circ}{W}_{m+2}\}$ from (3.2) as a basis of $T_{x_0}M$. Then there exists a unique pseudo-arc parametrized lightlike curve $C : x^i = x^i(u)$, $u\in[-\varepsilon,\varepsilon]$, such that $x^i(0)=x^i_0$, whose curvature functions are $k_1,\dots k_{2m}$ and whose lightlike Frenet field of frames $\{\xi,N,W_1,\dots,W_m\}$ satisfies*

$$\xi(0)=\overset{\circ}{W}_{m+1},\ N(0)=\overset{\circ}{W}_{m+2},\ W_\alpha(0)=\overset{\circ}{W}_\alpha,\quad \alpha\in\{1,\dots,m\}\,.$$

Proof. First we note that without loss of generality we may suppose M is the Minkowski space R^{m+2}_1. Then consider the system of differential equations

$$(3.5)\qquad \begin{aligned} W'_{m+1}(u) &= k_1W_1 \\ W'_{m+2}(u) &= k_2W_1 + k_3W_2 \\ W'_1(u) &= -k_2W_{m+1} - k_1W_{m+2} + k_4W_2 + k_5W_3 \\ &\;\;\vdots \\ W'_{m-2}(u) &= -k_{2m-5}W_{m-4} - k_{2m-4}W_{m-3} + k_{2m-2}W_{m-1} + k_{2m-1}W_m \end{aligned}$$

$$W'_{m-1}(u) = -k_{2m-3}W_{m-3} - k_{2m-2}W_{m-2} + k_{2m}W_m$$
$$W'_m(u) = -k_{2m-1}W_{m-2} - k_{2m}W_{m-1}\,,$$

and based on a well known result on the existence and uniqueness of its solutions, there exists a unique solution $(W_1, \ldots, W_{m+2})$ satisfying initial conditions $W_\alpha(0) = \overset{\circ}{W}_\alpha$, $\alpha \in \{1, \ldots, m+2\}$. Now we claim that $\{W_1(u), \ldots, W_{m+2}(u)\}$ is a quasi-orthonormal basis for any $u \in [-\varepsilon, \varepsilon]$. To this end, by direct calculations, using (3.4) we obtain

$$\frac{d}{du}\left(\sum_{\alpha=1}^{m} W^i_\alpha W^j_\alpha + W^i_{m+1}W^j_{m+2} + W^j_{m+1}W^i_{m+2}\right) = 0\,. \tag{3.6}$$

As for $u = 0$ we have (3.4), from (3.6) it follows

$$\sum_{\alpha=1}^{m} W^i_\alpha(u)W^j_\alpha(u) + W^i_{m+1}(u)W^j_{m+2}(u) + W^j_{m+1}(u)W^i_{m+2}(u) = g^{ij} \tag{3.7}$$

Further we construct the field of frames

$$\begin{cases} V_{m+1} = \dfrac{1}{\sqrt{2}}\left(W_{m+1} + W_{m+2}\right); \ V_{m+2} = \dfrac{1}{\sqrt{2}}\left(W_{m+1} - W_{m+2}\right) \\ V_\alpha = W_\alpha, \ \alpha \in \{1, \ldots, m\}\,. \end{cases} \tag{3.8}$$

Then (3.7) becomes

$$\sum_{\alpha=1}^{m+1} V^i_\alpha(u)V^j_\alpha(u) - V^i_{m+2}(u)V^j_{m+2}(u) = g^{ij}\,. \tag{3.9}$$

Following Bonnor [1], we define the matrix $[b^{ij}]$ as follows

$$\begin{cases} b^{\alpha\beta} = V^\alpha_\beta, \ \alpha, \beta \in \{1, \ldots, m+1\}; \ b^{\alpha m+2} = -\sqrt{-1}\,V^\alpha_{m+2} \\ b^{(m+2)\alpha} = \sqrt{-1}\,V^{m+2}_\alpha; \ b^{(m+2)(m+2)} = V^{m+2}_{m+2}\,. \end{cases} \tag{3.10}$$

It is easy to check that $[b^{ij}]$ is an orthogonal matrix. This implies $\{V_1, \ldots, V_{m+2}\}$ is an orthonormal basis with respect to the metric (3.1) of R^{m+2}_1. Hence $\{W_1, \ldots, W_{m+2}\}$ is a quasi-orthonormal basis for any $u \in [-\varepsilon, \varepsilon]$. The lightlike curve C is obtained by integrating the system

$$\frac{dx^i}{du} = W^i_{m+1}(u)\,. \tag{3.11}$$

It follows that C is pseudo-arc parametrized with curvature functions $\{k_1, \ldots, k_{2m}\}$ with respect to the quasi-orthonormal field of frames $\{\xi = W_{m+1}, N = W_{m+2}, W_1, \ldots, W_m\}$. The proof is complete.

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