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Stability of a quadratic functional equation on semigroups

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Dedicated to Professor Zoltán Daróczy on his seventieth birthday

Abstract. The stability problem of the functional equation of the form

$$f(x+2y)+f(x)=2f(x+y)+2f(y),$$

is investigated. We prove that if the norm of the difference between left-hand side and right-hand side of the equation is majorized by a function ω of two variables having some standard properties then there exists a unique solution F of our equation and the norm of differences between F and the given function f is controlled by a function depending on ω.

1. Introduction

In the last years the stability of quadratic functional equations are widely investigated. The most important quadratic functional equation has the following form

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1}$$

First stability result concerning this equation was obtained by P. W. Cholewa [2]. The theorem states that if g is a function transforming an abelian group G into a Banach space X and satisfying the inequality

$$\|g(x+y)+g(x-y)-2g(x)-2g(y)\| \le \delta, \quad x,y \in G, \tag{2}$$

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where δ is a given nonnegative constant, then there exists a unique quadratic function $f : G \to X$ such that

$$\|f(x) - g(x)\| \leq \frac{\delta}{2}, \quad x \in G. \tag{3}$$

In 2006 W. Fechner [3] considered a modified inequality (2) where on the right-hand-side of (2) $\delta = \delta(x, y)$ was a function satisfying some additionally conditions and he obtained estimation like (3) depending on this function δ.

Some authors investigated the stability of functional equations which are equivalent to equation (1) ([1], [7]). It make a sense because there are known examples of some equivalent functional equations: one of them stable in the sense of Hyers and Ulam and another one not stable in this sense. A nontrivial pair of functional equations of this type is Jensen functional equation (which is stable on the interval $(0, 1)$ [4]) and Hosszú functional equation (which is nonstable on the interval $(0, 1)$ [8]). The case of Hosszú functional equation is interesting also therefore, that it is stable on the space of all real numbers [6].

The natural domains to the considerations of quadratic functional equation are groups. But if we put $x + y$ instead of x in (1) we obtain

$$f(x + 2y) + f(x) = 2f(x + y) + 2f(y), \tag{4}$$

and this equation can be investigated for functions defining on semigroups. Moreover, it is easily seen that equations (1) and (4) are equivalent in the class of functions acting from a group to an another one. The equation (4) may be treated as a quadratic functional equation on a semigroup. In this paper we will prove the stability of functional equation (4).

2. Results

In our two theorems on the stability of functional equation (4) we will assume that $(S, +)$ is an abelian semigroup, X is a Banach space and g is a function acting from S into X. Let us put $S^* = S \setminus \{0\}$ and let $\omega : S^* \times S^* \to \mathbb{R}$ be a function satisfying the following assumptions

$$\lim_{k\to\infty} \rho^{-2k}\omega(\rho^k x, \rho^k y) = 0, \quad x, y \in S^*;$$

$$\sum_{k=0}^{\infty} \rho^{-2k}\omega((\rho^k u, \rho^k v) \text{ is convergent for all} \tag{5}$$

$$(u, v) \in \{(x, x),\ (x, 2x),\ (2x, x),\ (3x, x)\}, \quad x \in S^*,$$

where $\rho \in \{2, \frac{1}{2}\}$. For the simplicity we define a function φ by the following way

$$\varphi(x,y) = \frac{1}{2}[\omega(x,y) + \omega(x,2y) + \omega(3x,y) + 2\omega(2x,y)], \quad x,y \in S^*$$

and we observe that

$$\lim_{k\to\infty} \rho^{-2k}\varphi(\rho^k x, \rho^k y) = 0, \quad x,y \in S^*;$$

$$\sum_{k=0}^{\infty} \rho^{-2k}\varphi(\rho^k x, \rho^k x) \quad \text{is convergent for all} \quad x \in S^*. \tag{5'}$$

From a lemma which was originally proved in [5] for functions defined on a group, but its proof is literally the same in our case, follows the following lemma.

Lemma. *Let $g : S \to X$ be a function satisfying the inequality*

$$\left\| \sum_{i=1}^{r} \alpha_i g(\gamma_i x + \delta_i y) \right\| \le \varphi(x,y), \quad x,y \in S^*, \tag{6}$$

where r, γ_i, δ_i, $i \in \{1,\dots,r\}$ are given positive integers $\alpha_i, i \in \{1,\dots,r\}$ are given real constants. If there exist constants $K > 0$ and $\rho > 0$ such that

$$\begin{gathered}\left\|\rho^{-2(n+1)}g(\rho^{n+1}x) - \rho^{-2n}g(\rho^n x)\right\| \le K\rho^{-2n}\varphi(\rho^n x, \rho^n x),\\ x \in S^*, \quad n \in \mathbb{N}\cup\{0\}.\end{gathered} \tag{7}$$

and φ satisfies conditions (5') then for every $x \in S^*$ the sequence $(\rho^{-2n}g(\rho^n x))_{n\in\mathbb{N}}$ is convergent to a function $f : S^* \to X$ fulfilling the equation

$$\sum_{i=1}^{r} \alpha_i f(\gamma_i x + \delta_i y) = 0, \quad x,y \in S^* \tag{8}$$

and the estimation

$$\|g(x) - f(x)\| \le K\sum_{n=0}^{\infty} \rho^{-2n}\varphi(\rho^n x, \rho^n x), \quad x \in S^*. \tag{9}$$

Theorem 1. *Let S be an abelian semigroup, X be a Banach space, and let $g : S \to X$ be a function satisfying the following inequality*

$$\|g(x+2y) + g(x) - 2g(x+y) - 2g(y)\| \le \omega(x,y), \quad x,y \in S^*, \tag{10}$$

where $\omega : S^* \times S^* \to [0,\infty)$ fulfilled conditions (5) with $\rho = 2$. Then there exists a unique function $f : S \to X$ satisfying equation (4) and the estimation

$$\|f(x) - g(x)\| \le \sum_{k=0}^{\infty} \frac{1}{4^{k+1}}\varphi(2^k x, 2^k x), \quad x \in S^*. \tag{11}$$

Proof. According to (10) we obtain

$$\|g(3x) - 2g(2x) - g(x)\| \le \omega(x,x), \quad x \in S^*;$$
$$\|g(5x) + g(3x) - 2g(4x) - 2g(x)\| \le \omega(3x,x), \quad x \in S^*;$$
$$\|-g(5x) - g(x) + 2g(3x) + 2g(2x)\| \le \omega(x,2x), \quad x \in S^*,$$

whence

$$\|-2g(4x) - 4g(x) + 4g(3x)\| \le \omega(x,x) + \omega(x,2x) + \omega(3x,x), \quad x \in S^*. \tag{12}$$

Moreover, putting $2x$ instead of x, and x instead of y in (10) we get

$$\|2g(4x) + 2g(2x) - 4g(3x) - 4g(x)\| \le 2\omega(2x,x), \quad x \in S^*. \tag{13}$$

It follows from (12) and (13) that

$$\|g(2x) - 4g(x)\| \le \varphi(x,x), \quad x \in S^*, \tag{14}$$

and, consequently,

$$\left\| \frac{1}{4^{n+1}} g(2^{n+1}x) - \frac{1}{4^n} g(2^n x) \right\| \le \frac{1}{4^{n+1}} \varphi(2^n x, 2^n x), \quad x \in S^*.$$

On account of Lemma there exists a function $f : S \to X$ satisfying equation (4) and estimation (11).

Let $f_1 : S \to X$ be a function satisfying equation (4) and the estimation

$$\|g(x) - f_1(x)\| \le \sum_{k=0}^{\infty} \frac{1}{4^k} \varphi(2^k x, 2^k x), \qquad x \in S^*. \tag{15}$$

Quite similar as in the proof of (14), using (4) instead of (10), one can prove that

$$f(2x) = 4f(x) \text{as well as } f_1(2x) = 4f_1(x), \quad x \in S^*.$$

Now for arbitrary positive integer m we have

$$\begin{aligned} \|f(x) - f_1(x)\| &= \frac{1}{4^m} \|f(2^m x) - f_1(2^m x)\| \\ &\le \frac{1}{4^m} \left[\|f(2^m x) - g(2^m x)\| + \|g(2^m x) - f_1(2^m x)\| \right] \\ &\le \frac{2}{4^m} \sum_{k=0}^{\infty} \frac{1}{4^k} \varphi(2^{m+k} x, 2^{m+k} x) = 2 \sum_{k=m}^{\infty} \frac{1}{4^k} \varphi(2^k x, 2^k x) \end{aligned}$$

and hence $f(x) = f_1(x)$, $x \in S^*$. □

In a similar way one can prove the following theorem.

Theorem 2. *Let S be an abelian semigroup uniquely divisible by two and let X be a Banach space. If $g : S \to X$ is a function satisfying the following inequality*

$$\|g(x + 2y) + g(x) - 2g(x + y) - 2g(y)\| \leq \omega(x, y), \quad x, y \in S^*, \tag{16}$$

where $\omega : S^* \times S^* \to [0, \infty)$ fulfilled conditions (5) with $\rho = \frac{1}{2}$ then there exists a unique function $g : S \to X$ satisfying equation (4) and the estimation

$$\|f(x) - g(x)\| \leq \sum_{k=0}^{\infty} 4^k \varphi(2^{-k-1}x, 2^{-k-1}x), \quad x \in S^*.$$

Proof. Similarly as in the proof of Theorem 1 we obtain

$$\|g(2x) - 4g(x)\| \leq \varphi(x, x), \quad x \in S^*.$$

Putting $\frac{x}{2}$ instead of x we get

$$\left\| g(x) - 4g\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \quad x \in S^*,$$

whence

$$\left\| 4^{k+1} g\left(\frac{x}{2^{k+1}}\right) - 4^k g\left(\frac{x}{2^k}\right)\right\| \leq 4^k \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right), \quad x \in S^*.$$

The rest of the proof runs "mutatis mutandis" as in the proof of Theorem 1. □

Remark. Let $\theta \geq 0$ and let $\omega(x, y) = \theta(\|x\|^\alpha + \|y\|^\alpha)$ or $\omega(x, y) = \theta\|x\|^\beta\|y\|^\beta$, Theorems 1 and 2 can be applied particularly for these functions with $\alpha \neq 2$ and $\beta \neq 1$.

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