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Regularity theorem for a functional equation involving means

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Dedicated to Professor Zoltán Daróczy on the occasion of his 70th birthday

Abstract. We prove a result improving regularity of solutions of equation

$$\kappa x + (1-\kappa)y = \lambda\varphi^{-1}(\mu\varphi(x) + (1-\mu)\varphi(y)) + (1-\lambda)\psi^{-1}(\nu\psi(x) + (1-\nu)\psi(y)),$$

and leading to generalizations of some theorems established by D. Głazowska, W. Jarczyk, and J. Matkowski and by Z. Daróczy and Zs. Páles.

Given an interval $I \subset \mathbb{R}$, a continuous strictly monotonic function $\varphi : I \to \mathbb{R}$ and a real $\mu \in (0,1)$ we denote by A_μ^φ the quasi-arithmetic mean generated by φ and weighted by μ :

$$A_\mu^\varphi(x,y) = \varphi^{-1}(\mu\varphi(x) + (1-\mu)\varphi(y)).$$

In paper [5] D. Głazowska, W. Jarczyk, and J. Matkowski found all the quasi-arithmetic means $A_{1/2}^\varphi$ and $A_{1/2}^\psi$ such that the classical arithmetic mean A is an affine combination of them:

$$A = \lambda A_{1/2}^\varphi + (1-\lambda)A_{1/2}^\psi,$$

assuming that the generators φ, ψ are twice continuously differentiable. In other words, they determined all functions $\varphi, \psi : I \to \mathbb{R}$ of class $\mathcal{C}^2$ satisfying the

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functional equation

$$\frac{x+y}{2}=\lambda\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)+(1-\lambda)\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right). \tag{1}$$

The result of [5] was generalized by Z. Daróczy and Zs. Páles [3, Theorem 6], where the equation

$$\begin{aligned}\mu x+(1-\mu)y \\ =\lambda\varphi^{-1}\left(\mu\varphi(x)+(1-\mu)\varphi(y)\right)+(1-\lambda)\psi^{-1}\left(\mu\psi(x)+(1-\mu)\psi(y)\right)\end{aligned} \tag{2}$$

with given $\lambda\in\mathbb{R}\setminus\{0,1\}$ and $\mu\in(0,1)$ was solved in the class $\mathcal{C}^1$.

In the present paper we prove the theorem below which allows to generalize the results of both papers [5] and [3]. It shows that continuous functions satisfying the equation

$$\begin{aligned}\kappa x+(1-\kappa)y \\ =\lambda\varphi^{-1}\left(\mu\varphi(x)+(1-\mu)\varphi(y)\right)+(1-\lambda)\psi^{-1}\left(\nu\psi(x)+(1-\nu)\psi(y)\right),\end{aligned} \tag{3}$$

extending both of (1) and (2), are locally of much higher regularity. The Theorem provides a positive answer to a question posed recently by Z. Daróczy [1]. Results improving regularity of solutions of functional equations have a vast literature (cf. book [6] by A. Járai and the bibliography therein). Some of them will be used below.

The main result of this paper is the following regularity theorem concerning functional equation (3).

Theorem. *Let $I\subset\mathbb{R}$ be a non-trivial interval, $\kappa,\lambda\in\mathbb{R}\setminus\{0,1\}$ and let $\mu,\nu\in(0,1)$. If $\varphi,\psi:I\to\mathbb{R}$ are continuous strictly monotonic functions and the pair (φ,ψ) satisfies equation (3), then there exists a non-trivial interval $I_0\subset I$ such that $\varphi|_{I_0},\psi|_{I_0}$ are infinitely many times differentiable and $\varphi'(x)\neq 0$, $\psi'(x)\neq 0$ for every $x\in I_0$.*

In the proof we shall apply a modification of the method presented in [7]. In particular, we need the following result obtained by Zs. Páles (see [9, Corollary 6 and Example 2]), as well as Lemma 2 which was proved in [7]. The latter is also a consequence of L. Székelyhidi's results [10] (see also [2], [8]).

Lemma 1. *Let $J\subset\mathbb{R}$ be an open interval, $c\in(0,\infty)$, $\mu\in(0,1)$, and let $f:J\to\mathbb{R}$ be a strictly increasing function such that*

$$J\ni s\mapsto f(s)-cf\left(\mu s+(1-\mu)t\right)$$

is strictly monotonic for every $t \in J$. Then for every $s_0 \in J$ there exist numbers $\delta \in (0, \infty)$ and $K, L \in (0, \infty)$ such that $(s_0 - \delta, s_0 + \delta) \subset J$ and

$$K \leq \frac{f(s) - f(t)}{s - t} \leq L$$

for every $s, t \in (s_0 - \delta, s_0 + \delta)$, $s \neq t$.

Lemma 2. *Let $J \subset \mathbb{R}$ be an interval and let $\mu \in (0, 1)$, $\vartheta \in \mathbb{R}$. If $f : J \to \mathbb{R}$ satisfies*

$$f(\mu s + (1 - \mu)t) = \vartheta f(s) + (1 - \vartheta)f(t) \tag{4}$$

for all $s, t \in J$, then there exist an additive function $a : \mathbb{R} \to \mathbb{R}$ and a real b such that

$$f(s) = a(s) + b, \quad s \in J.$$

At first we prove the following fact.

Lemma 3. *Let $J \subset \mathbb{R}$ be an open interval, $\kappa, \lambda \in \mathbb{R} \setminus \{0, 1\}$, $\mu, \nu \in (0, 1)$, and let $f, g : J \to (0, \infty)$ satisfy the equation*

$$\begin{aligned} f(\mu s + (1 - \mu)t)[\kappa(1 - \nu)g(t) - (1 - \kappa)\nu g(s)] \\ = \lambda\mu(1 - \nu)f(s)g(t) - \lambda(1 - \mu)\nu f(t)g(s). \end{aligned} \tag{5}$$

If f is Lebesgue measurable and g is of the first Baire class, then f and g are infinitely many times differentiable on a non-trivial subinterval of J.

Proof. Putting $s = t$ in (5) it is easy to observe that

$$\kappa = \lambda\mu + (1 - \lambda)\nu. \tag{6}$$

At first assume that f is constant on a non-trivial subinterval of J. Then, by equation (5), we have

$$[(1 - \kappa) - \lambda(1 - \mu)]\nu g(s) = [\kappa - \lambda\mu](1 - \nu)g(t)$$

for s, t from the same subinterval. Hence, by (6), also g is constant there.

Now assume that g is constant on a non-trivial interval $J_0 \subset J$. Then, by (5), we have

$$\lambda\mu(1 - \nu)f(s) - \lambda(1 - \mu)\nu f(t) = [\kappa(1 - \nu) - (1 - \kappa)\nu]f(\mu s + (1 - \mu)t)$$

for all $s, t \in J_0$. Using (6) we can rewrite the above condition as

$$\mu(1 - \nu)f(s) - (1 - \mu)\nu f(t) = (\mu - \nu)f(\mu s + (1 - \mu)t), \quad s, t \in J_0. \tag{7}$$

If $\mu = \nu$ then, by (7), f is constant on J_0. Now we assume that $\mu \neq \nu$. Then (7) is equivalent to the condition

$$f(\mu s + (1-\mu)t) = \frac{\mu(1-\nu)}{\mu-\nu} f(s) - \frac{(1-\mu)\nu}{\mu-\nu} f(t), \quad s, t \in J_0.$$

Let $\vartheta := \frac{\mu(1-\nu)}{\mu-\nu}$. Then

$$f(\mu s + (1-\mu)t) = \vartheta f(s) + (1-\vartheta) f(t), \quad s, t \in J_0.$$

Applying Lemma 2 we obtain that there exist additive function $a : \mathbb{R} \to \mathbb{R}$ and number $b \in \mathbb{R}$ such that

$$f(s) = a(s) + b, \quad s \in J_0.$$

Thus, as f is Lebesgue measurable, it is continuous.

From that place we assume that neither f, nor g is constant on a non-trivial subinterval of J. Let

$$C(g) := \{v \in J : g \text{ is continuous at } v\}.$$

As g is of the first Baire class, $C(g)$ is a dense G_δ subset of J. We show that there exist $s_0, t_0 \in C(g)$, $s_0 \neq t_0$, such that

$$(1-\kappa)\nu g(s_0) \neq \kappa(1-\nu) g(t_0). \tag{8}$$

Suppose on the contrary that

$$(1-\kappa)\nu g(s) = \kappa(1-\nu) g(t)$$

for all different $s, t \in C(g)$. Then g is constant on $C(g)$, i.e. there exists a positive k such that

$$g(t) = k, \quad t \in C(g). \tag{9}$$

Therefore $(1-\kappa)\nu = \kappa(1-\nu)$, whence $\kappa = \nu$ and, by (6), $\mu = \nu$. Now equation (5) can be rewritten in the form

$$f(\mu s + (1-\mu)t)[g(t) - g(s)] = \lambda[f(s)g(t) - f(t)g(s)]. \tag{10}$$

Thus, by (9),

$$\lambda k(f(s) - f(t)) = 0, \quad s, t \in C(g),$$

whence f is constant on $C(g)$, i.e. there exists a positive l such that $f(t) = l$ for every $t \in C(g)$.

If there existed an $s_0 \in J$ such that $\mu s_0 + (1-\mu)t \in J \setminus C(g)$ for every $t \in C(g)$, then $C(g)$ would be homeomorphic with a subset of $J \setminus C(g)$. This, however, is impossible, as $C(g)$ is a dense G_δ subset of J and, consequently, $J \setminus C(g)$ is of the first Baire category. Therefore, for every $s \in J$ there exists a $t \in C(g)$ such that $\mu s + (1-\mu)t \in C(g)$. Now, if $s \in J$ and $t \in C(g)$ are such that $\mu s + (1-\mu)t \in C(g)$, then, by (10), we have

$$l[k - g(s)] = \lambda[kf(s) - lg(s)].$$

Hence

$$f(s) = \frac{kl - l(1-\lambda)g(s)}{k\lambda}, \quad s \in J.$$

Using again (10) we obtain

$$\begin{aligned}\frac{kl - l(1-\lambda)g(\mu s + (1-\mu)t)}{k\lambda}&[g(t) - g(s)] \\ &= \lambda\left(\frac{kl - l(1-\lambda)g(s)}{k\lambda}g(t) - \frac{kl - l(1-\lambda)g(t)}{k\lambda}g(s)\right), \quad s,\, t \in J,\end{aligned}$$

which, after some calculations, yields

$$[g(t) - g(s)][k - g(\mu s + (1-\mu)t)] = 0, \quad s, t \in J. \tag{11}$$

Since g is not constant on J, there exists a $v_0 \in J$ such that $m := g(v_0) \neq k$. Take arbitrary $v \in J$ and $\varepsilon > 0$ with $(v - \varepsilon, v + \varepsilon) \subset J$. As g is not constant on intervals, there exists an $s \in (v - \varepsilon, v + \varepsilon)$ such that

$$g(\mu s + (1-\mu)v_0) \neq k.$$

By (11) we have $g(s) = g(v_0) = m$. Therefore, in every neighbourhood of v there exists an s with $g(s) = m$ and, since $C(g)$ is dense in J, a point u such that $g(u) = k \neq m$. Thus g is not continuous at v and, consequently, $C(g) = \emptyset$, which is impossible. This proves the existence of different $s_0, t_0 \in C(g)$ satisfying (8).

According to (8) there exist open intervals U, V containing s_0, t_0, respectively, and such that for every $s \in U$ and $t \in V$ we have $(1-\kappa)\nu g(s) \neq \kappa(1-\nu)g(t)$. Making use of (5) we obtain

$$f(\mu s + (1-\mu)t) = \frac{\lambda\mu(1-\nu)f(s)g(t) - \lambda(1-\mu)\nu f(t)g(s)}{\kappa(1-\nu)g(t) - (1-\kappa)\nu g(s)}, \quad s \in U,\ t \in V.$$

Now we are going to apply [6, Th. 8.6] by A. Járai. To this aim put $n = 4$, $T := J$, $Z = Z_1 = \cdots = Z_4 = Y := \mathbb{R}$, $X_1 = X_3 = A_1 = A_3 := U$ and $X_2 = X_4 = A_2 = A_4 := V$. Fix an $\eta > 0$ with $(t_0 - \eta, t_0 + \eta) \subset V$ and define

$$D := \Big\{(v, y) \subset J \times U : |v - (\mu s_0 + (1-\mu)t_0)| < \frac{\eta}{2}(1-\mu) \\ \text{and } |y - s_0| < \frac{\eta}{2}\left(\frac{1}{\mu} - 1\right)\Big\}$$

and

$$W := \big\{(v, y, z_1, z_2, z_3, z_4) \in D \times \mathbb{R}^4 : \kappa(1-\nu)z_4 \neq (1-\kappa)\nu z_3\big\}.$$

Put also $f := f$, $f_1 := f|_U$, $f_2 := f|_V$, $f_3 := g|_U$, $f_4 := g|_V$ and define $g_1, g_3 : D \to U$, $g_2, g_4 : D \to V$ by

$$g_1(v, y) = g_3(v, y) = y, \qquad g_2(v, y) = g_4(v, y) = \frac{v - \mu y}{1-\mu},$$

and $h : W \to \mathbb{R}$ by

$$h(v, y, z_1, z_2, z_3, z_4) = \frac{\lambda\mu(1-\nu)z_1 z_4 - \lambda\nu(1-\mu)z_2 z_3}{\kappa(1-\nu)z_4 - \nu(1-\kappa)z_3}.$$

Put $K := [s_0 - \delta, s_0 + \delta]$, where $0 < \delta < \eta\big(\frac{1}{\mu} - 1\big)$ and $[s_0 - \delta, s_0 + \delta] \subset U$. Making use of [6, Theorem 8.6], applied to the Lebesgue measure, we infer that f is continuous on the interval

$$J_f := \Big\{v \in J : |v - (\mu s_0 + (1-\mu)t_0)| < \frac{\eta}{2}(1-\mu)\Big\}.$$

Fix an $s^* \in J_f$. Since f is not constant on intervals, there is a $t^* \in J_f$ such that $f(\mu s^* + (1-\mu)t^*) \neq \frac{\lambda\mu}{\kappa} f(s^*)$. By the continuity of f at t^* we have $f(\mu s^* + (1-\mu)t) \neq \frac{\lambda\mu}{\kappa} f(s^*)$ for t's from a non-trivial interval $J_g \subset J_f$. Then, by (5),

$$g(t) = \frac{\nu}{1-\nu} \cdot \frac{(1-\kappa)f(\mu s^* + (1-\mu)t) - \lambda(1-\mu)f(t)}{\kappa f(\mu s^* + (1-\mu)t) - \lambda\mu f(s^*)} g(s^*), \quad t \in J_g,$$

and, consequently, g is continuous on J_g.

Now we show that f is almost everywhere (with respect to the Lebesgue measure) differentiable on some non-trivial subinterval of J_g provided $\mu \neq \nu$. In that case equation (5) can be rewritten in the form

$$\nu g(s)[(1-\kappa)f(\mu s + (1-\mu)t) - \lambda(1-\mu)f(t)] \\ = (1-\nu)g(t)[\kappa f(\mu s + (1-\mu)t) - \lambda\mu f(s)].$$

Interchanging s by t here we obtain

$$\nu g(t)[(1-\kappa)f(\mu t+(1-\mu)s)-\lambda(1-\mu)f(s)] = (1-\nu)g(s)[\kappa f(\mu t+(1-\mu)s)-\lambda\mu f(t)]$$

for every $s,t\in J$. Multiplying these equalities by sides we have

$$\begin{aligned}(1-\nu)^2 g(s)g(t)&[\kappa f(\mu s+(1-\mu)t)-\lambda\mu f(s)][\kappa f(\mu t+(1-\mu)s)-\lambda\mu f(t)]\\ &=\nu^2 g(s)g(t)[(1-\kappa)f(\mu t+(1-\mu)s)-\lambda(1-\mu)f(s))]\\ &\cdot[(1-\kappa)f(\mu s+(1-\mu)t)-\lambda(1-\mu)f(t)],\end{aligned}$$

whence, dividing it by positive $g(s)g(t)$, we get

$$\begin{aligned}(1-\nu)^2&[\kappa f(\mu s+(1-\mu)t)-\lambda\mu f(s)][\kappa f(\mu t+(1-\mu)s)-\lambda\mu f(t)]\\ &=\nu^2[(1-\kappa)f(\mu t+(1-\mu)s)-\lambda(1-\mu)f(s))]\\ &\cdot[(1-\kappa)f(\mu s+(1-\mu)t)-\lambda(1-\mu)f(t)]\end{aligned}\tag{12}$$

for every $s,t\in J$. Put

$$\begin{aligned}k(s,t):=\lambda(1-\mu)\nu^2&[(1-\kappa)f(\mu s+(1-\mu)t)-\lambda(1-\mu)f(t)]\\ &-\lambda\mu(1-\nu)^2[\kappa f(\mu t+(1-\mu)s)-\lambda\mu f(t)]\end{aligned}$$

for every $s,t\in J$. Fix an $s_0\in J_g$. Then

$$\begin{aligned}k(s_0,s_0)=\lambda(1-\mu)\nu^2&[(1-\kappa)f(s_0)-\lambda(1-\mu)f(s_0)]\\ &-\lambda\mu(1-\nu)^2[\kappa f(s_0)-\lambda\mu f(s_0)],\end{aligned}$$

which, after using (6) and making some calculations, gives

$$k(s_0,s_0)=\lambda(1-\lambda)\nu(1-\nu)(\nu-\mu)f(s_0).$$

Since $f(s_0)>0$, $\mu\neq 1$, $\nu\neq 1$, and $\mu\neq\nu$, we have $k(s_0,s_0)\neq 0$. Thus there exists an $\varepsilon>0$ such that $(s_0-\varepsilon,s_0+\varepsilon)\subset J_g$ and $k(s,t)\neq 0$ for all $s,t\in(s_0-\varepsilon,s_0+\varepsilon)$. Let $J_0:=(s_0-\varepsilon,s_0+\varepsilon)$. By (12) we get

$$\begin{aligned}f(s)=&\frac{(1-\kappa)\nu^2 f(\mu t+(1-\mu)s)[(1-\kappa)f(\mu s+(1-\mu)t)-\lambda(1-\mu)f(t)]}{k(s,t)}\\ &-\frac{\kappa(1-\nu)^2 f(\mu s+(1-\mu)t)[\kappa f(\mu t+(1-\mu)s)-\lambda\mu f(t)]}{k(s,t)}\end{aligned}$$

for every $s, t \in J_0$.

Put $s = k = 1$, $n = 3$, $Z := \mathbb{R}$, $T := J_0$, $Y := \mathbb{R}$, $D := {J_0}^2$, $C := [s_0 - \vartheta\varepsilon, s_0 + \vartheta\varepsilon]$ with $\vartheta :=\max\{\mu, 1-\mu\}$, $W := D \times G$, where

$$G := \{(w_1, w_2, w_3) \in \mathbb{R}^3 : (1-\mu)\nu^2[(1-\kappa)w_2 - \lambda(1-\mu)w_1] \neq \mu(1-\nu)^2[\kappa w_3 - \lambda\mu w_1]\}.$$

Define $f := f|_{J_0}$, $g_1, g_2, g_3 : D \to \mathbb{R}$, by

$$g_1(s,t) = t, \quad g_2(s,t) = \mu s + (1-\mu)t, \quad g_3(s,t) = \mu t + (1-\mu)s, \tag{13}$$

and $h : W \to \mathbb{R}$ by

$$h(s, t, w_1, w_2, w_3) := \frac{(1-\kappa)\nu^2 w_3[(1-\kappa)w_2 - \lambda(1-\mu)w_1] - \kappa(1-\nu)^2 w_2[\kappa w_3 - \lambda\mu w_1]}{\lambda(1-\mu)\nu^2[(1-\kappa)w_2 - \lambda(1-\mu)w_1] - \lambda\mu(1-\nu)^2[\kappa w_3 - \lambda\mu w_1]}. \tag{14}$$

Then, according to [6, Th. 11.6] by A. Járai, f is locally Lipschitzian on J_0, and thus, on account of [4, Th. 3.1.9] it is almost everywhere differentiable on J_0.

Now take any positive integer p. We prove that f and g are p times continuously differentiable on a non-trivial subinterval of J_0. At first assume that $\mu \neq \nu$. Then, as $k(s_o, s_0) \neq 0$, we have $(f(s_0), f(s_0), f(s_0)) \in G$. Since G is open, there is an open interval P such that $f(s_0) \in P$ and $P^3 \subset G$. Using the continuity of f we find such an open interval J_1 that $s_0 \in J_1 \subset J_0$ and $f(J_1) \subset P$. Now let $s = k = 1$, $n = 3$, $Z := \mathbb{R}$, $Z_1 = Z_2 = Z_3 := P$, $Y = T = X_1 = X_2 = X_3 := J_1$, $D := {J_1}^2$, and take $r_1 = r_2 = r_3 = 1$. Define $f = f_1 = f_2 = f_3 := f|_{J_1}$, $g_1, g_2, g_3 : D \to \mathbb{R}$ by (12) and $h : D \times Z_1 \times Z_2 \times Z_3 \to \mathbb{R}$ by (14). According to [6, Th. 14.2] f is continuously differentiable on J_1. Now, using [6, Th. 15.2] $p-1$ times, we get by induction that f is p times continuously differentiable on J_1. As J_1 does not depend on p, this means that f is infinitely many times differentiable on J_1. It follows from (5) that

$$[\kappa(1-\nu)f(\mu s_0 + (1-\mu)t) - \lambda\mu(1-\nu)f(s_0)]g(t) = [(1-\kappa)\nu f(\mu s_0 + (1-\mu)t) - \lambda(1-\mu)\nu f(t)]g(s_0), \quad t \in J_1. \tag{15}$$

As f is not constant on non-trivial intervals we can find a $t \in J_1$ such that

$$\kappa(1-\nu)f(\mu s_0 + (1-\mu)t) - \lambda\mu(1-\nu)f(s_0) \neq 0.$$

By the continuity of f this is true for t's running through a subinterval of J_1. Consequently, we can calculate $g(t)$ by (15) on that subinterval. Clearly, g is infitely many times differentiable there.

If $\mu = \nu$ then, by (6), we have $\kappa = \mu$, and thus equation (5) takes the form

$$f(\mu s + (1-\mu)t)[g(t) - g(s)] = \lambda[f(s)g(t) - f(t)g(s)].$$

Now it is enough to use [3, Th. 5 and 2]. □

The following fact seems to be of interest on its own.

Lemma 4. *Let $I \subset \mathbb{R}$ be an open interval, $\mu \in (0,1)$, and let $\varphi : I \to \mathbb{R}$ be a continuous strictly monotonic function. Assume that the mean A_μ^φ is differentiable with respect to one of the variables. Then φ is differentiable on a non-trivial interval and φ' does not vanish wherever it exists. If, in addition, the partial derivative of A_μ^φ is continuous in the other variable on a non-trivial interval, then φ is continuously differentiable on a non-trivial interval.*

Proof. Assume, for instance, that A_μ^φ is differentiable with respect to the first variable.

Since φ^{-1} is strictly monotonic, it is differentiable almost everywhere with respect to the Lebesgue measure. Fix any point $u_0 \in \varphi(I)$ of the differentiability of φ^{-1}. We prove that φ^{-1} is differentiable in the open interval $\mu u_0 + (1-\mu)\varphi(I)$ and the derivative of φ^{-1} does not vanish wherever it exists.

Take any point $v \in \varphi(I)$ and then any $u \in \varphi(I)\backslash\{u_0\}$ such that $\mu u + (1-\mu)v \in \mu u_0 + (1-\mu)\varphi(I)$. Then we have

$$\begin{aligned}
&\frac{\varphi^{-1}(\mu u + (1-\mu)v) - \varphi^{-1}(\mu u_0 + (1-\mu)v)}{(\mu u + (1-\mu)v) - (\mu u_0 + (1-\mu)v)} \\
&= \frac{A_\mu^\varphi(\varphi^{-1}(u), \varphi^{-1}(v)) - A_\mu^\varphi(\varphi^{-1}(u_0), \varphi^{-1}(v))}{\mu(u - u_0)} \\
&= \frac{1}{\mu} \cdot \frac{A_\mu^\varphi(\varphi^{-1}(u), \varphi^{-1}(v)) - A_\mu^\varphi(\varphi^{-1}(u_0), \varphi^{-1}(v))}{\varphi^{-1}(u) - \varphi^{-1}(u_0)} \cdot \frac{\varphi^{-1}(u) - \varphi^{-1}(u_0)}{u - u_0}.
\end{aligned}$$

Now letting u tend to u_0 we see that φ^{-1} is differentiable at $\mu u_0 + (1-\mu)v$ and

$$(\varphi^{-1})'(\mu u_0 + (1-\mu)v) = \frac{1}{\mu}\, \partial_1 A_\mu^\varphi(\varphi^{-1}(u_0), \varphi^{-1}(v)) \cdot (\varphi^{-1})'(u_0) \tag{16}$$

for all $v \in \varphi(I)$. If $(\varphi^{-1})'$vanished anywhere, then, by (16), it would be zero on a non-trivial interval, which is impossible as φ^{-1} is one-to-one. The desired properties of the function φ follows directly from what we have just proved about φ^{-1}.

The additional assertion is a direct consequence of formula (16). □

Proof of the Theorem. Replacing I with its interior we may assume that I is open. Without loss of generality we may also confine ourselves to the case of strictly increasing φ and ψ. Moreover, replacing, if necessary, κ with $1-\kappa$ (consequently, μ with $1-\mu$ and ν with $1-\nu$) and by interchanging x and y, we may assume that κ is positive. Of course, at least one of the numbers λ and $1-\lambda$ is positive. Assume, for instance, the first case. Let $J := \varphi(I)$. Clearly, J is an open interval.

At first we show that φ and φ^{-1} are locally Lipschitzian and their derivatives do not vanish wherever they exist. Putting $s = \varphi(x)$ and $t = \varphi(y)$ in (3) we get

$$\begin{aligned}(1-\lambda)\psi^{-1}\left(\nu\psi(\varphi^{-1}(s)) + (1-\nu)\psi(\varphi^{-1}(t))\right) \\ = \kappa\varphi^{-1}(s) + (1-\kappa)\varphi^{-1}(t) - \lambda\varphi^{-1}(\mu s + (1-\mu)t)\end{aligned}$$

for every $s, t \in J$. Since the left-hand side is strictly monotonic as a function of s, so does the right-hand side. Hence

$$J \ni s \mapsto \varphi^{-1}(s) - \frac{\lambda}{\kappa}\varphi^{-1}\left(\mu s + (1-\mu)t\right)$$

is strictly monotonic for every $t \in J$. For every $v_0 \in J$, by Lemma 1, we can find $\delta \in (0, \infty)$ and $K, L \in (0, \infty)$ such that $(v_0 - \delta, v_0 + \delta) \subset J$ and

$$K \le \frac{\varphi^{-1}(u) - \varphi^{-1}(v)}{u - v} \le L, \quad u, v \in (v_0 - \delta, v_0 + \delta),\ u \ne v.$$

Then also for every $x_0 \in I$ there exist $\delta > 0$ and $K, L > 0$ such that

$$\frac{1}{L} \le \frac{\varphi(x) - \varphi(y)}{x - y} \le \frac{1}{K}, \quad x, y \in (x_0 - \delta, x_0 + \delta),\ x \ne y.$$

In particular, it follows that if the function φ is differentiable at a point $x_0 \in I$, then $\varphi'(x_0) \ne 0$ and if the function φ^{-1} is differentiable at $v_0 \in \varphi(I)$, then $(\varphi^{-1})'(v_0) \ne 0$.

Now we show that φ is differentiable on I. For every $v \in J$ put

$$U(v) = \frac{1}{1-\mu}(J - v) \cap \frac{1}{\mu}(v - J);$$

observe that $U(v)$ is an open interval containing 0. Given any $v \in J$ and $u \in U(v)$ define also

$$V(u) = (J - (1-\mu)u) \cap (J + \mu u);$$

clearly $V(u)$ is an open interval and $v \in V(u)$. Putting $x = \varphi^{-1}(v + (1-\mu)u)$ and $y = \varphi^{-1}(v - \mu u)$ in (3) we get

$$\begin{aligned}\lambda\varphi^{-1}(v) &= \kappa\varphi^{-1}(v + (1-\mu)u) + (1-\kappa)\varphi^{-1}(v - \mu u) \\ &\quad - (1-\lambda)\psi^{-1}\left(\nu\psi(\varphi^{-1}(v + (1-\mu)u)) + (1-\nu)\psi(\varphi^{-1}(v - \mu u))\right) \quad (17)\end{aligned}$$

for every $v \in J$ and $u \in U(v)$.

Take any $v_0 \in J$ and define functions $f_1, f_2 : U(v_0) \to I$ by

$$f_1(u) = \varphi^{-1}(v_0 + (1-\mu)u), \quad f_2(u) = \varphi^{-1}(v_0 - \mu u).$$

For $i = 1, 2$ put

$$N_i = \{u \in U(v_0) : f_i \text{ is not differentiable at } u\}.$$

By the monotonicity of f_1, f_2 the sets N_1, N_2 are of Lebesgue measure 0 and, consequently, so is their union N. Since φ and φ^{-1} are locally Lipschitzian, also the function A_μ^φ has that property, and thus, by Rademacher's theorem [4, Theorem 3.1.9], A_μ^φ is almost everywhere differentiable on I^2. In particular, the set

$$C=\left\{(x, y) \in I^2\colon A_\mu^\varphi(\cdot, y) \text{ is differentiable at } x \text{ and } A_\mu^\varphi(x, \cdot) \text{ is differentiable at } y\right\}$$

is of full Lebesgue measure in I^2. As $(f_1, f_2)(U(v_0))$ is the product of two open intervals and the functions f_1, f_2 are locally Lipschitzian, the set $(f_1, f_2)^{-1}(C)$ has a positive measure; otherwise $C \cap (f_1, f_2)(U(v_0)) = (f_1, f_2)[(f_1, f_2)^{-1}(C)]$ would be of measure zero. Hence it follows that the set $(f_1, f_2)^{-1}(C) \setminus N$ is non-empty. Take any $u_0 \in (f_1, f_2)^{-1}(C) \setminus N$. Then f_1, f_2 are differentiable at u_0 and the functions $A_\mu^\varphi(\cdot, f_2(u_0))$ and $A_\mu^\varphi(f_1(u_0), \cdot)$ are differentiable at $f_1(u_0)$ and $f_2(u_0)$, respectively.

Now define functions $g_1, g_2 : V(u_0) \to I$ by

$$g_1(v) = \varphi^{-1}(v + (1-\mu)u_0), \quad g_2(v) = \varphi^{-1}(v - \mu u_0).$$

Observe that $g_1(v_0) = f_1(u_0)$ and $g_2(v_0) = f_2(u_0)$. Therefore the functions $A_\mu^\varphi(\cdot, g_2(v_0))$ and $A_\mu^\varphi(g_1(v_0), \cdot)$ are differentiable at the points $g_1(v_0)$ and $g_2(v_0)$, respectively, whence, according to (3), $A_\nu^\psi(\cdot, g_2(v_0))$ and $A_\nu^\psi(g_1(v_0), \cdot)$ are differentiable at $g_1(v_0)$ and $g_2(v_0)$, respectively. Moreover, as f_1 is differentiable at u_0, the function φ^{-1} is differentiable at $v_0 + (1-\mu)u_0$, and thus g_1 is differentiable at v_0. Similarly, we infer that the function g_2 has the same property. Consequently, the function $V(u_0) \ni v \mapsto A_\nu^\psi(g_1(v), g_2(v))$ is differentiable at v_0. Now (17) gives

$$\lambda\varphi^{-1}(v) = \kappa g_1(v) + (1-\kappa)g_2(v) - (1-\lambda)A_\nu^\psi(g_1(v), g_2(v)), \quad v \in V(u_0),$$

and we get the differentiability of φ^{-1} at v_0. As v_0 is an arbitrary point of J and the derivative of φ^{-1} does not vanish, φ is differentiable on I.

According to (3) and applying Lemma 4 to ψ and ν instead of φ and μ, respectively, we find a non-empty open interval $I_0 \subset I$ such that ψ is differentiable in I_0; clearly also φ is differentiable in I_0.

Define functions $f, g : I_0 \to (0, \infty)$ by

$$f(s) = \varphi'(\varphi^{-1}(s)), \quad g(s) = \psi'(\varphi^{-1}(s)).$$

We show that the pair (f, g) satisfies equation (5). Indeed, differentiating both sides of equality (3) with respect to x we get

$$\frac{\lambda\mu\varphi'(x)}{\varphi'(\varphi^{-1}(\mu\varphi(x) + (1-\mu)\varphi(y)))} + \frac{(1-\lambda)\nu\psi'(x)}{\psi'(\psi^{-1}(\nu\psi(x) + (1-\nu)\psi(y)))} = \kappa \tag{18}$$

for all x, $y \in I_0$. On the other hand, differentiating equality (3) with respect to y we have

$$\frac{\lambda(1-\mu)\varphi'(y)}{\varphi'(\varphi^{-1}(\mu\varphi(x) + (1-\mu)\varphi(y)))} + \frac{(1-\lambda)(1-\nu)\psi'(y)}{\psi'(\psi^{-1}(\nu\psi(x) + (1-\nu)\psi(y)))} = 1-\kappa \tag{19}$$

for all x, $y \in I_0$. Multiplying equality (18) by $(1-\nu)\psi'(y)$ and (19) by $-\nu\psi'(x)$ and adding the obtained equalities by sides we have

$$\frac{\lambda\mu(1-\nu)\varphi'(x)\psi'(y) - \lambda(1-\mu)\nu\varphi'(y)\psi'(x)}{\varphi'(\varphi^{-1}(\mu\varphi(x) + (1-\mu)\varphi(y)))} = \kappa(1-\nu)\psi'(y) - (1-\kappa)\nu\psi'(x)$$

for all $x, y \in I_0$, whence, setting here $x = \varphi^{-1}(s)$ and $y = \varphi^{-1}(t)$, we see that equality (5) holds for every $s, t \in \varphi(I_0)$. Since φ^{-1} is locally Lipschitzian and φ' is measurable $\varphi' \circ \varphi^{-1}$ is Lebesgue measurable. Moreover, ψ' is of the first Baire class and φ^{-1} is continuous whence $\psi' \circ \varphi^{-1}$ is of the first Baire class. Therefore, due to Lemma 3, we infer that f, g are infinitely many times differentiable on a non-empty subinterval of $\varphi(I_0)$. This competes the proof. □

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