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Open problems originated in our research work with Zoltán Daróczy

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Dedicated to my friend, Zoltán Daróczy on his 70th anniversary

Abstract. Some open problems on Rényi–Parry and other expansions are presented

§ 1. Introduction

In this paper I shall pose some research problems, the largest part of which were formulated in some of our joint papers written by Daróczy. Some problems are stated here in more general form than they were formulated originally.

§ 2. Rényi–Parry expansions, additive functions

A. Rényi [1] and W. Parry [2] investigated the following expansion. Let $q > 1$, $q \neq$ integer, $k = [q]$, $\theta = 1/q$, $\mathcal{A}_k = \{0, 1 \dots, k\}$, $I_1 = [0, 1)$; $I_2 = (0, 1]$.

Regular expansion of $x \in I_1$**.** Let $\varepsilon_1(x) = [qx]$ $(\in \mathcal{A}_k)$, and $x_1 = \{qx\}$ $(\in I_1)$. Then $x = \varepsilon_1(x)\theta + \theta x_1$, and this procedure can be continued:

$$x = \varepsilon_1(x)\theta + \varepsilon_2(x)\theta^2 + \dots, \varepsilon_j(x) \in \mathcal{A}_k, \quad j \in \mathbb{N}. \tag{2.1}$$

(2.1) is the regular expansion of x.

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Quasi regular expansion of $x \in I_2$. Let

$$\delta_1(x) = \begin{cases} [qx], & \text{if } \{qx\} \neq 0 \\ [qx]-1, & \text{if } \{qx\} = 0 \end{cases}$$

Then $x = \delta_1(x)\theta + \theta x_1$, $\delta_1(x) \in \mathcal{A}_k$, $x_1 \in I_2$.
Continuing,

$$x = \delta_1(x)\theta + \delta_2(x)\theta^2 + \dots \tag{2.2}$$

is the quasi regular expansion of x.

Let $\underline{\varepsilon}(x) = \varepsilon_1(x)\varepsilon_2(x)\dots$; $\underline{\delta}(x) = \delta_1(x)\delta_2(x)\dots$ be the sequences of digits in the regular, resp. quasi regular expansion of x. It is clear that $\underline{\varepsilon}(x) = \underline{\delta}(x)$ if $\underline{\varepsilon}(x)$ is of "infinite type," i.e. if $\varepsilon_\nu(x) \neq 0$ holds for infinitely many n. Thus the equality holds for all but countable many $x \in (0,1)$.
Since $1 = k\theta + \theta\{q\}$, $\{q\} \in (0,1)$, therefore $\{q\}$ has regular, and quasi regular expansion, also.

Let

$$\begin{aligned} &1 = l_1\theta + l_2\theta^2 + \dots, \quad l_\nu = \varepsilon_\nu(1), \\ &\underline{l} = l_1 l_2 \dots \end{aligned} \tag{2.3}$$

be the regular, and

$$\begin{aligned} &1 = t_1\theta + t_2\theta^2 + \dots \\ &\underline{t} = t_1 t_2 \dots \end{aligned} \tag{2.4}$$

be the quasi regular expansion of 1.

Let σ be the *shift operator* over $\mathcal{A}_k^{\mathbb{N}}$, i.e. if $\underline{a} = a_1 a_2 \dots$, $a_n \in \mathcal{A}_k$, then $\sigma(\underline{a}) = a_2 a_3 \dots$. W. Parry [2] proved that for $\underline{a} \in \mathcal{A}_k^{\mathbb{N}}$ there exists $x \in I_1$ such that $\underline{\varepsilon}(x) = \underline{a}$, if and only if

$$\sigma^n(\underline{a}) < \underline{t} \quad (n \in \mathbb{N}_0), \tag{2.5}$$

where $\underline{t}$ is defined in (2.4), and "$<$" is the lexicographic ordering.

Additive functions with respect to the Rényi–Parry (RP) expansions

Let $F : I_1 \to \mathbb{C}$ (or $I_1 \to \mathbb{R}$). We say that F is additive with respect to the RP expansion, if

$$F(0) = 0, \quad \sum_{l=1}^{\infty} \sum_{a=1}^{k} |F(a\theta^l)| < \infty,$$

and for each $x \in [0,1)$,

$$F(x) = \sum_{n=1}^{\infty} F(\varepsilon_n(x)\theta^n).$$

In our paper [14] we tried to give all the continuous additive functions. We considered only the case $\theta \in \left(\frac{1}{2}, 1\right)$. The result obtained in [14] can be formulated for each $\theta \in (0,1)$. Namely the next assertion can be proved.

Theorem 1. *If $F : I_1 \to \mathbb{C}$ is a continuous additive function, then $F(j\theta^n) = jF(\theta^n)$ ($j \in \mathcal{A}_k$, $n \in \mathbb{N}$). The sequence $u_n \in \mathbb{C}$ ($n = 1, 2, \dots$) generates a continuous additive function such that $u_n = F(\theta^n)$, if and only if*

$$u_m = t_1 u_{m+1} + t_2 u_{m+2} + \dots \quad (m \in \mathbb{N}). \tag{2.6}$$

Let $G(z) = 1 - t_1 z - t_2 z^2 - \dots$. Then $G(z)$ is analytic in $|z| < 1$. Let ϱ be a root of $G(z) = 0$ in $|z| < 1$, and let m be its multiplicity, i.e. $G(z) = (z-\varrho)^m H(z)$, $H(\varrho) \neq 0$. It is clear that $u_\nu = \nu^j \varrho^\nu$ ($\nu \in \mathbb{N}$) generates continuous additive functions for $j = 0, \dots, m-1$. Such types of continuous additive functions, and even the elements of the closed space generated by such functions called to be elementary continuous additive functions, also.

We formulated our conjecture that every continuous additive function is elementary. We present this conjecture in a more general form in § 5.

Theorem 2. *Let F be a continuous additive function which is differentiable in some point. Then $F(x) = cx$, c is constant.*

This theorem is proved in the case $\theta \in \left(\frac{1}{2}, 1\right)$ in [14]. We shall return to this question in § 3.

§ 3. On a subspace in l_1

Let $\mathbb{C}^\infty$ be the space of sequences $\underline{c} = (c_0, c_1, \dots)$ where $c_0, c_1, \dots \in \mathbb{C}$. The shift operator $\sigma : \mathbb{C}^\infty \to \mathbb{C}^\infty$ is defined by $\sigma(\underline{c}) = (c_1, c_2, \dots)$.

Let $t_0 = 1$, $t_\nu \in \mathbb{C}$ ($\nu = 1, \dots$) be bounded, $\underline{t} = (t_0, t_1, \dots)$. Let l_1 be the linear space of the sequences $\underline{b} \in \mathbb{C}^\infty$ for which $\sum_\nu |b_\nu| < \infty$. The scalar product of a bounded sequence $\underline{c}$ and a $\underline{b} \in l_1$ is defined by

$$\underline{c}\,\underline{b} = \underline{b}\,\underline{c} = \sum_{\nu=0}^{\infty} b_\nu c_\nu.$$

Let

$$H_{\underline{t}} = \{\underline{b} \in l_1 \mid \sigma^l(\underline{b})\underline{t} = 0,\ l = 0,1,2,\dots\}. \tag{3.1}$$

It is clear that $H_{\underline{t}}$ is a closed linear subspace of l_1.

Let $H_{\underline{t}}^{(0)}(\subseteq H_{\underline{t}})$ be the set of those $\underline{b} \in H_{\underline{t}}$ for which

$$|b_\nu| \le C(\varepsilon, \underline{b})\dot{e}^{-\varepsilon\nu} \quad (\nu \ge 0) \tag{3.2}$$

holds with suitable constants $\varepsilon > 0$ and $C(\varepsilon, \underline{b}) < \infty$.

Let

$$R(z) = t_0 + t_1 z + \dots, \tag{3.3}$$

$|t_\nu|$ be bounded. Then $R(z)$ is regular in $|z| < 1$. Assume that ϱ is a root of R of multiplicity m, $|\varrho| < 1$. Then $b_\nu = \nu^j \varrho^\nu$ $(\nu = 0,1,\dots)$ satisfies $\sigma^l(\underline{b})\underline{t} = 0$ $(l = 0,1,\dots)$ if $j = 0,\dots,m-1$; therefore $\underline{b} \in H_{\underline{t}}^{(0)}$. Let $H_{\underline{t}}^{(e)}$ be the space of the finite linear combinations of such solutions, furthermore let $\overline{H}_{\underline{t}}^{(e)}$ be the closure of $H_{\underline{t}}^{(e)}$.

It is obvious that $\overline{H}_{\underline{t}}^{(e)} \subseteq H_{\underline{t}}$.

Theorem 3. *We have*

$$H_{\underline{t}}^{(e)} = H_{\underline{t}}^{(0)}.$$

Proof. It is clear that $H_{\underline{t}}^{(e)} \subseteq H_{\underline{t}}^{(0)}$. We shall prove that $H_{\underline{t}}^{(0)} \subseteq H_{\underline{t}}^{(e)}$, i.e. that if $\sigma^l(\underline{b})\underline{t} = 0$ $(l = 0,1,2,\dots)$, and (3.2) holds, then there exist $\varrho_1,\dots,\varrho_k$ suitable roots of $R(z)$, $|\varrho_s| \le e^{-\varepsilon}$ $(s = 1,\dots,k)$, such that

$$b_\nu = \sum_{s=1}^{k} p_s(\nu)\varrho_s^\nu \quad (\nu = 0,1,2,\dots),$$

p_s are polynomials, $\deg p_s \le m_s - 1$, where m_s is the multiplicity of the root ϱ_s of $R(z)$.

Let $\underline{b}$ be a solution of

$$\sigma^l(\underline{b})\underline{t} = 0 \quad (l = 0,1,2,\dots), \text{ satisfying (3.2).} \tag{3.4}$$

Let $\varepsilon_1 > 0$ be an arbitrary small number, $\varrho_1,\dots,\varrho_p$ be all the roots of $R(z)$ in the disc $|z| < e^{-\varepsilon} + \varepsilon_1$. Let m_s be the multiplicity of ϱ_s, i.e. $R^{(j)}(\varrho_s) = 0$, $j = 0,\dots,m_s - 1$, $R^{(m_s)}(\varrho_s) \ne 0$.

Let

$$B(z) = \sum_{\nu=0}^{\infty} \frac{b_\nu}{z^\nu}.$$

Then

$$R(z)B(z) = \left(\sum_{u=0}^{\infty} t_u z^u\right)\left(\sum_{v=0}^{\infty} b_v z^{-v}\right) = \sum_{r=-\infty}^{\infty} \kappa_r z^r,$$

where

$$\kappa_r = \sum_{\substack{u-v=r\\ u,v\geq 0}} t_u b_v.$$

Let us observe that $\kappa_r = 0$, if $r < 0$, and $\kappa_r = \mathcal{O}(1)$ for $r > 0$. Thus

$$R(z)B(z) = K(z) = \kappa_0 + \kappa_1 z + \dots,$$

the right hand side is regular in $|z| < 1$. It is clear that $B(z)$ is regular in $|z| > e^{-\varepsilon}$, and it is bounded in $|z| > e^{-\varepsilon} + \varepsilon_2$, where $0 < \varepsilon_2 < \varepsilon_1$ is an arbitrary constant.

Let $\varphi(z) = \prod_{j=1}^{p}(z - \varrho_j)^{m_j}$. Then $\frac{\varphi(z)}{R(z)}$ is bounded in $|z| \leq e^{-\varepsilon} + \varepsilon_2$. Thus

$$\varphi(z)B(z) = \frac{\kappa(z)\varphi(z)}{R(z)}$$

is bounded in $|z| \leq e^{-\varepsilon} + \varepsilon_2$, and it is $\mathcal{O}(|\varphi(z)|)$ as $z \to \infty$. It means that

$$\varphi(z)B(z) = q(z),$$

where q is a polynomial of degree $\leq \deg \varphi$. Thus $B(z) = \frac{q(z)}{\varphi(z)}$ whence Theorem 3 immediately follows. $\square$

Conjecture 1. $\overline{H}_{\underline{t}}^{(e)} = H_{\underline{t}}$.

A weaker form of it is

Conjecture 2. *Assume that* $R(z) \neq 0$ *in* $|z| < 1$. *Then* $H_{\underline{t}} = \{\underline{0}\}$.

§ 4. A variant of the Rényi–Parry expansion

Let $q > 1$, $\theta = 1/q$, $q \neq$ integer, $\mathcal{B} = \{\dots, q^n, \dots, q, 1, \theta, \theta^2, \dots\}$. Let $k = [q]$. We expand a nonnegative number x according to the following rules. If $x = 0$, then let $\varepsilon_\nu(0) = 0$, $(\nu \in \mathbb{Z})$. Let $x > 0$. Let n_1 be that integer for which $q^{n_1} \leq x < q^{n_1+1}$. Let $\varepsilon_\nu(x) = 0$ for $\nu \geq n_1 + 1$, $\varepsilon_{n_1}(x) = \left[\frac{x}{q^{n_1}}\right]$. Then $\varepsilon_{n_1}(x) \leq k$. Write $x = \varepsilon_{n_1}(x)q^{n_1} + x_1$. Then $x_1 < q^{n_1}$, and we continue this process. This expansion is the so called regular expansion of x.

Let Φ be the set of double infinite sequences $\underline{a} = \{a_\nu\}_{\nu=-\infty}^{\infty}$ for which $a_\nu \in \mathcal{A}_k$, and $a_\nu = 0$ if $\nu \ge n_0$ with a suitable integer n_0. Sometimes we write $\underline{a} = \dots a_n \dots a_1 a_0 a_{-1} \dots a_{-m} \dots$.

Let σ be the shift operator defined over Φ. Namely $\sigma(\underline{a}) = \underline{a}'$ if $a'_n = a_{n-1}$ $(n \in \mathbb{Z})$.

Let $\underline{\varepsilon}(x) = \{\varepsilon_\nu(x)\}_{\nu=-\infty}^{\infty} \in \Phi$ be the sequence of the digits of x.

It is clear that $\underline{\varepsilon}(\theta x) = \sigma(\underline{\varepsilon}(x))$, thus we can assert that a sequence $\underline{a} \in \varPhi$ represents an $x \ge 0$ in the form $\underline{a} = \underline{\varepsilon}(x)$ if $a_{n_0-1}a_{n_0-2}\dots = \underline{\tilde{a}}$ satisfies the Parry condition $\sigma^j(\underline{\tilde{a}}) < \underline{t}$ $(j = 0, 1, \dots)$, where n_0 is so defined that $a_\nu = 0$ for $\nu \ge n_0$. The quasi regular expansion is defined similarly.

We say that a function $F : [0, \infty) \to \mathbb{C}$ (or $\mathbb{R}$) is additive with respect to this expansion, if

$$F(0) = 0, \quad \sum_{-\infty}^{-1} \max_{j=1,\dots,k} |F(jq^l)| < \infty,$$

and for $x \in [0, \infty)$

$$F(x) = \sum_{n=-\infty}^{\infty} F(\varepsilon_n(x) q^n).$$

One can characterize the continuous additive functions as follows:

If F is continuous and additive, then $F(jq^n) = jF(q^n)$ $(j = 0, \dots, k)$. Let $\{e_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers, such that $\sum_{n=1}^{\infty} |e_n| < \infty$. Then the additive function F, generated by $F(0) = 0$; $F(jq^n) = jF(q^n) = je_{-n}$ $(j = 1, \dots, k;\ n \in \mathbb{Z})$ is continuous if and only if

$$e_n = t_1 e_{n+1} + t_2 e_{n+2} + \dots \tag{4.1$)_n$}$$

holds for every $n \in \mathbb{Z}$. $\underline{t} = t_1 t_2 \dots$ is the quasi regular expansion of 1.

Let us assume that $(4.1)_n$ holds for every $n > m$. Then there is a unique e_m for which $(4.1)_m$ holds true.

Consequently, *if F is a continuous additive function over $[0, 1)$, then it can be extended uniquely as a continuous additive function over $[0, \infty)$.*

§ 5. Smooth interval filling sequences

Let $\lambda_1 > \lambda_2 > \dots$ be a sequence of real numbers, $L_N = \lambda_{N+1} + \lambda_{N+2} + \dots$. Assume that $L_0 < \infty$. Let $k > 0$ be a fixed integer. *We say that λ_n is an interval filling sequence of order k if each $x \in [0, kL_0]$ can be written as $x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n$, where $\varepsilon_n \in \{1, \dots, k\}$ $(= \mathcal{A}_k)$.*

One can see that $\{\lambda_n\}$ is an interval filling sequence of order k if and only if

$$\lambda_n \le kL_n \quad (n = 1, 2, \dots) \tag{5.1}$$

holds. We define the digits $\varepsilon_n(x)$ of the regular expansion and the digits $\delta_n(x)$ of the quasi regular expansion of x according to the following rules.

Let $x \in [0, kL_0]$, and $\varepsilon_1(x)$ be the largest integer among the elements of $\mathcal{A}_k$, for which $0 \le x - \varepsilon_1(x)\lambda_1$ $(=: x_1)$.

Then $0 \le x_1 \le kL_1$. Let $\varepsilon_2(x)$ be the largest integer in $\mathcal{A}_k$ for which $0 \le x_1 - \varepsilon_2(x)\lambda_2$ $(=: x_2)$. Then $0 \le x_2 \le kL_2$.

Let us continue this process:

$$x = \varepsilon_1(x)\lambda_1 + \varepsilon_2(x)\lambda_2 + \dots.$$

The number 0 does not have quasi regular expansion. Let $0 < y \le kL_0$, and $\delta_1(y)$ be the largest integer among the elements of $\mathcal{A}_k$, for which $0 < y - \delta_1(y)\lambda_1 = y_1$. Then $0 < y_1 \le kL_1$, and we can continue this process.

We shall say that a function $F : [0, kL_0] \to \mathbb{C}$ is "additive" if $F(0) = 0$, $\sum_{n=0}^{\infty} \max_{j \le k} |F(j\lambda_n)| < \infty$, furthermore

$$F(x) = \sum_{n=1}^{\infty} F(\varepsilon_n(x)\lambda_n).$$

We proved: the additive function F is continuous if and only if

a) $F(j\lambda_n) = jF(\lambda_n)$ $(j = 1, \dots, k)$
   $F(0) = 0$,

b) $F(\lambda_n) = \sum_{l=1}^{\infty} \delta_{n+l}(\lambda_n) F(\lambda_{n+l})$

hold true.

This assertion is an easy modification of our theorem proved in [15].

We say that $\{\lambda_n\}$ is a smooth sequence if there exists an integer T for which $\lambda_{n+T} \le \frac{1}{2}\lambda_n$ $(n \in \mathbb{N})$. In [15] we proved: *Let $\{\lambda_n\}$ be a smooth, interval filling sequence of order k. Let F be a continuous additive function with respect to $\{\lambda_n\}$. Assume that F is differentiable on a set of positive Lebesgue measure. Then $F(x) = cx$ with some constant c.*

In the same paper we formulated our

Conjecture 3. *Let $\{\lambda_n\}$ be a smooth, interval filling sequence of order k. Let F be a continuous additive function with respect to $\{\lambda_n\}$. Assume that F is differentiable in one arbitrary chosen point. Then $F(x) = cx$, c is a constant.*

Let $0 < x < 1$, x be fixed, $\lambda_n = \log(1 + x^n)$.
Let $q = 1/x$, $k = [q]$.
It is clear that

$$1 + x < \prod_{n=2}^{\infty} (1 + x^n)^k,$$

and in general that

$$1 + x^h < \prod_{n=h+1}^{\infty} (1 + x^n)^k,$$

which guarantees that λ_n is an interval filling sequence of order k. It is a smooth sequence also, since $\frac{\lambda_n}{x^n} \to 1$ $(n \to \infty)$. Let $L_0 = \sum \lambda_n$. We have that if $y \in [0, kL_0]$, then

$$y = \sum e_n(y)\lambda_n,$$

i.e. if $u \in [1, e^{kL_0}]$, then it can be written as

$$u = \prod (1 + x^n)^{\varepsilon_n(u)}, \quad \varepsilon_n(u) \in \mathcal{A}_k.$$

Let G be a function defined on $[1, e^{kL_0}]$ taking positive real values such that $G(0) = 1$, $\sum_{n \geq 1} |\log G(1 + x^n)| < \infty$, and assume that

$$G(u) = \prod_{n=1}^{\infty} G(1 + x^n)^{\varepsilon_n(u)}.$$

We say that G is multiplicative. It is clear that G is multiplicative if and only if $\log G(u) = F(\log u)$ is additive, and G is continuous if and only if F is continuous. Thus, if G is continuous and multiplicative and it is differentiable on a set of positive measure, then $F(\log u) = c \log u$, i.e. $G(u) = u^c$, $c \in \mathbb{R}$.

§ 6. On completely additive functions

Let $\{\lambda_n\}$ be a sequence of positive numbers for which $\lambda_n \geq \lambda_{n+1}$ $(n \in \mathbb{N})$, and $L_n := \lambda_{n+1} + \lambda_{n+2} + \cdots < \infty$ $(n \in \mathbb{N})$ holds true.

We say that the function $F : [0, L_0] \to \mathbb{R}$ is completely additive, if $F(0) = 0$, and

$$F\Big(\sum \varepsilon_n \lambda_n\Big) = \sum_{n=1}^{\infty} \varepsilon_n F(\lambda_n) \tag{6.1}$$

for every $\varepsilon_n \in \{0, 1\}$, $n = 1, 2, \ldots$. We assume furthermore that $\sum |F(\lambda_n)| < \infty$.

In our joint paper with Z. Daróczy and T. Szabó [16] we proved that only the linear functions are completely additive.

We note that in our paper [16] we assumed that the sequence $\{\lambda_n\}$ is strictly monotonic, but this assumption is unimportant. The argumentation remains valid under the condition "$\lambda_n \geq \lambda_{n+1}$".

We think that a similar theorem can be proved under some condition on $\{\lambda_n\}$ weaker than "interval filling".

Conjecture 4. *Let $\{\lambda_n\}$ be a sequence of positive numbers for which $\lambda_n \geq \lambda_{n+1}$ $(n = 1, 2, \dots)$ such that $L_n = \lambda_{n+1} + \cdots < \infty$. Assume that $H = \big\{x \mid x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n,\ \varepsilon_n \in \{0, 1\}\big\}$ contains an interval. Assume furthermore, that $\{a_n\}_{n=1}^{\infty} \in l_1$ is such a sequence for which*

$$\sum \delta_n \lambda_n = 0 \quad \delta_n \in \{-1, 0, 1\}$$

always implies that

$$\sum_{n=1}^{\infty} \delta_n a_n = 0.$$

Then $a_n/\lambda_n =$ constant $(n = 1, 2, \dots)$.

Let $\lambda_n := \theta^n$, $\frac{1}{2} < \theta < 1$, $L_0 = \lambda_1 + \lambda_2 + \cdots = \frac{\theta}{1-\theta}$. Let t be defined on the set $\{-1, 0, 1\} : t(0) = 2$, $t(1) = t(-1) = 1$.

For some sequence $\varepsilon_1, \dots, \varepsilon_N \in \{-1, 0, 1\}$ let

$$\tau(\varepsilon_1, \dots, \varepsilon_N) = \prod_{j=1}^{N} t(\varepsilon_j).$$

We say that a sequence $\varepsilon_1, \dots, \varepsilon_N$ is continuable if

$$|\varepsilon_1 \theta + \cdots + \varepsilon_N \theta^N| \leq \theta^N L_0.$$

Let $m_N(\theta) = \sum \tau(\varepsilon_1, \dots, \varepsilon_N)$, where the summation is extended over the continuable sequences. One can easily see that

$$m_N(\theta) \geq c(4\theta)^N,$$

$c > 0$ constant.

Let $\mathcal{F}$ be a set of sequences $\underline{\varepsilon} = \varepsilon_1 \varepsilon_2 \dots,\ \varepsilon_\nu \in \{-1, 0, 1\}$. Let $\mathcal{F}_N$ be the set of those sequences $\delta_1, \dots, \delta_N \in \{-1, 0, 1\}^N$ which can be continued by suitable $\varepsilon_\nu \in \{-1, 0, 1\}$ $(\nu = N+1, N+2, \dots)$ such that $\delta_1 \dots, \delta_N \varepsilon_{N+1} \varepsilon_{N+2} \dots \in \mathcal{F}$. Let $\pi_N(\Theta, \mathcal{F}) = \#(\mathcal{F}_N)$.

Conjecture 5. *Let $\mathcal{F}$ be such a set of $\underline{\varepsilon} \in \{-1,0,1\}^{\mathbb{N}}$, for which $\pi_N(\theta \mid \mathcal{F}) \to \infty$ $(N \to \infty)$. Let $\underline{a} = a_1 a_2 \cdots \in l_1$ be such a sequence for which*

$$\sum \varepsilon_n a_n = 0$$

whenever $\underline{\varepsilon} \in \mathcal{F}$.
Then $a_n/\theta^n =$ constant.

§ 7. A modification of the Rényi–Parry expansion

Let $1 < q < 2$, $\theta = 1/q$, $\mathcal{A}_1 = \{0,1\}$. Let

$$\eta \in \left[\theta, \frac{\theta^2}{1-\theta}\right], \tag{7.1}$$

$I_\eta^{(1)} = [0, q\eta)$, $I_\eta^{(2)} = (0, q\eta]$.

Regular expansion of level η :
Let $x \in I_\eta^{(1)}$. Then

$$x = \varepsilon_1(x)\theta + \theta x_1, \quad \varepsilon_1(x) \in \mathcal{A}_1,\ x_1 \in I_\eta^{(1)};$$

$$\varepsilon_1(x) = \begin{cases} 0, & \text{if } x \in [0,\eta) \\ 1, & \text{if } x \in [\eta, q\eta) \end{cases}$$

$$x = \sum \varepsilon_n(x)\theta^n. \tag{7.2}$$

Quasi regular expansion of level η :
Let $x \in I_\eta^{(2)}$. Then

$$x = \delta_1(x)\theta + \theta x_1, \quad \delta_1(x) \in \mathcal{A}_1,\ x_1 \in I_\eta^{(2)};$$

$$\delta_1(x) = \begin{cases} 0, & \text{if } x \in (0,\eta] \\ 1, & \text{if } x \in (\eta, q\eta] \end{cases}$$

$$x = \sum \delta_n(x)\theta^n. \tag{7.3}$$

Remark. The regular expansion of level η of 1 is defined now as follows: Write $q\eta = 1 \cdot \theta + \theta z$. Then $0 < z < q\eta$. Let

$$z = t_2\theta + t_3\theta^2 + \dots$$

be the regular expansion of level η of z.

We say that $q\eta = t_1\theta + t_2\theta^2 + \dots$, $t_1 = 1$ is the regular expansion of level η of $q\eta$. Let $\underline{t} = t_1t_2\dots$. Let $\eta = \pi_1\theta + \pi_2\theta^2 + \dots$ be the regular expansion of η, $\underline{\pi} = \pi_1\pi_2\dots$. Observe that the quasi regular expansion of level η of η is $0\cdot\theta + t_1\theta^2 + t_2\theta^3 + \dots$. Thus $\underline{\varepsilon}(q\eta) = \underline{t}$, $\underline{\varepsilon}(\eta) = \underline{\pi}$, $\underline{\delta}(\eta) = 0\underline{t}$.

Let $\mathcal{E} := \{\underline{\varepsilon}(x) \mid x \in [0, q\eta)\}$.

Let furthermore $\mathcal{F}$ be the set of those sequences $\underline{f} = f_1f_2\dots \in \{0,1\}^\infty$ for which

(1) $\sigma^j(\underline{f}) < \underline{t}$ $(j = 0, 1, 2, \dots)$,

(2) if $f_\nu = 1$, then $\sigma^{\nu-1}(\underline{f}) = f_\nu f_{\nu+1}\cdots \geq \underline{\pi}$.

Theorem 4. *We have $\mathcal{E} = \mathcal{F}$.*

Proof. The relation $\mathcal{E} \subseteq \mathcal{F}$ is obvious. Let $x = \varepsilon_1(x)\theta + \dots$, $x \in I_\eta^{(1)}$. If $y_1, y_2 \in I_\eta^{(1)}$, $0 \leq y_1 < y_2 \leq q\eta$, then $\underline{\varepsilon}(y_1) < \underline{\varepsilon}(y_2)$, thus $\underline{\varepsilon}(x) < \underline{t}$. Since $x_n = \varepsilon_{n+1}(x)\theta + \dots < q\eta$, therefore $\sigma^n(\underline{\varepsilon}(x)) < \underline{t}$. If $\varepsilon_n(x) = 1$, then $x_{n-1} = \varepsilon_n(x)\theta + \dots \geq \eta$, consequently $\sigma^{n-1}(\underline{x}) \geq \underline{\pi}$. Thus $\mathcal{E} \subseteq \mathcal{F}$.

Let $\underline{f} \in \mathcal{F}$, $y = f_1\theta + f_2\theta^2 + \dots$. We shall prove that $y < q\eta$, and that, if $f_k = 1$, then $f_k\theta + \dots \geq \eta$. Hence it would follow that $\underline{\varepsilon}(y) = \underline{f}$. Let $y_h = f_{h+1}\theta + f_{h+2}\theta^2 + \dots$.

Let $f_j = t_j$ $(j = 1, \dots, k_1 - 1)$, $f_{k_1} = 0$, $t_{k_1} = 1$, where $k_1 = 1$ is allowed. There exists such a finite k_1. Furthermore, let $f_{k_1+j} = t_j$ for $j = 1, \dots, k_2 - 1$, $f_{k_2} = 0$, $t_{k_2} = 1$, and so on. We allow the choice $k_\nu = 1$. In this case $j = 1, \dots, k_\nu - 1$ is an empty condition.

Thus we have

$$\begin{aligned} y &= (t_1\theta + \dots + t_{k_1-1}\theta^{k_1-1}) + \theta^{k_1}(t_1\theta + \dots + t_{k_2-1}\theta^{k_2-1}) \\ &\quad + \theta^{k_1+k_2}(t_1\theta + \dots + t_{k_3-1}\theta^{k_3-1}) + \dots . \end{aligned}$$

If $t_k = 1$, then $t_k\theta + t_{k+1}\theta^2 + \dots \geq \eta$, and so

$$t_1\theta + \dots + t_{k-1}\theta^{k-1} \leq q\eta - \theta^{k-1}\eta = q\eta(1 - \theta^k).$$

Thus

$$y \leq q\eta(1 - \theta^{k_1}) + (q\eta)\theta^{k_1}(1 - \theta^{k_2}) + \dots = q\eta.$$

Since $q\eta - y = t_{k_1}\theta^{k_1} + t_{k_2}\theta^{k_1+k_2} + \dots > 0$, therefore $y < q\eta$.

The estimation from below is similar.

Let $f_j = \pi_j$ $(j = 1, \dots, k_1 - 1)$, $f_{k_1} = 1$, $\pi_{k_1} = 0$ ($k_1 = 1$ is allowed). Let $f_{k_1+j} = \pi_j$ $(j = 1, \dots, k_2 - 1)$, $f_{k_1+k_2} = 1$, $\pi_{k_2} = 0$ ($k_2 = 1$ is allowed), and so on.

If k is such an integer for which $\pi_k = 0$, then $\eta = \pi_1\theta + \dots + \pi_{k-1}\theta^{k-1} + \theta^{k-1}\xi$, $\xi < \eta$, and so

$$\pi_1\theta + \dots + \pi_{k-1}\theta^{k-1} > \eta(1 - \theta^{k-1}).$$

Therefore

$$y > \eta(1 - \theta^{k_1-1}) + \eta\theta^{k_1-1}(1 - \theta^{k_2-1}) + \dots > \eta.$$

Hence the assertion easily follows. □

It is highly probable, that the following assertion is true:

Conjecture 6. *Let $\eta_1 < \eta_2$, $\eta_1, \eta_2 \in [\theta, \theta L]$. Let $\mathcal{H}(\eta_1, \eta_2)$ be the set of those $x \in [0, L]$ for which their expansions of level η_1 and of level η_2 are the same. Then the Lebesgue measure of $\mathcal{H}(\eta_1, \eta_2)$ is zero.*

§ 8. Univoque numbers in numeration systems generated by Rényi–Parry expansions

Let $\theta \in (0, 1)$, $q = 1/\theta$, $q \neq$ integer, $k = [q]$, $\mathcal{A} = \{0, \dots, k\}$. We shall use the notation: $\overline{j} = k - j$ $(j \in \mathcal{A})$.

Let $1 = l_1\theta + l_2\theta^2 + \dots$ be the quasi regular expansion of 1.

The sequence $\{G_n\}_{n=1}^{\infty}$, of positive integers is defined by recursion:

$$G_1 = 1, \quad G_{n+1} = l_1 G_n + \dots + l_n G_1 + 1 \quad (n \in \mathbb{N}). \tag{8.1}$$

Let

$$H_m = G_1 + \dots + G_m. \tag{8.2}$$

Then every $n \leq G_{m+1} - 1$ can be expanded by the greedy algorithm as

$$n = \varepsilon_m G_m + \dots + \varepsilon_1 G_1. \tag{8.3}$$

Furthermore the digits $\varepsilon_\nu \in \mathcal{A}$.

Such kind of numeration systems have been investigated by Pethő and Tichy [6], and Grabner and Tichy [7].

One can prove easily that the sequence $(\varepsilon_m, \dots, \varepsilon_1)$ represents an integer $n < G_{m+1}$ in the form (8.3) if and only if $\varepsilon_m\theta + \dots + \varepsilon_1\theta^m = \eta < 1$, and if

$$\varepsilon_1 \dots \varepsilon_m 0^\infty \quad \text{is the regular expansion of} \quad \eta.$$

Corollary 1. *The number of regular sequences of type $\varepsilon_1 \dots \varepsilon_N 0^\infty$ is G_{N+1}.*

Let us define $H(z) = 1 - l_1 z - \dots$. Then $H(\theta) = 0$, H is regular in $|z| < 1$, and $H(z) \neq 0$ in $|z| \leq \theta + \varepsilon$, if $z \neq \theta$, and $\varepsilon > 0$ is sufficiently small. From (8.1) we get easily that

$$G_N = cq^N + \mathcal{O}(\Lambda^N), \quad \Lambda = \frac{q}{1+\varepsilon_1} \tag{8.4}$$

$\varepsilon_1 > 0$ is a suitable constant.

Let $U(z) = G_1 + G_2 \cdot z + G_3 \cdot z^2 + \dots$. Then

$$U(z)H(z) = G_1\dot{1} + (G_2 - l_1 G_1)z + (G_3 - l_1 G_2 - l_2 G_1)z^2 + \dots = \frac{1}{1-z},$$

and so

$$\begin{aligned} U(z) &= \frac{1}{H(z)(1-z)} = \frac{1}{(z-\theta)}\eta(z), \\ \eta(z) &= \frac{z-\theta}{H(z)(1-z)}. \end{aligned}$$

Since $H'(\theta) \neq 0$, therefore $\eta(z)$ is regular in $|z| \leq \theta + \varepsilon$. Hence (8.4) immediately follows.

Let $\mathcal{A}^*$ be the set of finite sequences (words) over $\mathcal{A}$, $\Phi : \mathbb{N}_0 \to \mathcal{A}^*$ be the mapping defined as follows: $\Phi(0) =$ empty word $= \Lambda$; if $n \in [G_m, G_{m+1} - 1)$, then $\Phi(n) = \varepsilon_m \dots \varepsilon_1$, according to (8.3). Let $\mathcal{R} = \{\Phi(n) \mid n \in \mathbb{N}_0\}$ be the set of the so called regular sequences.

As we mentioned earlier, the following assertion holds true.

Lemma 1. *A non-empty sequence $\varepsilon_m \dots \varepsilon_1 \in \mathcal{A}^m$ belongs to $\mathcal{R}$, if and only if $\varepsilon_m \neq 0$, and*

$$\varepsilon_j \dots \varepsilon_1 \leq l_1 \dots l_j \quad (j = 1, \dots, m). \tag{8.5}$$

Characterization of the univoque numbers

Let $\mathcal{E}$ be the set of those $n \in \mathbb{N}_0$ which have only one expansion as $n = \sum e_j G_j$, $e_j \in \mathcal{A}$. We say that $\mathcal{E}$ is the set of univoque numbers.

The lazy algorithm is defined as follows. Let $n \in (kH_{r-1}, kH_r]$, and δ_r be the smallest integer for which $n_1 := n - \delta_r G_r \leq kH_{r-1}$. Clearly, $\delta_r \in \mathcal{A}$. Continue this process with n_1 instead of n, and iterate. Finally we obtain

$$n = \delta_r G_r + \dots + \delta_1 G_1.$$

Let t be the smallest integer for which $l_t \neq k$. Then $l_1 = \dots = l_{t-1} = k$, consequently

$$G_1 = 1, \quad G_2 = (k+1), \ \dots, \quad G_t = (k+1)^{t-1}. \tag{8.6}$$

This implies easily that n is univoque, if $n < G_{t+1}$.

We can see also that $kH_j = G_{j+1} - 1$ $(j = 1, \dots, t-1)$, $G_{t+1} \leq kH_t$, $G_{s+1} < kH_s$, if $s > t$.

Let $m > t$, n be an element of $\mathcal{E}$ in the interval $[G_m, G_{m+1} - 1)$. If $n \in [G_m, kH_{m-1}]$, then $\varepsilon_m(n) \geq 1$, $\delta_m(n) = 0$, where ε_m is the coefficient of G_m in the regular expansion of n, and δ_m is the same in the lazy expansion of n. Continuing, n should belong to one of the intervals:

$$(kH_{m-1}, 2G_m), \quad (G_m + kH_{m-1}, 3G_m), \dots, ((k-1)G_m + kH_m, G_{m+1}), \tag{8.7}$$

where some interval above is considered to be empty, if the left-end point is larger than or equal to the right-end point.

Example. It is easy to see that if $\theta \in \left(\frac{\sqrt{5}-1}{2}, 1\right)$, then all of the intervals in (8.7) are empty at least for every large m. As a conclusion we obtain that $\mathcal{E}$ is a finite set.

Let $n \in \mathcal{E}$, $n \in [G_m, G_{m+1})$, $\varepsilon_m(n) \neq 0$ its first digit $(n = \varepsilon_m(n)G_m + \dots + \varepsilon_1(n)G_1)$. Then $n_1 := n - \varepsilon_m(n)G_m \in \mathcal{E}$ as well. Since $kH_m > n > kH_{m-1}$, therefore

$$kH_m - n = (k - \varepsilon_m(n))G_m + \dots + (k - \varepsilon_1(n))G_1 \tag{8.8}$$

is the regular expansion of $kH_m - n$, if we ignore on the right hand side the formal

$$\overline{\varepsilon}_m(n)G_m + \dots + \overline{\varepsilon}_{m-h}(n)G_{m-h}$$

sum, if $\overline{\varepsilon}_m(n) = \dots = \overline{\varepsilon}_{m-h}(n) = 0$.

Hence we obtain that if $n \in \mathcal{E}$, $n \in (G_m, G_{m+1})$, then

$$\begin{cases} \varepsilon_j(n) \dots \varepsilon_1(n) \leq l_1 \dots l_j \\ \overline{\varepsilon}_j(n) \dots \overline{\varepsilon}_1(n) \leq l_1 \dots l_j \\ (j = 1, \dots, m) \end{cases} \tag{8.9}$$

should be satisfied.

Let $\underline{e} = e_m \dots e_1 \in \mathcal{A}^m$, $e_m \neq 0$ be such a sequence for which

$$\begin{cases} e_j \dots e_1 \leq l_1 \dots l_j \\ \overline{e}_j \dots \overline{e}_1 \leq l_1 \dots l_j \\ (j = 1, \dots, m) \end{cases} \tag{8.10}$$

are satisfied.

Let

$$n = e_m G_m + \cdots + e_1 G_1. \tag{8.11}$$

We shall prove that $n \in \mathcal{E}$.

(8.11) is the regular expansion of n. Consider

$$kH_m - n = \overline{e}_m G_m + \cdots + \overline{e}_1 G_1. \tag{8.12}$$

From the second inequality of (8.10) it is clear that the right hand side of (8.12) is the regular expansion of (8.12), and so (8.11) is also the lazy expansion of n.

We proved the following.

Theorem 5. *The sequence $e_m \dots e_1 (\in \mathcal{A}^m)$, $e_m \neq 0$ generates a univoque number by* (8.11) *if and only if the relations* (8.10) *hold true.*

Some simple cases:

8.1. *The case $\theta \in (\theta_1, 1)$, where $\theta_1 := \frac{\sqrt{5}-1}{2}$.*
Then there exists some integer s such that $l_1 = \cdots = l_{2s-1} = 1$, $l_2 = \cdots = l_{2s} = 0$, $l_{2s+1} = 0$. Here, as everywhere in this paper, l_ν are the digits in the quasi regular expansion of 1 with the base θ.
We can see that the sequence $e_m \dots e_1 (\in \mathcal{A}_m)$, $e_m \neq 0$ can not generate univoque number n ($n = e_m G_m + \cdots + e_1 G_1$) if $m \geq 2s+2$. This is clear, since (8.10) can not be satisfied.
We have $\mathcal{F} = \{1;\ (10)^h 1\ (h = 0, \dots, s-1);\ (10)^h\ (h = 1, \dots, s)\}$.

8.2. *The case $\theta = \theta_1 = \frac{\sqrt{5}-1}{2}$*
Then $\mathcal{F} = \{1, (10)^s, (10)^s 1,\ s = 1, 2, \dots\}$.
The proof is obvious.

8.3. *The case $k \geq 2$, $\theta_k < \theta < 1/k$, where $\theta_k = \frac{\sqrt{k^2+4}-k}{2}$.*
In this case, if $l_1 = k$, then $l_2 = 0$. If $l_u = k$, then $l_{u+1} = 0$ due to the Parry condition, that $\sigma^{u-1}(\underline{l}) < \underline{l}$ $(u = 2, 3, \dots)$. Since $\underline{l} \neq (k0)^\infty$, therefore

$$l_1 \dots l_{2s+1} = (k0)^s l_{2s+1}, \quad l_{2s+1} < k.$$

First we observe that $e = e_m \dots e_1$ with $e_m = k$ cannot be in $\mathcal{F}$, if $m \geq 2s+2$. Indeed, $e_m = k$ implies $e_{m-1} = 0 = \overline{k}$, $\overline{e}_{m-2} = 0$, i.e. $e_{m-2} = k$, and so on. Thus $\underline{e} = k0k0\dots$, but this contradicts to (8.10), if $m \geq 2s+2$.
Let $\underline{e} \in \mathcal{F}$ be a sequence of length m with leading term $e_m \neq 0, k$. Let h be the largest index for which $e_h = 0$. Then $\overline{e}_h \dots \overline{e}_1 \in \mathcal{R}$, $\overline{e}_h = k$, therefore $h \leq 2s+1$.

Theorem 6. *Let $A_0 = \{1, \dots, k-1\}$, $\tilde{A}_0$ be the set of finite words over A_0, the empty word is excluded. There exists a finite set*

$$\mathcal{K} = \{\beta_l = \varepsilon_{u_l} \dots \varepsilon_1, \quad \varepsilon_{u_l} = k\}$$

such that

$$\mathcal{F} := \{0\} \cup \mathcal{K} \cup \tilde{A}_0 \cup (\tilde{A}_0\mathcal{K}) \cup (\tilde{A}_0\overline{\mathcal{K}})$$

where $\overline{\mathcal{K}} = \{\overline{\beta} \mid \beta \in \mathcal{K}\}$.
Furthermore, the length of the β_l in $\mathcal{K}$ are not larger than $2s+1$.

The theorem is obvious.

8.4. *The case $k \geq 2$, $\theta = \theta_k$*
Then, obviously

$$\mathcal{F} = \tilde{A}_0\mathcal{H},$$
$$\mathcal{H} = \{(k0)^s, (k0)^s k, (0k)^s, (0k)^s 0, \ s = 0, 1, 2, \dots\},$$

where $A_0, \tilde{A}_0$ are defined in 8.3.

8.5 *The case $l_1 \dots l_p\, l_{p+1} \dots l_{p+q} = k^p 0^q$, $q \geq p+1$.*
If $\underline{e} = e_m \dots e_1 \in \mathcal{F}$, and $e_m = e_{m-1} = \dots = e_{m-p+1} = k$, then $m \leq 2p$.
Let $\mathcal{F}_1$ be the set of those regular sequences with leading digit $e_m \neq 0$, in which no p consecutive k, and no p consecutive 0 occur. Then $\mathcal{F}_1 \subseteq \mathcal{F}$, and

$$\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_1\mathcal{T},$$

where

$$\mathcal{T} = \{k^p 0^s, 0^p k^s, \quad s = 0, \dots, p\}.$$

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