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On the embeddability of commuting continuous injections in iteration semigroups

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Dedicated to Professor Zoltán Daróczy on the occasion of his 70th birthday

Abstract. Let $f, g : (a,b) \to (a,b)$ be commuting continuous injections, iteratively incommensurable and such that $f < g < \mathrm{id}$. We consider the problem of the embeddability of the mappings f and g in an iteration semigroup (semiflow). Among others we show that if f and g are continuously differentiable in an open interval $(a, a+\delta)$ and $f' > 0$, $g' > 0$ are of finite variation in $(a, a+\delta)$, then there exists a unique continuous iteration semigroup $\{h^t : t \geq 0\}$ of continuous functions such that $h^1 = f$ and $g \in \{h^t : t \geq 0\}$. We also consider the problem of the embeddability of convex and concave functions.

1. Introduction

Let $\{h^t : I \to I : t \geq 0\}$ be a continuous iteration semigroup (semiflow) on an open interval $I \subset \mathbb{R}$. It is obvious that the functions $f = h^1$ and $g = h^s$ commute for every $s \geq 0$. If $s \notin \mathbb{Q}$ and f has no fixed points then $f^n(x) \neq g^m(x)$ for all $n, m \in \mathbb{N}$ and $x \in I$. We consider the inverse problem - when the mappings with the above given properties are embeddable in a continuous iteration semigroup? The answer to this question depends on the regularity of the functions f and g. The problem when f and g have a common fixed point and have a suitable regularity at this point has been considered by many authors (see eg. [5], p. 214).

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Here we present another approach to the problem. Namely, we consider the regularity on open intervals without fixed points. The received assumptions leads to results different than previous one. This problem was considered in paper [8]. Here, among others, we generalize some results of this paper.

2. Preliminaries

Let $I = (a, b) \subset \mathbb{R}$ be an open interval. On given functions f and g we assume general hypothesis

$$(H) \qquad f, g : I \mapsto I \quad \text{are continuous, strictly increasing and such that}$$

$$f < g < \mathrm{id} \quad \text{and} \quad f \circ g = g \circ f.$$

Let us start with reminding some basic facts on commuting functions. It is known that for every $x \in (a, b)$ there exists a unique sequence $\{m_k(x)\}$ of positive integers such that

$$f^{m_k(x)+1}(x) \leq g^k(x) < f^{m_k(x)}(x)$$

and there exists the finite limit

$$\lim_{k\to\infty} \frac{m_k(x)}{k} =: s(f, g)$$

and this limit does not depend on x (see [9]). This limit $s(f, g)$ is said to be the iterative index of f and g.

Index $s(f, g) \notin \mathbb{Q}$ if and only if f and g are iteratively incommensurable, i.e. for every $x \in I$ and every $n, m \in \mathbb{N}$ $f^n(x) \neq g^m(x)$.

One can show (see [3], [9]) that

$$s(f, g) = inf \left\{ \frac{n}{m} : n, m \in \mathbb{N}, f^n < g^m \right\} \tag{1}$$

and

$$s(g, f) = \frac{1}{s(f, g)}. \tag{2}$$

Define

$$\mathcal{N}(x) := \{(n, m) \in \mathbb{N} \times \mathbb{N} : f^n(x) \in g^m(I)\}$$

and

$$\mathcal{N} := \{(n, m) \in \mathbb{N} \times \mathbb{N} : f^n[I] \subset g^m[I]\}.$$

It is obvious that if $x < y$, then $\mathcal{N} \subset \mathcal{N}(x) \subset \mathcal{N}(y)$ and in the case where f and g are bijections $\mathcal{N} = \mathcal{N}(x) = \mathbb{N} \times \mathbb{N}$.

Lemma 1. *If $s(f,g) \notin \mathbb{Q}$ and f is not surjective, then $(n,m) \in \mathcal{N}$ if and only if $s(f,g) < \frac{n}{m}$.*

Proof. Let $(n,m) \in \mathcal{N}$. Then $f^n[I] \subset g^m[I]$. Iteratively incommensurability of f and g implies that $f^n < g^m$. Hence, by (1),

$$s(f,g) < \frac{n}{m}.$$

Conversely, let $s(f,g) < \frac{n}{m}$. Suppose, in the contrary, that the assertion is not true. Then, again by incommensurability of f and g, we get $g^m < f^n$ and, in consequence, by (1),

$$s(g,f) < \frac{m}{n}.$$

By (2) we have $\frac{n}{m} < s(f,g)$ which contradicts the assumption. The proof is finished. $\square$

Define

$$L(f,g) := \{g^{-m} \circ f^n(x) : (n,m) \in \mathcal{N}(x)\}^d,$$

where A^d denotes the set of all limit points of A.

Proposition 1 (see [4]). *The set $L(f,g)$ does not depend on the choice of x and $cl(L(f,g) \setminus \{b\})$ is either a Cantor set, i.e. a perfect and nowhere dense set or $cl(L(f,g) \setminus \{b\}) = [a,b]$. In the case when f and g are bijections*

$$L(f,g) = \{g^m \circ f^n(x) : n,m \in \mathbb{Z}\}^d.$$

Suppose $b < \infty$. Let $b' > b$ and $J := (a,b')$. Let f and g satisfy hypothesis (H) and $f(b-) := \lim_{x\to b^-} f(x) < b$. Then for every homeomorphic extension $\overline{f}$ of f onto J such that $\overline{f}(x) < x$, $x \in J$ there exists a unique homeomorphic extension $\overline{g}$ of g mapping J onto itself such that $\overline{g} \circ \overline{f} = \overline{f} \circ \overline{g}$. Moreover, $\overline{g}(x) < x$ for $x \in J$ and $s(\overline{f}, \overline{g}) = s(f,g)$

Proposition 2 (see [4]). *Let f, g satisfy hypothesis (H) on I and $s(f,g) \notin \mathbb{Q}$. Then for every interval $J = (a,b')$, where $b' > b$ and every homeomorphic extension $\overline{f}$ of f without fixed points, mapping J onto itself*

$$L(\overline{f}, \overline{g}) \cap [a,b) = L(f,g) \setminus \{b\}.$$

Let us consider the following system of inequalities

$$\begin{cases} \varphi(f(x)) \le \varphi(x) + 1 \\ \varphi(g(x)) \ge \varphi(x) + s \end{cases}, \quad x \in I \tag{3}$$

Proposition 3 (see [4])**.** *Let f, g satisfy hypothesis (H) and $s(f,g) \notin \mathbb{Q}$.*

(i) *If $s = s(f,g)$ then system* (3) *has a unique up to an additive constant continuous solution. This solution is decreasing and satisfies the system of Abel equations*

$$\begin{cases} \varphi(f(x)) = \varphi(x) + 1 \\ \varphi(g(x)) = \varphi(x) + s(f,g) \end{cases}, \quad x \in I. \tag{4}$$

The solution φ is invertible if and only if $\operatorname{Int} L(f,g) \neq \emptyset$.

(ii) *If $s > s(f,g)$ then system* (3) *has no continuous solutions.*

(iii) *If $s < s(f,g)$ and* $\operatorname{Int} L(f,g) \neq \emptyset$, *then system* (3) *has infinitely many continuous and strictly decreasing solutions. They depend on an arbitrary function.*

Definition 1. A one parameter family $\{h^t : t \geq 0\}$ of continuous functions $h^t : I \to I$ such that $h^t \circ h^s = h^{t+s}$, $t, s \geq 0$ is said to be an iteration semigroup. If for every $x \in I$ the mapping $t \to h^t(x)$ is continuous then an iteration semigroup is said to be continuous.

Remark 1 (see [8])*.* If $f : I \to I$ is continuous, strictly increasing, not surjective and such that $f < \mathrm{id}$, then every iteration semigroup $\{h^t : t \geq 0\}$ such that $h^1 = f$ is continuous.

Definition 2. The mappings f and g are embeddable in an iteration semigroup $\{h^t : t \geq 0\}$ if there exist $r, s \geq 0$ such that $f = h^r$ and $g = h^s$.

Remark 2 (see [8])*.* If $\{h^t : t \geq 0\}$ is a continuous iteration semigroup such that at least one iterate has no fixed points, then

$$h^t(x) = \varphi^{-1}(\varphi(x) + t), \quad x \in I, \quad t \geq 0, \tag{5}$$

where $\varphi : I \to \mathbb{R}$ is a continuous injection.

3. Results

We can now formulate our results.

Theorem 1. *Let f, g satisfy hypothesis (H) and $s(f,g) \notin \mathbb{Q}$. If $\{h^t : t \geq 0\}$ is a continuous iteration semigroup such that*

$$h^r \leq f < g \leq h^s \quad \textit{for some} \;\; r, s \geq 0, \tag{6}$$

then $\frac{s}{r} \leq s(f,g)$. *If* $\frac{s}{r} = s(f,g)$ *then* $f = h^r$ *and* $g = h^s$. *Moreover,*

$$h^t(x) = \lim_{k\to\infty} g^{-m_k} \circ f^{n_k}(x), \quad x \in I,\ t \geq 0 \tag{7}$$

for every sequence $\{(n_k, m_k)\}$ *such that* $\lim_{k\to\infty} rn_k - sm_k = t$.

Proof. Let $\{h^t : t \geq 0\}$ be a continuous iteration semigroup satisfying (6). Since $h^r \leq f$ we infer, by (H), that h^r has no fixed points in I. By Remark 2 there exists a continuous and strictly decreasing function $\varphi : I \to \mathbb{R}$ such that $\lim_{x\to a+} \varphi(x) = \infty$ and h^t is given by (5). Hence, by (6), we have

$$\varphi(x) + r = \varphi(h^r(x)) \geq \varphi(f(x)) > \varphi(g(x)) \geq \varphi(h^s(x)) = \varphi(x) + s, \quad x \in I.$$

Thus

$$\frac{1}{r}\varphi(f(x)) \leq \frac{1}{r}\varphi(x) + 1$$

and

$$\frac{1}{r}\varphi(g(x)) \geq \frac{1}{r}\varphi(x) + \frac{s}{r}.$$

By Proposition 3(ii) we have $\frac{s}{r} \leq s(f,g)$.
If $\frac{s}{r} = s(f,g)$ then, by Proposition 3(i), $\frac{1}{r}\varphi$ satisfies system (4). Hence $f(x) = \varphi^{-1}(\varphi(x) + r) = h^r(x)$ and $g(x) = \varphi^{-1}(\varphi(x) + s) = h^s(x)$ for $x \in I$, since φ is invertible.
Fix $t > 0$ and let $\{(n_k, m_k)\}$ be a sequence such that $rn_k - sm_k \to t$. Such a sequence exists since the set $\{n - s(f,g)m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$ (see [6]). Since $t > 0$ we may assume that $n_k - s(f,g)m_k > 0$, $k \in \mathbb{N}$. By Lemma 1 $(n_k, m_k) \in \mathcal{N}$. Hence $g^{-m_k} \circ f^{n_k}$ are well defined on the whole interval I. Since $g^{m_k} \circ h^{rn_k - sm_k} = h^{sm_k} \circ h^{rn_k - sm_k} = h^{rn_k} = f^{n_k}$, the continuity of the semigroup $\{h^t : t \geq 0\}$, implies that

$$h^t(x) = \lim_{k\to\infty} h^{rn_k - sm_k}(x) = \lim_{k\to\infty} g^{-m_k}(f^{n_k}(x)), \quad x \in I. \qquad \square$$

Theorem 2. *Let* f *and* g *satisfy hypothesis* (H), $s(f,g) \notin \mathbb{Q}$ *and* $\mathrm{Int}(L(f,g)) \neq \emptyset$. *If* $r, s > 0$ *are such that* $\frac{s}{r} \leq s(f,g)$, *then there exists a continuous iteration semigroup* $\{h^t : t \geq 0\}$ *such that* (6) *holds. If* $\frac{s}{r} < s(f,g)$, *there are infinitely many such semigroups. If* $\frac{s}{r} = s(f,g)$, *this semigroup is uniquely determined and is given by the formula* (7).

Proof. Let $\frac{s}{r} \leq s(f,g)$. Then, by Proposition 3, the system

$$\begin{cases} \varphi(f(x)) \leq \varphi(x) + 1 \\ \varphi(g(x)) \geq \varphi(x) + \frac{s}{r} \end{cases}, \quad x \in I \tag{8}$$

has continuous and strictly decreasing solution φ. Put $\psi := r\varphi$ and define $h^t(x) := \psi^{-1}(\psi(x) + t), x \in I$. Hence $f(x) \geq \psi^{-1}(\psi(x) + r) = h^r(x)$ and $g(x) \leq \psi^{-1}(\psi(x) + s) = h^s(x)$. If $\frac{s}{r} < s(f,g)$ then, by Proposition 3(iii), there are infinitely many continuous semigroups $\{h^t : t \geq 0\}$ such that (6) holds. If $\frac{s}{r} = s(f,g)$ then, by Proposition 3(i), there exists a unique continuous semigroup $\{h^t : t \geq 0\}$ such that (6) holds and it is given by (7). $\square$

As a simple consequence of the previous theorem we obtain the following

Corollary 1. *Let f and g satisfy (H), $s(f,g) \notin \mathbb{Q}$ and $s(f,g) = \frac{s}{r}$ for some $r, s > 0$. If $\operatorname{Int}(L(f,g)) \neq \emptyset$ then there exists a unique continuous iteration semigroup $\{h^t : t \geq 0\}$ such that $f = h^r$ and $g = h^s$. If $\operatorname{Int}(L(f,g)) = \emptyset$ then there is no continuous semigroup with above property.*

Theorem 3. *Let f and g satisfy (H) and $s(f,g) \notin \mathbb{Q}$. If there exists $c \in I$ such that $f, g \in \operatorname{Diff}^1$ in (a,c) and f', g' are of finite variation in (a,c), then $\operatorname{Int}(L(f,g)) \neq \emptyset$.*

Proof. Let $x_0 \in (0,c)$. Then f' and g' are of finite variation in $(a, x_0]$, that is $Var f'|_{[a',x_0]} < \infty$, $Var g'|_{[a',x_0]} < \infty$ for every $a < a' < x_0$. Let $\alpha_0 : [f(x_0), x_0] \to (0,\infty)$ be an arbitrary function of class C^1 such that $\alpha_0(f(x_0))f'(x_0) = \alpha_0(x_0)$ and $\int_{f(x_0)}^{x_0} \alpha_0(t)dt = 1$. Define an extension α of α_0 on the interval $(0, x_0]$

$$\alpha(x) := \frac{\alpha_0(f^{-n}(x))}{f^{n\prime}(f^{-n}(x))} = \frac{\alpha_0(f^{-n}(x))}{f'(f^{-1}(x)) \cdot \ldots \cdot f'(f^{n-1}(x))}, \quad x \in I_n, n \in \mathbb{N},$$

where $I_n = [f^{n+1}(x_0), f^n(x_0)]$. Clearly

$$\alpha(f(x))f'(x) = \alpha(x), \quad x \in (a, x_0].$$

Note that every function $\alpha|_{I_n}$ is of finite variation since $\alpha_0 \circ f^{-n}, f' \circ f^{-1}, \ldots, f' \circ f^{n-1}$ are of finite variation on I_n and $f' > 0$ (see [7], p. 374). Hence α is also of finite variation. Put

$$\gamma_-(x) := \int_x^{x_0} \alpha(t)dt, \quad \text{for } x \in (a, x_0].$$

It is easy to verify that $\gamma_-(f(x)) = \gamma_-(x) + 1$, $x \in (a, x_0]$. We extend γ_- on I in the following way. Let $\gamma_+ : [x_0, b] \to (-\infty, 0]$ be an arbitrary function of class C^2 such that $\gamma_+' < 0$, $\gamma_+(x_0) = 0$, $\gamma_+'(x_0) = -\alpha(x_0)$ and $\lim_{x \to b-} \gamma_+(x) = -\infty$. Put

$$\gamma(x) := \begin{cases} \gamma_-(x), & x \in (a, x_0] \\ \gamma_+(x), & x \in [x_0, b) \end{cases}$$

and

$$\overline{f}(x) := \begin{cases} f(x), & x \in (a, x_0] \\ \gamma^{-1}(\gamma(x)+1), & x \in [x_0, b) \end{cases}.$$

We have

$$\gamma'(x) := \begin{cases} \alpha(x), & x \in (a, x_0] \\ \gamma'_+(x), & x \in [x_0, b) \end{cases}.$$

Since α is of finite variation and γ'_+ is of class C^1, γ' is of finite variation.
We have also

$$\overline{f}'(x) := \begin{cases} f'(x), & x \in (a, x_0] \\ \dfrac{\gamma'(x)}{\gamma'(\overline{f}(x))}, & x \in [x_0, b) \end{cases},$$

so we infer that $\overline{f}'$ is of finite variation since f' and γ' are of finite variation. Note that γ and $\overline{f}$ are of class C^1 in I and $\overline{f}$ maps I onto itself. Moreover

$$\gamma^{-1}(x+1) = \overline{f}(\gamma^{-1}(x)), \quad x \in \mathbb{R}. \tag{9}$$

Let $\overline{g}$ be a homeomorphic extension of $g|_{(a,x_0]}$ mapping I onto itself such that

$$\overline{f} \circ \overline{g} = \overline{g} \circ \overline{f}. \tag{10}$$

Let us note that $\overline{g}$ is of class C^1 and $\overline{g}'$ is of finite variation. In fact, put $I_n := [\overline{f}^{n+1}(x_0), \overline{f}^{n}(x_0)]$ for $n \in \mathbb{Z}$. We have $\cup_{n=-1}^{\infty} I_n = [x_0, b)$. By (10) we have $\overline{g}(x) = \overline{f}^{n} \circ g \circ \overline{f}^{-n}(x)$ for $x \in I_n$ and for every negative integers n. Hence $\overline{g}_{|I_n}$ is of class C^1 and, consequently, $\overline{g}$ is of class C^1 (see [5], Theorem 4.2). Moreover $Var\overline{g}'|_{I_n} = Var\Lambda_n|_{I_0}$, where $\Lambda_n(x) = \frac{(\overline{f}^{n})'(g(x))\cdot g'(x)}{(\overline{f}^{n})'(x)}$, $n \in \mathbb{N}$. Since $\overline{f}'$ and g' are of finite variation $Var\Lambda_n|_{I_0} < \infty$. Hence $\overline{g}'$ is of finite variation.
Put

$$h := \gamma \circ \overline{g} \circ \gamma^{-1}. \tag{11}$$

Note that

$$h(x+1) = h(x) + 1, \qquad x \in \mathbb{R}. \tag{12}$$

In fact, by (11), (9) and (10), we have

$$\begin{aligned} h(x+1) &= \gamma \circ \overline{g} \circ \gamma^{-1}(x+1) = \gamma \circ \overline{g} \circ \overline{f}(\gamma^{-1}(x)) = \gamma \circ \overline{f} \circ \overline{g} \circ \gamma^{-1}(x) \\ &= (\gamma \circ \overline{f} \circ \gamma^{-1}) \circ (\gamma \circ \overline{g} \circ \gamma^{-1}(x)) = h(x) + 1, \quad x \in \mathbb{R}. \end{aligned}$$

Obviously h is of class C^1. Moreover h' is of finite variation since $h'(x) = \Gamma(\gamma^{-1}(x))$, where $\Gamma(x) = \frac{\gamma'(\overline{g}(x))\overline{g}'(x)}{\gamma'(x)}$ and γ' and $\overline{g}'$ are of finite variation.

Define the mapping $H : \mathbb{S}^1 \to \mathbb{S}^1$, where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, by the formula

$$H(e^{2\pi it}) := e^{2\pi ih(t)}, \quad t \in \mathbb{R}. \tag{13}$$

Obviously $H \in \mathrm{Diff}^1(\mathbb{S}^1)$ and H' is of bounded variation, since $h'(t+1) = h'(t)$, $t \in \mathbb{R}$ and $Var h'|_{[0,1]} < \infty$. Let us note that H has no periodic points. In fact, suppose that $H^n(z_0) = z_0$ for some $n \in \mathbb{N}$ and $z_0 \in \mathbb{S}^1$. Then, by (13), $h^n(t_0) = t_0 + m$ for an $m \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$ such that $e^{2\pi it_0} = z_0$. By (11) we have $\overline{g}^n = \gamma^{-1} \circ h^n \circ \gamma$ and, by (9), $\overline{f}^m(x) = \gamma^{-1}(\gamma(x) + m)$, hence for $y_0 = \gamma^{-1}(t_0)$ we have $\overline{g}^n(y_0) = \overline{f}^m(y_0)$. Let $k \in \mathbb{N}$ be such that $f^k(y_0) < x_0$. Then, by (10), we have $\overline{g}^n(f^k(y_0)) = \overline{f}^m(f^k(y_0))$ so $g^n(u_0) = f^m(u_0)$ for $u_0 = f^k(y_0)$. But this is a contradiction since $s(f,g) \notin \mathbb{Q}$. Hence the rotation number $\varrho(H)$ of the diffeomorphism H is irrational (see eg. [1], Ch. 3). Further, by Denjoy's theorem (see eg. [1], Ch. 5), H is topologically conjugate to the rotation

$$Q(z) := ze^{2\pi i\varrho(H)}, \quad z \in \mathbb{S}^1,$$

that is there exists a homeomorphism $\Psi : \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$\Psi \circ H = Q \circ \Psi. \tag{14}$$

Since $\Psi : \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism there exists a homeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ such that $\Psi(e^{2\pi it}) = e^{2\pi i\psi(t)}$, $t \in \mathbb{R}$ and

$$\psi(t+1) = \psi(t) + 1, \quad t \in \mathbb{R}. \tag{15}$$

By (14) we obtain that

$$\psi(h(t)) = \psi(t) + \varrho(H) + k \tag{16}$$

for a $k \in \mathbb{Z}$. Putting $\phi := \psi \circ \gamma$, by (9), (15) and (11), (16) we get $\phi(\overline{f}(x)) = \phi(x) + 1$ and $\phi(\overline{g}(x)) = \phi(x) + \varrho(H) + k$. Hence $\phi(\overline{g}^m \circ \overline{f}^n(x)) = \phi(x) + ms + n$, where $s = \varrho(H) + k$. Since ϕ is a homeomorphism and the set $\{ms + n : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$,

$$L(\overline{f}, \overline{g}) = \{\overline{g}^m \circ \overline{f}^n(x) : n, m \in \mathbb{Z}\}^d = clI.$$

By Proposition 2, $\operatorname{Int} L(f,g) \neq \emptyset$. $\square$

Theorem 4. *If f and g satisfy (H), $s(f,g) \notin \mathbb{Q}$ and in a neighborhood $U = (a,c)$ of a $f,g \in \mathrm{Diff}^1(U)$ and f', g' are of bounded variation in U, then there exists a unique iteration semigroup $\{h^t : t \geq 0\}$ such that $h^1 = f$ and $g \in \{h^t : t \geq 0\}$.*

Proof. By Theorem 3 and Proposition 3(i) system (4) has a continuous strictly decreasing solution φ. Hence $f(x) = \varphi^{-1}(\varphi(x)+1) = h^1(x)$ and $g(x) = \varphi^{-1}(\varphi(x)+s(f,g)) = h^{s(f,g)}(x)$. The uniqueness follows from Remark 2 and Proposition 3(i). □

The assumption that f' and g' are of finite variation in a neighborhood of left end of interval I is essential. If we assume only that $f,g \in \mathrm{Diff}^1(I)$ then the thesis of previous theorem is not true. More precisely this shows the following

Remark 3. For every $f \in \mathrm{Diff}^\infty(I)$ such that $f(x) < x$, $x \in I$ there exists $g \in \mathrm{Diff}^1(I)$ commuting with f such that $s(f,g) \notin \mathbb{Q}$ and f and g are not embeddable in any continuous iteration semigroup.

Proof. In paper [2] Denjoy gave an example of $H \in \mathrm{Diff}^1(\mathbb{S}^1)$ with irrational rotation number such that $\{H^n(z) : n \in \mathbb{Z}\}^d =: \Delta \neq \mathbb{S}^1$. Let h be a lift of H. Hence $h \circ e = e \circ h$, where $e(x) = x+1$. We may assume that f maps I onto itself. Let $\gamma \in \mathrm{Diff}^\infty(I)$ be such that $\gamma(f(x)) = \gamma(x)+1$, $x \in I$. Define $g := \gamma^{-1} \circ h \circ \gamma$. Obviously $g \in \mathrm{Diff}^1(I)$. We have

$$f \circ g = (\gamma^{-1} \circ e \circ \gamma) \circ (\gamma^{-1} \circ h \circ \gamma) = \gamma^{-1} \circ h \circ e \circ \gamma = \gamma^{-1} \circ e \circ h \circ \gamma = g \circ f.$$

It is easy to see that $s(f,g) = \varrho(H)$, where $\varrho(H)$ is the rotation number of H. Suppose that f and g are embeddable in a continuous iteration semigroup. Then, by Proposition 3(i), there exists a homeomorphism $\varphi : I \to \mathbb{R}$ such that $\varphi(f(x)) = \varphi(x)+1$ and $\varphi(g(x)) = \varphi(x)+\varrho(H)$. Put $\psi := \varphi \circ \gamma^{-1}$. Hence $\psi(h(x)) = \psi(x)+\varrho(H)$ and $\psi(x+1) = \psi(x)+1$, $x \in \mathbb{R}$. This means that the function $\Psi : \mathbb{S}^1 \to \mathbb{S}^1$ defined by

$$\Psi(e^{2\pi i t}) = e^{2\pi i \psi(t)}$$

is a homeomorphic solution of the equation

$$\Psi(H(z)) = e^{2\pi i \varrho(H)}\Psi(z), \quad z \in \mathbb{S}^1.$$

Hence $\Psi[\Delta] = cl\{e^{2\pi i n \varrho(H)} : n \in \mathbb{Z}\}\Psi(z) = \mathbb{S}^1$, but this is a contradiction. □

Let us note that convex and concave functions f and g satisfying (H) have positive and monotone derivatives so f' and g' are of finite variation. Hence as a direct consequence of Theorem 4 we obtain the following

Corollary 2. *If f and g satisfy (H), $s(f,g) \notin \mathbb{Q}$ and f and g are of class C^1 and convex or concave in a neighborhood of a, then there exists a unique iteration semigroup $\{h^t : t \geq 0\}$ such that $h^1 = f$ and $g \in \{h^t : t \geq 0\}$.*

If in a semigroup there are two functions from which one is convex and the second is concave, then the semigroup has a very simple form. This shows the next

Theorem 5. *If f and g of class C^1 satisfy (H), $s(f,g) \notin \mathbb{Q}$ and f is concave (convex) and g is convex (concave), then a continuous iteration semigroup in which f and g are embeddable contains only affine functions.*

Proof. By Corollary 2 there exists a continuous iteration semigroup $\{h^t : t \geq 0\}$ such that $f = h^r$ and $g = h^s$. Let f be convex and g be concave. Fix $t > 0$. By Theorem 1 there exists a sequence $\{(n_k, m_k)\}$ of integers such that $rn_k - sm_k \to t$ and $h^t = \lim_{k\to\infty} g^{-m_k} \circ f^{n_k}$. The functions $g^{-m_k} \circ f^{n_k}$ are convex thus h^t is convex for every $t > 0$. In particular $h^s = g$ is convex and in consequence affine as a concave function. Now let $n > 0$. We have

$$\underbrace{h^{s/n} \circ h^{s/n} \circ \ldots \circ h^{s/n}}_{n \text{ times}} = g.$$

Let us note that $h^{\frac{s}{n}}$ is affine. In fact, suppose that $h^{\frac{s}{n}}$ is not concave. Then there exist $x_0, y_0 \in I$ and $\lambda \in [0,1]$ such that $h^{\frac{s}{n}}(\lambda x_0 + (1-\lambda)y_0) < \lambda h^{\frac{s}{n}}(x_0) + (1-\lambda)h^{\frac{s}{n}}(y_0)$. Since $h^{\frac{n-1}{n}s}$ is convex and strictly increasing we have $h^s(\lambda x_0 + (1-\lambda y_0)) = h^{\frac{n-1}{n}s}(h^{\frac{s}{n}}(\lambda x_0 + (1-\lambda y_0))) < h^{\frac{n-1}{n}s}(\lambda h^{\frac{s}{n}}(x_0) + (1-\lambda)h^{\frac{s}{n}}(y_0)) \leq \lambda h^s(x_0) + (1-\lambda)h^s(y_0)$, but this is a contradiction since $h^s = g$ is concave. Thus $h^{\frac{s}{n}}$ is affine for every $n \in \mathbb{N}$ and, consequently, $h^{\frac{m}{n}s}$ are affine for every $n, m \in \mathbb{N}$. Since, for every $t > 0$, there exists a sequence $\{w_n\}$ with terms in $\mathbb{Q}$ such that $w_n \to t$, we have that $h^t = \lim_{n\to\infty} h^{w_n s}$ is affine as a limit of affine functions.
If f is concave and g is convex the proof runs analogously. □

Theorem 6. *If f, g of class C^1 satisfy (H) and one of them is concave and the second is convex, then f and g are affine.*

Proof. If $s(f,g) \notin \mathbb{Q}$ Theorem 6 is a direct consequence of Theorem 5. Now let $s(f,g) \in \mathbb{Q}$. Then there exists $x_0 \in I$ such that $f^n(x_0) = g^m(x_0)$ with some $n, m \in \mathbb{N}$ (see [9]). Hence

$$f^n(f^k(x_0)) = g^m(f^k(x_0)), \quad k \in \mathbb{N}. \tag{17}$$

Let f be convex and g be concave. Then f^n is convex and g^m is concave.
Let us note that if P is convex and Q is concave and for $x_1 < x_2 < x_3$ $P(x_1) =$

$Q(x_1)$, $P(x_2) = Q(x_2)$ and $P(x_3) = Q(x_3)$, then $P = Q$ in $[x_1, x_3]$. By (17) for $k = 0, 1$ and $k = p$, and by the property mentioned above we infer that $g^m = f^n$ for $x \in [f^p(x_0), x_0]$ for every $p \in \mathbb{N}$ and, consequently, $g^m = f^n$ in $(a, x_0]$ since $\lim_{p\to\infty} f^p(x_0) = a$.
Fix $\overline{x} \in [x_0, b)$. There exists $k \in \mathbb{N}$ such that $f^k(\overline{x}) < x_0$. Hence

$$f^n(\overline{x}) = f^n(f^{-k}(f^k(\overline{x}))) = f^{-k}(f^n(f^k(\overline{x}))) = f^{-k}(g^m(f^k(\overline{x}))) = g^m(\overline{x}).$$

Thus $f^n = g^m$ in I and, consequently, $f^n = g^m$ is affine in $(a, x_0]$. Put $F := f^n$ and $G := g^m$. It is obvious that $F < \mathrm{id}$, $G < \mathrm{id}$ and $A := F_{|(a,x_0]} = G_{|(a,x_0]}$ is affine. Let $\overline{x} > x_0$ be such that $F(\overline{x}) < x_0$ and $G(\overline{x}) < x_0$. Since $F \circ G = G \circ F$ we have $A(G(\overline{x})) = A(F(\overline{x}))$, hence $G(\overline{x}) = F(\overline{x})$, that is $g^m(\overline{x}) = f^n(\overline{x})$. By the proved part of theorem we obtain that f and g are affine in $(a, \overline{x}]$. Repeating this procedure of extension suitable many times we get that f and g are affine in (a, b). $\square$

According to Remark 1 in the case where f is not surjective we can strengthen all Theorems omitting the assumption of continuity of a semigroup. More precisely we have

Remark 4. If the function f is not surjective then in all theorems phrase "continuous iteration semigroup" can be replaced by "iteration semigroup".

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