Publ. Math. Debrecen
75/1-2 (2009), 137–148

Limits of random iterates

By RAFAŁ KAPICA (Katowice) and JANUSZ MORAWIEC (Katowice)

Dedicated to Professor Zoltán Daróczy on his 70th birthday

Abstract. Given a probability space $(\Omega, \mathcal{A}, P)$, a nonempty subset X of a separable Banach space Y and an rv-function $f : X \times \Omega \to X$, we assume that the sequence of iterates of f converges to a function $\xi : X \times \Omega^\infty \to Y$. We give conditions on f and types of convergence which imply continuity of ξ with respect to the first variable. A possible application of obtained results to iterative equations is presented.

1. Introduction

Throughout this paper we assume that $(\Omega, \mathcal{A}, P)$ is a probability space, X is a non-empty subset of a separable Banach space $(Y, \|\cdot\|)$. By $\mathcal{B}(X)$ we denote the σ-algebra of all Borel subsets of X. Following [6] we say that $f : X \times \Omega \to X$ is a *random valued vector function* (shortly *an rv-function*) if it is measurable with respect to the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{A}$. Given an rv-function f define the sequence $(f^m)_{m\in\mathbb{N}}$ of its *iterates* as follows (see [6]; cf. [7]):

$$f^1(x, \omega_1, \omega_2, \dots) = f(x, \omega_1), \quad f^{m+1}(x, \omega_1, \omega_2, \dots) = f(f^m(x, \omega_1, \omega_2, \dots), \omega_{m+1}),$$

for all $x \in X$ and $(\omega_1, \omega_2, \dots) \in \Omega^\infty$. Since, in fact, $f^m(\cdot, \omega)$ depends only on the first m coordinates of $\omega = (\omega_1, \omega_2, \dots)$, we may (and we do) consider this iterate

Mathematics Subject Classification: 37H99, 37N99, 39B12.
Key words and phrases: Random-valued vector functions, sequences of iterates, iterative equations, continuous solutions.
This research was supported by the Silesian University Mathematics Department (Discrete Dynamical Systems and Iteration Theory program – the first author, and Functional Equations program – the second author).

as a function defined on $X \times \Omega^\infty$ or, alternatively, on $X \times \Omega^m$. It may be shown that these iterates form a random dynamical system (see [2]) and a homogeneous Markov chain (see [18]). The basic property of iterates of rv-functions says that they are rv-functions on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. Further properties of iterates of rv-functions were studied in [3], [14] in the scalar case and in [10], [12] in the vector case.

Iteration is one of the basic technique for solving functional equations in a single variable which usually leads to solutions expressed by limits of iterates (see [4], [15], [16]). It turns out that the above definition of iterates of rv-functions can be successfully adopted in this technique (see [5]). However, we still do not know much on regularity of limits of convergent iterates of rv-functions.

Assume that for every $x \in X$ the sequence $(f^m(x,\cdot))_{m\in\mathbb{N}}$ of an rv-function f converges, in some sense, to a function $\xi(x,\cdot)$. In general $\xi : X \times \Omega^\infty \to Y$ is measurable with respect to the second variable. The problem is: What we need assume on f and which type of convergence should holds to get an additional information on ξ. In the present paper we focus on the problem of continuity of ξ with respect to the first variable.

The paper is organized as follows. At the beginning we introduce definitions of some types of continuity of rv-functions. In the third section we show that if the sequence of iterates of a given rv-function consists of continuous functions, then its limit function ξ so is, if suitable kind of convergence holds. Next, in the section four, we study conditions under which continuity of a given rv-function implies continuity of its sequence of iterates. In the last section we give a possible application of obtained results to iterative equations.

2. Notation

In the remainder of this paper we assume that $p \in [1, +\infty)$ is fixed.

Let $(f_m)_{m\in\mathbb{N}_0}$ be a sequence of measurable functions, acting from Ω to X. By $P\text{-}\lim_{m\to\infty} f_m = f_0$ we denote the convergence in probability P and, if moreover, $f_m \in L^p(\Omega, \mathcal{A}, P)$ for all $m \in \mathbb{N}_0$, then by $L^p\text{-}\lim_{m\to\infty} f_m = f_0$ we denote the convergence in L^p. Here and later we consider integrability in the Bochner sense.

Motivated by [5] we introduce a few kinds of regularity of rv-functions.

We say that an rv-function $f : X \times \Omega \to X$ is:

- *P-continuous at* $x_0 \in X$ if $P\text{-}\lim_{j\to\infty} f(x_j,\cdot) = f(x_0,\cdot)$ for every sequence $(x_j)_{j\in\mathbb{N}}$ of points from X convergent to x_0;
- *P-continuous* if f is P-continuous at every point from X;

– *Uniformly P-continuous* if for any $\varepsilon, \beta > 0$ there exists a $\delta > 0$ such that $P(\|f(x,\cdot) - f(y,\cdot)\| \geq \beta) \leq \varepsilon$ for all $x, y \in X$ with $\|x - y\| \leq \delta$.

Assume that $f : X \times \Omega \to X$ is an rv-function such that $f(x,\cdot) \in L^p$ for all $x \in X$. We say that f is:

– *L^p-continuous at $x_0 \in X$* if $L^p\text{-}\lim_{j\to\infty} f(x_j,\cdot) = f(x_0,\cdot)$ for every sequence $(x_j)_{j\in\mathbb{N}}$ of points from X convergent to x_0;

– *L^p-continuous* if f is L^p-continuous at every point from X;

– *Uniformly L^p-continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_\Omega \|f(x,\cdot) - f(y,\cdot)\|^p dP \leq \varepsilon$ for all $x, y \in X$ with $\|x - y\| \leq \delta$.

It is clear that every uniformly P-continuous rv-function is P-continuous, every L^p-continuous at x_0 rv-function is P-continuous at x_0, and every uniformly L^p-continuous rv-function is uniformly P-continuous and L^p-continuous.

Remark 2.1. Assume that $f : X \times \Omega \to X$ is an rv-function and there exists an integrable function $\psi : \Omega \to [0, +\infty)$ such that $\|f(x,\cdot)\|^p \leq \psi$ for all $x \in X$.

(i) If f is P-continuous at $x_0 \in X$, then f is L^p-continuous at x_0.

(ii) If f is uniformly P-continuous, then f is uniformly L^p-continuous.

Proof. We will prove assertion (i) only. The proof of assertion (ii) is similar.

Fix $\varepsilon > 0$ and a sequence $(x_j)_{j\in\mathbb{N}}$ convergent to x_0. By P-continuity of f at x_0 we choose a $j_0 \in \mathbb{N}$ such that $\int_{\|f(x_j,\cdot)-f(x_0,\cdot)\|^p \geq \varepsilon} \psi dP \leq \varepsilon$ for all $j \geq j_0$. Hence

$$\int_\Omega \|f(x_j,\cdot) - f(x_0,\cdot)\|^p dP \leq 2^{p+1} \int_{\|f(x_j,\cdot)-f(x_0,\cdot)\|^p \geq \varepsilon} \psi dP + \varepsilon \leq (2^{p+1} + 1)\varepsilon$$

for all $j \geq j_0$. □

We say that a sequence $(f_m)_{m\in\mathbb{N}}$ of rv-functions is P-continuous at x_0, P-continuous, etc., if f_m is P-continuous at x_0, P-continuous, etc., for all $m \in \mathbb{N}$.

We finish this section with two examples. In both of them we assume that $(\Omega, \mathcal{A}, P) = ((0,1), \mathcal{B}(0,1), l_1|_{\mathcal{B}(0,1)})$.

Example 2.2. Consider a function $g : (0, +\infty) \times (0,1) \to (0, +\infty)$ given by $g(x,\omega) = \frac{\omega}{x}$. A simple verification shows that g is L^p-continuous, but $g^m(x,\cdot) \notin L^p$ for all $x \in (0,1)$ and $m \geq 2$.

The second example shows that L^p-continuity is stronger than P-continuity.

Example 2.3. Consider a function $f : [0, +\infty) \times (0, 1) \to [0, +\infty)$ given by

$$f(x,\omega) = \begin{cases} x^{-\frac{1}{p}}\chi_{[0,x)}(\omega), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

It is easy to check that f is P-continuous at 0. Since $\int_0^1 |f(x,\cdot)|^p dP = 1$ for all $x \neq 0$, we conclude that $f(x,\cdot) \in L^p$ for all $x \in [0, +\infty)$ and that f is not L^p-continuous at 0.

For $p = 1$ we can get, by induction,

$$f^m(x,\omega_1,\ldots,\omega_m) = \begin{cases} x^{(-1)^n}, & \text{if } x \neq 0 \text{ and } \omega_k < x^{(-1)^{k+1}} \text{ for all } k \in \{1,\ldots,n\}, \\ 0, & \text{otherwise,} \end{cases}$$

for all $m \in \mathbb{N}$. It follows easily that for every $m \in \mathbb{N}$ the iterate f^{2m+1} is not L^1-continuous at 0, whereas the sequence $(f^{2m})_{m\in\mathbb{N}}$ is uniformly L^1-continuous and the sequence $(f^m)_{m\in\mathbb{N}}$ is P^∞-continuous.

3. Continuity of limits

Throughout this section we assume that $f : X \times \Omega \to X$ and $\xi : X \times \Omega^\infty \to Y$ are rv-functions. We show that continuity of $(f^m)_{m\in\mathbb{N}}$ can be transferred to continuity of its limit function if suitable local uniform convergence holds with respect to the first variable; i.e., every point $x \in X$ has a neighbourhood on which convergence is uniform.

Proposition 3.1. *Assume that $(f^m)_{m\in\mathbb{N}}$ is P^∞-continuous at $x_0 \in X$.*

(i) *If*

for every $\varepsilon, \beta > 0$ and for each sequence $x_j \to x_0$ there exist $m, j_0 \in \mathbb{N}$ such that, for all $j \geq j_0$, we have $P^\infty(\|f^m(x_j,\cdot) - \xi(x_j,\cdot)\| \geq \beta) \leq \varepsilon$, (3.1)

then ξ is P^∞-continuous at x_0.

(ii) *If ξ is P^∞-continuous at x_0 and if*

for every $\varepsilon, \beta > 0$ there exists m such that $P^\infty(\|f^m(x_0,\cdot) - \xi(x_0,\cdot)\| \geq \beta) \leq \varepsilon$, (3.2)

then (3.1) *holds.*

Proof. Fix $\varepsilon, \beta > 0$ and a sequence $(x_j)_{j\in\mathbb{N}}$ convergent to x_0.

(i) Applying (3.1), to a sequence $(y_j)_{j\in\mathbb{N}}$ defined by $y_{2n-1} = x_0$ and $y_{2n} = x_n$ for all $n \in \mathbb{N}$, we conclude that there exist $m, j_0 \in \mathbb{N}$ such that

$$P^\infty(\|f^m(x_0,\cdot) - \xi(x_0,\cdot)\| \geq \beta) \leq \varepsilon, \tag{3.3}$$

$$P^\infty(\|f^m(x_j,\cdot) - \xi(x_j,\cdot)\| \geq \beta) \leq \varepsilon \quad \text{for all } j \geq j_0. \tag{3.4}$$

By P^∞-continuity of f^m at x_0 we choose a $j_1 \geq j_0$ such that

$$P^\infty(\|f^m(x_j,\cdot) - f^m(x_0,\cdot)\| \geq \beta) \leq \varepsilon \quad \text{for all } j \geq j_1. \tag{3.5}$$

Now (3.4), (3.5) and (3.3) imply

$$P^\infty(\|\xi(x_j,\cdot) - \xi(x_0,\cdot)\| \geq 3\beta) \leq P^\infty(\|\xi(x_j,\cdot) - f^m(x_j,\cdot)\| \geq \beta)$$
$$+P^\infty(\|f^m(x_j,\cdot) - f^m(x_0,\cdot)\| \geq \beta) + P^\infty(\|f^m(x_0,\cdot) - \xi(x_0,\cdot)\| \geq \beta) \leq 3\varepsilon$$

for all $j \geq j_1$.

(ii) Applying (3.2) we choose an $m \in \mathbb{N}$ such that (3.3) holds. By P^∞-continuity of f^m at x_0 we choose a $j_1 \in \mathbb{N}$ such that (3.5) holds, and by P^∞-continuity of ξ at x_0 we choose a $j_2 \geq j_1$ such that

$$P^\infty(\|\xi(x_j,\cdot) - \xi(x_0,\cdot)\| \geq \beta) \leq \varepsilon \quad \text{for all } j \geq j_2. \tag{3.6}$$

From (3.5), (3.3) and (3.6) we conclude that $P^\infty(\|f^m(x_j,\cdot) - \xi(x_j,\cdot)\| \geq 3\beta) \leq 3\varepsilon$ for all $j \geq j_2$. □

If $(f^m)_{m\in\mathbb{N}}$ is P^∞-continuous at x_0 and if $P^\infty\text{-}\lim_{m\to\infty} f^m(x_0,\cdot) = \xi(x_0,\cdot)$, then Proposition 3.1 shows that P^∞-continuity of ξ at x_0 is equivalent to condition (3.1). Hence we have the following corollary.

Corollary 3.2. *Assume that $(f^m)_{m\in\mathbb{N}}$ is P^∞-continuous. If $P^\infty\text{-}\lim_{m\to\infty} f^m(x,\cdot) = \xi(x,\cdot)$ locally uniformly, then ξ is P^∞-continuous.*

Note that in the case where X has an ordered structure the sequence $(f^m(x,\cdot))_{m\in\mathbb{N}}$ forms a submartingale provided the mean $m(x) = \mathbb{E}f(x,\cdot)$ satisfies $m(x) \geq x$ for all $x \in X$ (see [10]; cf. [5]). Consequently, convergence of iterates follows from a submartingale convergence theorem (see [17]). However, a uniform convergence theorem holds only for positive parts of $(f^m(x,\cdot) - \xi(x,\cdot))_{m\in\mathbb{N}}$ (see [9]).

The next proposition may be proved in the same way as Proposition 3.1.

Proposition 3.3. *Assume that* $(f^m)_{m\in\mathbb{N}}$ *is* L^p*-continuous at* $x_0 \in X$ *and* $\xi(x,\cdot) \in L^p$ *for all* $x \in X$.

(i) *If*

$$\begin{aligned}&\text{for every } \varepsilon > 0 \text{ and for each sequence } x_j \to x_0 \text{ there exist } m, j_0 \in \mathbb{N}\\ &\text{such that, for all } j \ge j_0, \text{ we have } \int_{\Omega^\infty} \|f^m(x_j,\cdot)-\xi(x_j,\cdot)\|^p dP^\infty \le \varepsilon,\end{aligned} \tag{3.7}$$

then ξ *is* L^p*-continuous at* x_0.

(ii) *If* ξ *is* L^p*-continuous at* x_0 *and if for every* $\varepsilon > 0$ *there exists an* $m \in \mathbb{N}$ *such that* $\int_{\Omega^\infty} \|f^m(x_0,\cdot) - \xi(x_0,\cdot)\|^p dP^\infty \le \varepsilon$, *then* (3.7) *holds.*

Corollary 3.4. *Assume that* $(f^m)_{m\in\mathbb{N}}$ *is* L^p*-continuous. If* $L^p\text{-}\lim_{m\to\infty} f^m(x,\cdot) = \xi(x,\cdot)$ *locally uniformly, then* ξ *is* L^p*-continuous.*

Concerning uniform continuity we have the following propositions.

Proposition 3.5. *Assume that* $(f^m)_{m\in\mathbb{N}}$ *is uniformly* P^∞*-continuous. If* $P^\infty\text{-}\lim_{m\to\infty} f^m(x,\cdot) = \xi(x,\cdot)$ *uniformly, then* ξ *is uniformly* P^∞*-continuous.*

Proposition 3.6. *Assume that* $(f^m)_{m\in\mathbb{N}}$ *is uniformly* L^p*-continuous. If* $L^p\text{-}\lim_{m\to\infty} f^m(x,\cdot) = \xi(x,\cdot)$ *uniformly, then* ξ *is uniformly* L^p*-continuous.*

4. Continuity of iterates

In this section we are interested in conditions under which sequences of iterates of continuous rv-functions are continuous. For this purpose we will formulate a more general problem.

Until the end we assume that $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{P})$ is a probability space, $\widetilde{X}$ is a nonempty subset of a separable Banach space $(\widetilde{Y}, \|\cdot\|)$, $f : X \times \Omega \to X$ and $g : \widetilde{X} \times \widetilde{\Omega} \to \widetilde{X}$ are rv-functions, $\Phi : \widetilde{X} \to X$ is a continuous function. Define a Φ-composition function $f \circ_\Phi g : \widetilde{X} \times \widetilde{\Omega} \times \Omega \to X$ putting

$$f \circ_\Phi g(\widetilde{x}, \widetilde{\omega}, \omega) = f(\Phi(g(\widetilde{x}, \widetilde{\omega})), \omega).$$

It is easy to check that $f \circ_\Phi g$ is an rv-function with respect to the product space $(\widetilde{\Omega} \times \Omega, \widetilde{\mathcal{A}} \otimes \mathcal{A}, \widetilde{P} \otimes P)$. Clearly, $f \circ_{Id_X} f$ coincides with the second iterate f^2 of f. The problem reads: Under which assumptions the Φ-composition function is continuous?

We begin with a result which can be proved by adapting the proof of Lemma 2.2 from [5]; that lemma is just the first part of our Corollary 4.2 in the case where $X = [0, 1]$.

Theorem 4.1. *If g is $\widetilde{P}$-continuous at $\widetilde{x}_0 \in \widetilde{X}$ and if f is P-continuous at points of the set $\Phi \circ g(\{\widetilde{x}_0\} \times \widetilde{\Omega})$, then $f \circ_\Phi g$ is $\widetilde{P} \otimes P$-continuous at $\widetilde{x}_0$.*

If there exists a $\widetilde{P} \otimes P$-integrable function $\psi : \widetilde{\Omega} \times \Omega \to [0, +\infty]$ such that $\|f \circ_\Phi g(\widetilde{x}, \cdot)\|^p \le \psi$ for all $\widetilde{x} \in \widetilde{X}$, then from Remark 2.1(i) we conclude that $\widetilde{P} \otimes P$-continuity in the assertion of Theorem 4.1 can be replaced by L^p-continuity. Note that such a ψ exists if f is bounded, and in particular, if X is bounded.

Corollary 4.2. *If f is P-continuous, then $(f^m)_{m \in \mathbb{N}}$ is P^∞-continuous. Moreover, if f is bounded, then $(f^m)_{m \in \mathbb{N}}$ is L^p-continuous.*

Concerning uniform continuity in probability we have the following result.

Theorem 4.3. *Assume that there exists an $a > 0$ such that*

$$\|\Phi(\widetilde{x}) - \Phi(\widetilde{y})\| \le a \|\widetilde{x} - \widetilde{y}\| \qquad \text{for all } \widetilde{x}, \widetilde{y} \in g(\widetilde{X} \times \widetilde{\Omega}). \tag{4.1}$$

If g is uniformly $\widetilde{P}$-continuous and if f is uniformly P-continuous, then $f \circ_\Phi g$ is uniformly $\widetilde{P} \otimes P$-continuous.

Proof. Fix $\varepsilon, \beta > 0$. By uniform P-continuity of f we choose a $\gamma > 0$ such that $P(\|f(x, \cdot) - f(y, \cdot)\| \ge \beta) \le \varepsilon$ for all $x, y \in X$ with $\|x - y\| \le a\gamma$. This jointly with (4.1) gives

$$P(\|f \circ_\Phi g(\widetilde{x}, \widetilde{\omega}, \cdot) - f \circ_\Phi g(\widetilde{y}, \widetilde{\omega}, \cdot)\| \ge \beta) \le \varepsilon \qquad \begin{array}{l} \text{for all } \widetilde{x}, \widetilde{y} \in \widetilde{X}, \widetilde{\omega} \in \widetilde{\Omega} \\ \text{with } \|g(\widetilde{x}, \widetilde{\omega}) - g(\widetilde{y}, \widetilde{\omega})\| \le \gamma. \end{array}$$

Now, by uniform $\widetilde{P}$-continuity of g we choose a $\delta \le \gamma$ such that

$$\widetilde{P}(\|g(\widetilde{x}, \cdot) - g(\widetilde{y}, \cdot)\| \ge \gamma) \le \varepsilon \text{ for all } \widetilde{x}, \widetilde{y} \in \widetilde{X} \text{ with } \|\widetilde{x} - \widetilde{y}\| \le \delta.$$

Fix $\widetilde{x}, \widetilde{y} \in \widetilde{X}$ such that $\|\widetilde{x} - \widetilde{y}\| \le \delta$. Then, by the Fubini theorem, we have

$$\begin{aligned} &(\widetilde{P} \otimes P)(\|f \circ_\Phi g(\widetilde{x}, \cdot) - f \circ_\Phi g(\widetilde{y}, \cdot)\| \ge \beta) \\ &\quad = \int_{\|g(\widetilde{x}, \widetilde{\omega}) - g(\widetilde{y}, \widetilde{\omega})\| \ge \gamma} P(\|f \circ_\Phi g(\widetilde{x}, \widetilde{\omega}, \cdot) - f \circ_\Phi g(\widetilde{y}, \widetilde{\omega}, \cdot)\| \ge \beta) d\widetilde{P}(\widetilde{\omega}) \\ &\qquad + \int_{\|g(\widetilde{x}, \widetilde{\omega}) - g(\widetilde{y}, \widetilde{\omega})\| < \gamma} P(\|f \circ_\Phi g(\widetilde{x}, \widetilde{\omega}, \cdot) - f \circ_\Phi g(\widetilde{y}, \widetilde{\omega}, \cdot)\| \ge \beta) d\widetilde{P}(\widetilde{\omega}) \\ &\quad \le \widetilde{P}(\|g(\widetilde{x}, \cdot) - g(\widetilde{y}, \cdot)\| \ge \gamma) + \int_{\|g(\widetilde{x}, \widetilde{\omega}) - g(\widetilde{y}, \widetilde{\omega})\| < \gamma} \varepsilon d\widetilde{P}(\widetilde{\omega}) \le 2\varepsilon, \end{aligned}$$

which completes the proof. $\square$

As an immediate consequence of Theorem 4.3 and Remark 2.1(ii) we get the following corollaries.

Corollary 4.4. *Assume that* (4.1) *holds with some* $a > 0$ *and there exists a* $\widetilde{P} \otimes P$*-integrable function* $\psi : \widetilde{\Omega} \times \Omega \to [0, +\infty]$ *such that* $\|f \circ_\Phi g(\widetilde{x}, \cdot)\|^p \le \psi$ *for all* $\widetilde{x} \in \widetilde{X}$. *If* g *is uniformly* $\widetilde{P}$*-continuous and if* f *is uniformly* P*-continuous, then* $f \circ_\Phi g$ *is uniformly* L^p*-continuous.*

Corollary 4.5. *If* f *is uniformly* P*-continuous, then* $(f^m)_{m\in\mathbb{N}}$ *is uniformly* P^∞*-continuous. Moreover, if* f *is bounded, then* $(f^m)_{m\in\mathbb{N}}$ *is uniformly* L^p*-continuous.*

Now we pass to L^p-continuity.

Theorem 4.6. *Assume that there exist* $a, b, c, d, \alpha, \beta \ge 0$ *such that*

$$\int_\Omega \|f(x, \cdot)\|^p dP \le a\|x\|^\alpha + b \quad \text{for all } x \in \Phi \circ g(\widetilde{X} \times \widetilde{\Omega}), \tag{4.2}$$

$$\|\Phi(\widetilde{x})\|^\alpha \le c\|\widetilde{x}\|^\beta + d \quad \text{for all } \widetilde{x} \in g(\widetilde{X} \times \widetilde{\Omega}), \tag{4.3}$$

and for every countable and bounded set $K \subset \widetilde{X}$ *the function*

$$\text{“}\widetilde{\Omega} \ni \widetilde{\omega} \longmapsto \sup\left\{\|g(\widetilde{x}, \widetilde{\omega})\|^\beta : \widetilde{x} \in K\right\} \in [0, +\infty]\text{”} \quad \text{is } \widetilde{P}\text{-integrable.} \tag{4.4}$$

If g *is* $\widetilde{P}$*-continuous at* $\widetilde{x}_0 \in \widetilde{X}$ *and if* f *is* L^p*-continuous at points of the set* $\Phi \circ g(\{\widetilde{x}_0\} \times \widetilde{\Omega})$, *then* $f \circ_\Phi g$ *is* L^p*-continuous at* $\widetilde{x}_0$.

Proof. By (4.2), (4.3) and (4.4) we have $f \circ_\Phi g(\widetilde{x}, \cdot) \in L^p$ for all $\widetilde{x} \in \widetilde{X}$.

Fix a sequence $(\widetilde{x}_j)_{j\in\mathbb{N}}$ convergent to $\widetilde{x}_0$ and for every $j \in \mathbb{N}$ put

$$I_j = \int_{\widetilde{\Omega}\times\Omega} \|f \circ_\Phi g(\widetilde{x}_j, \cdot) - f \circ_\Phi g(\widetilde{x}_0, \cdot)\|^p d(\widetilde{P} \otimes P).$$

The proof will be completed if we show that every strictly increasing sequence $(j_k)_{k\in\mathbb{N}}$ of positive integers has a subsequence $(j'_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} I_{j'_k} = 0. \tag{4.5}$$

Fix a strictly increasing sequence $(j_k)_{k\in\mathbb{N}}$ of positive integers. By $\widetilde{P}$-continuity of g we choose its subsequence $(j'_k)_{k\in\mathbb{N}}$ such that $g(\widetilde{x}_{j'_k}, \cdot)$ converges to $g(\widetilde{x}_0, \cdot)$ almost everywhere and put $A = \{\widetilde{\omega} \in \widetilde{\Omega} : \lim_{k\to\infty} g(\widetilde{x}_{j'_k}, \widetilde{\omega}) = g(\widetilde{x}_0, \widetilde{\omega})\}$. Clearly, $\widetilde{P}(A) = 1$. Next, for every $k \in \mathbb{N}$ define a function $\psi_k : A \to [0, +\infty]$ putting

$$\psi_k(\widetilde{\omega}) = \int_\Omega \|f \circ_\Phi g(\widetilde{x}_{j'_k}, \widetilde{\omega}, \cdot) - f \circ_\Phi g(\widetilde{x}_0, \widetilde{\omega}, \cdot)\|^p dP.$$

It is clear that

$$I_{j'_k} = \int_A \psi_k d\widetilde{P}. \tag{4.6}$$

Continuity of Φ implies $\lim_{k\to\infty} \Phi(g(\widetilde{x}_{j'_k},\widetilde{\omega})) = \Phi(g(\widetilde{x}_0,\widetilde{\omega}))$ for all $\widetilde{\omega} \in A$, and then L^p-continuity of f yields

$$\lim_{k\to\infty} \psi_k(\widetilde{\omega}) = 0 \quad \text{for all } \widetilde{\omega} \in A. \tag{4.7}$$

Put $K = \{\widetilde{x}_{j'_k} : k \in \mathbb{N}\} \cup \{\widetilde{x}_0\}$. Fix $k \in \mathbb{N}$ and $\widetilde{\omega} \in A$. From (4.2) and (4.3) we get

$$\begin{aligned}\psi_k(\widetilde{\omega}) &\le 2^p \int_\Omega \|f \circ_\Phi g(\widetilde{x}_{j'_k},\widetilde{\omega},\cdot)\|^p dP + 2^p \int_\Omega \|f \circ_\Phi g(\widetilde{x}_0,\widetilde{\omega},\cdot)\|^p dP \\ &\le 2^p ac\big(\|g(\widetilde{x}_{j'_k},\widetilde{\omega})\|^\beta + \|g(\widetilde{x}_0,\widetilde{\omega})\|^\beta\big) + 2^{p+1}(ad+b) \\ &\le 2^{p+1} ac \sup\big\{\|g(\widetilde{x},\widetilde{\omega})\|^\beta : \widetilde{x} \in K\big\} + 2^{p+1}(ad+b).\end{aligned}$$

This jointly with (4.4), (4.6) and (4.7) imply (4.5). □

Applying Theorem 4.6 to $g = f$ with $\alpha = \beta = p$ we get the following corollary.

Corollary 4.7. *Assume that there exist $a, b \ge 0$ such that $\int_\Omega \|f(x,\cdot)\|^p dP \le a\|x\|^p + b$ for all $x \in X$ and for every bounded set $K \subset X$ there exists an integrable function $\psi : \Omega \to [0,+\infty)$ such that $\|f(x,\cdot)\|^p \le \psi$ for all $x \in K$. If f is L^p-continuous, then $(f^m)_{m\in\mathbb{N}}$ is L^p-continuous.*

Observe that if f and g are non expansive in L^p and if Φ is non expansive, then $f \circ_\Phi g$ does. In particular, we have the following proposition.

Proposition 4.8. *If $f(x,\cdot) \in L^p$ for all $x \in X$ and if there exists an $a \ge 0$ such that $\int_\Omega \|f(x,\cdot) - f(y,\cdot)\|^p dP \le a\|x-y\|^p$ for all $x, y \in X$, then $(f^m)_{m\in\mathbb{N}}$ is uniformly L^p-continuous.*

5. Iterative equations

Fix an rv-function $\xi : X \times \Omega^\infty \to X$ and define a function $\pi : X \times \mathcal{B}(X) \to [0,1]$ putting

$$\pi(x,B) = P^\infty(\xi(x,\cdot) \in B). \tag{5.1}$$

It is easy to see that for every $x \in X$ the function $\pi(x,\cdot)$ is a probability measure. Assume now that ξ is P^∞-continuous at $x_0 \in X$. Fix a continuous and bounded function $g : X \to \mathbb{R}$ and a sequence $(x_j)_{j\in\mathbb{N}}$ convergent to x_0. Then P^∞-$\lim_{j\to\infty} g\circ\xi(x_j,\cdot) = g\circ\xi(x_0,\cdot)$ and $g\circ\xi$ is bounded. Hence $\lim_{j\to\infty}\int_X g(x)\pi(x_j,dx) = \int_X g(x)\pi(x_0,dx)$, and in consequence, the sequence $(\pi(x_j,\cdot))_{j\in\mathbb{N}}$ converges weakly to $\pi(x_0,\cdot)$. This jointly with Corollary 3.2 and [11, Theorem 2] gives following proposition concerning solutions of iterative equations.

Proposition 5.1. *Assume that f is P-continuous and for every $x\in X$ the sequence $(f^m(x,\cdot))_{m\in\mathbb{N}}$ converges locally uniformly in probability to a random variable $\xi(x,\cdot)$, and $\pi(x,\cdot)$ is the measure given by* (5.1). *If $\pi(x,\cdot)\neq\pi(y,\cdot)$ for some $x,y\in X$, then there exists a continuous and bounded function $g: X\to\mathbb{R}$ such that the function $\varphi: X\to\mathbb{R}$ defined by*

$$\varphi(x) = \int_X g(y)\pi(x,dy)$$

is a bounded, continuous and non-constant solution of the equation

$$\varphi(x) = \int_\Omega \varphi(f(x,\cdot))dP.$$

The next example shows a possible application of Proposition 5.1.

Example 5.2. Fix measurable functions $L, M: \Omega\to\mathbb{R}$ and continuous functions $F, G: \mathbb{R}\to\mathbb{R}$. Define a function $f: \mathbb{R}^2\times\Omega\to\mathbb{R}^2$ putting

$$f(x,y,\omega) = (G(x)L(\omega), F(x)M(\omega)+y).$$

It is clear that f is P-continuous. Iterating f we get

$$f^m(x,y,\omega) = \left(G_m(x,\omega), \sum_{k=1}^{m} F\Big(G_{k-1}(x,\omega)\Big)M_k(\omega)+y\right)$$

for all $(x,y,\omega)\in\mathbb{R}^2\times\Omega^\infty$, where $G_k: \mathbb{R}\times\Omega^\infty\to\mathbb{R}$ and $M_k: \Omega^\infty\to\mathbb{R}$ are defined by $G_0(x,\omega)=x$, $G_k(x,\omega) = G(G_{k-1}(x,\omega))L(\omega_k)$, $M_k(\omega) = M(\omega_k)$ for all $k\in\mathbb{N}$.

Assume now that $|G(x)|\le|x|$ for all $x\in\mathbb{R}$ and there exists an $\alpha>0$ such that $|F(x)|\le\alpha|x|$ for all $x\in\mathbb{R}$. Then

$$\begin{aligned}&\|f^{m+n}(x,y,\omega) - f^m(x,y,\omega)\|\\&\qquad = |G_{m+n}(x,\omega) - G_m(x,\omega)| + \left|\sum_{k=m+1}^{m+n} F\Big(G_{k-1}(x,\omega)\Big)M_k(\omega)\right|\end{aligned}$$

$$\leq |x|\left(\prod_{k=1}^{m+n}|L_k(\omega)|+\prod_{k=1}^{n}|L_k(\omega)|\right)+\alpha|x|\sum_{k=m+1}^{m+n}|M_k(\omega)|\prod_{i=1}^{k-1}|L_i(\omega)| \quad (5.2)$$

for all $m,n\in\mathbb{N}$ and $(x,y,\omega)\in\mathbb{R}^2\times\Omega^\infty$, where $L_i:\Omega^\infty\to\mathbb{R}$ is defined by $L_i(\omega)=L(\omega_i)$ for all $i\in\mathbb{N}$. According to the Kolmogorov law of large numbers the first summand in (5.2) converges to zero provided $P(L=0)=0$ and $-\infty<\mathbb{E}\log|L|<0$. If additionally $\mathbb{E}\log\max\{|M|,1\}<+\infty$, then we get the desired convergence of $(f^m(x,y,\cdot))_{m\in\mathbb{N}}$ (see [13]; cf. [8]). It is not difficult to check that the convergence is locally uniform and that the limit function in not constant in (x,y). Proposition 5.1 now shows that the equation

$$\varphi(x,y)=\int_\Omega \varphi(G(x)L,F(x)M+y)dP$$

has a bounded, continuous and non-constant solution $\varphi:\mathbb{R}^2\to\mathbb{R}$.

Remark 5.3. Assume additionally that in Example 5.2 functions $L,M:\Omega\to\mathbb{R}$ are in L^p. Then f is L^p-continuous, and, by Corollary 4.7, we conclude that $(f^m(x,y,\cdot))_{m\in\mathbb{N}}$ is L^p-continuous. Moreover, if $\mathbb{E}|L|^p<1$, then $(f^m(x,y,\cdot))_{m\in\mathbb{N}}$ converges in L^p (see [19]; cf. [1]); this convergence is locally uniform. Corollary 3.4 now implies L^p-continuity of the limit function.

References

[1] G. Alsmeyer, A. Iksanov and U. Rösler, On distributional properties of perpetuities, http://arxiv.org/abs/0803.3716.

[2] L. Arnold, Random Dynamical Systems, Lecture Notes in Math. 1609, *Springer, Berlin*, 1995.

[3] K. Baron, On the convergence of sequences on iterates of random-valued functions, *Aequationes Math.* **32** (1987), 240–251.

[4] K. Baron and W. Jarczyk, Recent results on functional equations in a single variable, perspectives and open problems, *Aequationes Math.* **61** (2001), 1–48.

[5] K. Baron and W. Jarczyk, Random-valued functions and iterative functional equations, *Aequationes Math.* **67** (2004), 140–153.

[6] K. Baron and M. Kuczma, Iteration of random-valued functions on the unit interval, *Colloq. Math.* **37** (1977), 263–269.

[7] Ph. Diamond, A stochastic functional equation, *Aequationes Math.* **15** (1977), 225–23.

[8] A. K. Grincevičjus, On the continuity of the distribution of a sum of dependent variables connected with independent walks on lines, *Teor. Probability Appl.* **19** (1974), 163–168.

[9] J. Hoffmann-Jørgensen, Uniform convergence of martingales, Probability in Banach Spaces, 7 (Oberwolfach, 1988), 127–137, Progr. Probab., 21, *Birkhäuser*, *Boston, MA*, 1990.

[10] R. Kapica, Convergence of sequences of iterates of random-valued vector functions, *Colloq. Math.* **97** (2003), 1–6.

[11] R. Kapica, Sequences of iterates of random-valued vector functions and solutions of related equations, *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber.* II **213** (2004), 113–118.

[12] R. Kapica, Theorems of Thron's type for random-valued vector functions and the Krein-Rutman theorem, *Ann. Polon. Math.* **85** (2005), 13–24.

[13] H. Kesten, Random difference equations and renewal theory for products of random matrices, *Acta Math.* **131** (1973), 207–248.

[14] M. Kuczma, Normalizing factors for iterates of random valued functions, *Pr. Nauk. Uniw. Śl. Katow.* **6** (1975), 67–72.

[15] M. Kuczma, Functional Equations in a Single Variable, Monografie Matematyczne 46 PWN, *Polish Scientific Publishers, Warszawa*, 1968.

[16] M. Kuczma, B. Choczewski and R. Ger, Iterative Functional Equations, in: Encyclopedia of Mathematics and its Applications, Vol. 32, *Cambridge University Press*, 1990.

[17] M. Loéve, Probability Theory II, Graduate Texts in Mathematics 46, *Springer-Verlag, New York – Heidelberg – Berlin*, 1978.

[18] S. P. Meyn and R. L. Tweedie, Markov Chains and Stochastic Stability, Communications and Control Engineering Series, *Springer-Verlag, London*, 1993.

[19] W. Vervaat, On a stochastic difference equation and a representation of non-negative infinitely divisible random variables, *Adv. Appl. Prob.* **11** (1979), 750–783.

RAFAŁ KAPICA
INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
BANKOWA 14, PL-40-007 KATOWICE
POLAND

E-mail: rkapica@math.us.edu.pl

JANUSZ MORAWIEC
INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
BANKOWA 14, PL-40-007 KATOWICE
POLAND

E-mail: morawiec@math.us.edu.pl

(Received July 9, 2008; revised March 16, 2009)