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On a decomposition of the plane for a flow of free mappings

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Dedicated to Professor Zoltán Daróczy on the occasion of his 70th birthday

Abstract. We consider a flow of the plane which has no fixed points. We present a method for finding a countable family of maximal parallelizable regions of the flow which cover the plane. Moreover, we describe the relations between parallelizing homeomorphisms defined on the maximal parallelizable regions using solutions of appropriate functional equations.

1. Introduction

Let $\{f^t : t \in \mathbb{R}\}$ be a flow such that f^t for $t \in \mathbb{R} \setminus \{0\}$ is a *free mapping*, i.e. a homeomorphism of the plane onto itself without fixed points which preserves orientation. It follows from the Jordan theorem that each orbit C of $\{f^t : t \in \mathbb{R}\}$ divide the plane into two simply connected regions. Note that each of them is invariant under f^t for $t \in \mathbb{R}$. Thus two different orbits C_p and C_q of points p and q, respectively, divide the plane into three simply connected invariant regions, one of which contains both C_p and C_q in its boundary. We will call this region by the *strip* between C_p and C_q and denote by D_{pq}.

For any distinct orbits C_{p_1}, C_{p_2}, C_{p_3} of $\{f^t : t \in \mathbb{R}\}$ one of the following two possibilities must be satisfied: exactly one of the orbits C_{p_1}, C_{p_2}, C_{p_3} is contained in the strip between the other two or each of the orbits C_{p_1}, C_{p_2}, C_{p_3} is contained in the strip between the other two. In the first case if C_{p_j} is the orbit which lies

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in the strip between C_{p_i} and C_{p_k}, we will write $C_{p_i}|C_{p_j}|C_{p_k}$ ($i,\ j,\ k \in \{1,2,3\}$ and i, j, k are different). In the second case we will write $|C_{p_i}, C_{p_j}, C_{p_k}|$ (see [2]).

Put

$$J^+(q) := \{p \in \mathbb{R}^2 : \text{there exist a sequence } (q_n)_{n\in\mathbb{N}} \text{ and a sequence } (t_n)_{n\in\mathbb{N}} \\ \text{such that } q_n \to q,\ t_n \to +\infty,\ f^{t_n}(q_n) \to p \text{ as } n \to +\infty\},$$

$$J^-(q) := \{p \in \mathbb{R}^2 : \text{there exist a sequence } (q_n)_{n\in\mathbb{N}} \text{ and a sequence } (t_n)_{n\in\mathbb{N}} \\ \text{such that } q_n \to q,\ t_n \to -\infty,\ f^{t_n}(q_n) \to p \text{ as } n \to +\infty\}.$$

The set $J(q) := J^+(q) \cup J^-(q)$ is called the *first prolongational limit set* of q. Let us observe that $p \in J(q)$ if and only if $q \in J(p)$ for any $p, q \in \mathbb{R}^2$. For a subset $H \subset \mathbb{R}^2$ we define

$$J(H) := \bigcup_{q\in H} J(q).$$

One can observe that for each $p \in \mathbb{R}^2$ the set $J(p)$ is invariant.

An invariant region $M \subset \mathbb{R}^2$ is said to be *parallelizable* if there exists a homeomorphism ψ mapping M onto $\mathbb{R}^2$ such that

$$f^t(x) = \psi^{-1}(\psi(x) + (t,0)) \quad \text{for } x \in M,\ t \in \mathbb{R}.$$

The homeomorphism ψ occurring in this equality will be called a *parallelizing homeomorphism* of M. It is known that a region M is parallelizable if and only if there exists a *topological line* K (i.e. a homeomorphic image of a straight line) that is a closed set in M such that K has exactly one common point with every orbit of $\{f^t : t \in \mathbb{R}\}$ contained in M (see [1], p. 49 and e.g. [6]). Such a set K we will call a *section* in M.

It is known that a region M is parallelizable if and only if $J(M) \cap M = \emptyset$ (see [1], p. 46 and 49). Hence for every parallelizable region M we have $J(M) \subset \operatorname{fr} M$. If M is a maximal parallelizable region (i.e. M is not contained properly in any parallelizable region), then $J(M) = \operatorname{fr} M$ (see [8]).

Now we collect the results from [5] and [7] which are needed in this paper.

Proposition 1.1 (see [7]). *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$. Then $\operatorname{fr} M$ is invariant.*

Proposition 1.2 (see [5]). *Let M be a parallelizable region and let $q \in \operatorname{fr} M$. Then $\operatorname{cl} M \setminus C_q$ is contained in one of the components of $\mathbb{R}^2 \setminus C_q$.*

Proposition 1.3 (see [7]). *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$. Let $p \in M$ and H be a component of $\mathbb{R}^2 \setminus C_p$. Then for all distinct orbits C_{q_1}, C_{q_2} contained in $\operatorname{fr} M \cap H$ the relation $|C_{q_1}, C_{q_2}, C_p|$ holds.*

Proposition 1.4 (see [7]). *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$. Let $p \in \operatorname{fr} M$ and $q_1, q_2 \in M$. Assume that $q_1, q_2 \in J(p)$. Then $C_{q_1} = C_{q_2}$.*

2. The form of flows of free mappings

In this section we describe the form of an arbitrary flow of free mappings. The proof of the theorem of this section is based on the idea of W. Kaplan (see [2]).

Let $\alpha = (p_1, \dots, p_n)$ be a sequence of integers. Then, for any integer k by $\alpha * k$ will be denoted the concatenation of the sequences α and the one-element sequence k (for one-element sequences we omit parentheses), i.e the sequence $(p_1, \dots, p_n, k)$.

A class A^+ of finite sequences α of positive integers will be termed *admissible* if the following conditions hold:

(1) A^+ contains the sequence: 1, and no other one-element sequence;

(2) if $\alpha * k$ is in A^+ and $k > 1$, then so also is $\alpha * (k-1)$;

(3) if $\alpha * 1$ is in A^+, then so also is α.

A class A^- of finite sequences α of negative integers will be termed *admissible* if the following conditions hold:

(1) A^- contains the sequence: -1, and no other one-element sequence;

(2) if $\alpha * k$ is in A^- and $k < -1$, then so also is $\alpha * (k+1)$;

(3) if $\alpha * -1$ is in A^-, then so also is α.

The set $A := A^+ \cup A^-$, where A^+, A^- are some admissible classes of finite sequences of positive, negative integers, respectively, will be said to be *admissible class of finite sequences.*

Lemma 2.1. *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of free mappings. Let $p \in \mathbb{R}^2$. Then there exists an at most countable family of maximal parallelizable regions $\{M_j : j \in J\}$, where J is the set of all positive integers or $J = \{1, \dots, N\}$ for some positive integer N, such that $p \in M_1$ and for each positive integer n the set $\operatorname{cl} B(p, n)$, where $B(p, n)$ is the ball centered at p with radius n, is covered by a finite subfamily $\{M_1, \dots, M_{j_n}\}$ of $\{M_j : j \in J\}$. Moreover, $j_n \le j_{n+1}$ for every n.*

Proof. On account of the Whitney–Bebutov Theorem (see [1], p. 52), for each $q \in \mathbb{R}^2$ there exists a parallelizable region M_0^q containing q. Then there exists a maximal parallelizable region M^q such that $M_0^q \subset M^q$ (see [8]). Thus for each $q \in \mathbb{R}^2$ we can choose a maximal parallelizable region M^q containing q.

Consider the ball $B(p,1)$ centered at p with radius 1. Then the family $\{M^q : q \in \operatorname{cl} B(p,1)\}$ cover the set $\operatorname{cl} B(p,1)$. By the Heine–Borel Theorem we can choose a finite number of elements of the family $\{M^q : q \in \operatorname{cl} B(p,1)\}$ which cover $\operatorname{cl} B(p,1)$. Denote these regions by $M_1, \ldots, M_{j_1}$, where j_1 is a positive integer. Without loss of generality we can assume that $p \in M_1$.

Suppose that we have chosen a finite sequence $M_1, \ldots, M_{j_1}, \ldots, M_{j_n}$ of distinct elements of the family $\{M^q : q \in \mathbb{R}^2\}$ such that for every positive integer $k \leq n$

$$\operatorname{cl} B(p,k) \subset \bigcup_{j=1}^{j_k} M_j$$

and the sequence $j_1, j_2, \ldots, j_n$ is nondecreasing. If the regions $M_1, \ldots, M_{j_1}, \ldots, M_{j_n}$ cover the set $\operatorname{cl} B(p, n+1)$, then we put $j_{n+1} = j_n$. Otherwise, we consider the family

$$\{M_j : j = 1, \ldots, j_n\} \cup \{M^q : q \in \operatorname{cl} B(p, n+1) \setminus \bigcup_{j=1}^{j_n} M_j\}.$$

By compactness of $\operatorname{cl} B(p, n+1)$ we can choose from this family a finite number of distinct maximal parallelizable regions $M_{j_n+1}, \ldots, M_{j_{n+1}} \notin \{M_j : j = 1, \ldots, j_n\}$ such that

$$\operatorname{cl} B(p, n+1) \subset \bigcup_{j=1}^{j_{n+1}} M_j.$$

Therefore we have the finite sequence $M_1, \ldots, M_{j_{n+1}}$ covering $\operatorname{cl} B(p, n+1)$ such that $M_1, \ldots, M_{j_1}, \ldots, M_{j_n}$ are its initial elements. Hence $j_n \leq j_{n+1}$. □

Theorem 2.2. *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of free mappings. Then there exist a family of orbits $\{C_\alpha : \alpha \in A\}$ and a family of maximal parallelizable regions $\{M_\alpha : \alpha \in A\}$, where $A = A^+ \cup A^-$ is an at most countable admissible class of finite sequences, such that*

$$C_\alpha \subset M_\alpha \quad \text{for } \alpha \in A, \qquad \bigcup_{\alpha \in A} M_\alpha = \mathbb{R}^2$$

and

$$f^t(x) = \psi_\alpha^{-1}(\psi_\alpha(x) + (t,0)) \quad \text{for } x \in M_\alpha, t \in \mathbb{R}$$

for arbitrarily chosen parallelizing homeomorphism ψ_α of M_α. Moreover, the families can be constructed in such a way that

$$M_\alpha \cap M_{\alpha * i} \neq \emptyset \qquad \text{for } \alpha * i \in A,$$

$$C_{\alpha*i} \subset J(M_\alpha) \qquad \textit{for } \alpha * i \in A,$$
$$|C_\alpha, C_{\alpha*i_1}, C_{\alpha*i_2}| \quad \textit{for } \alpha * i_1, \alpha * i_2 \in A, \quad i_1 \neq i_2,$$
$$C_\alpha | C_{\alpha*i} | C_{\alpha*i*l} \qquad \textit{for } \alpha * i * l \in A.$$

Remark 2.3. The construction of the set of indices A and the families $\{C_\alpha : \alpha \in A\}$ and $\{M_\alpha : \alpha \in A\}$ occurring in Theorem 2.2 starts from the orbit $C_1 = C_{-1}$ of an arbitrary point $p \in \mathbb{R}^2$ and the maximal parallelizable region M_1 occurring in Lemma 2.1 (we take $M_{-1} = M_1$ and the same parallelizing homeomorphism $\psi_1 = \psi_{-1}$ of M_1). Having constructed an $\alpha \in A$ and C_α, M_α, we index bijectively the set of all orbits contained in

$$\operatorname{fr} M_\alpha \cap H_\alpha,$$

where H_1, H_{-1} are components of $\mathbb{R}^2 \setminus C_1$ and for all $\alpha = \beta * l \in A$ the set H_α is the component of $\mathbb{R}^2 \setminus C_\alpha$ which has no common point with M_β, by sequences of the form $\alpha * k$ starting from $k = 1$ and taking subsequent positive integers k if $\alpha \in A^+$ and starting from $k = -1$ and taking subsequent negative integers k if $\alpha \in A^-$. We enlarge the set A by all these sequences $\alpha * k$ and for each orbit $C_{\alpha*k}$ indexed by $\alpha * k$ we take as $M_{\alpha*k}$ an element of the subfamily $\{M_j : j = 1, \ldots, j_{m_{\alpha*k}}\}$ of the family occurring in Lemma 2.1 that contains $C_{\alpha*k}$, where $m_{\alpha*k}$ is the smallest integer which is greater or equal to the distance of the orbit $C_{\alpha*k}$ from p. Moreover, we only consider such parallelizing homeomorphism $\psi_\alpha : M_\alpha \to \mathbb{R}^2$ that $\psi_\alpha(C_\alpha) = \mathbb{R} \times \{0\}$ and

$$\psi_\alpha(M_\alpha \cap H_\alpha) = \mathbb{R} \times (0, +\infty) \quad \text{if } \alpha \in A^+$$

and

$$\psi_\alpha(M_\alpha \cap H_\alpha) = \mathbb{R} \times (-\infty, 0) \quad \text{if } \alpha \in A^-.$$

Example 2.4. The flow $\{f^t : t \in \mathbb{R}\}$ depicted in Figure 1 is the one of the simplest flows of free mappings for which the set of indices A occurring in Theorem 2.2 is not uniquely determined. The topological lines pictured in Figure 1 are invariant under the flow. The free mapping f^1 moves points one unit of arc length along these lines.

The lines denoted by C_1, $C_{(1,1)}$, $C_{(1,1,1)}$ are contained in the maximal parallelizable regions $M_1 = M_{-1}$, $M_{(1,1)}$, $M_{(1,1,1)}$, where M_1 is the open half-plane which consists of all points that lie above the line $C_{(1,1)}$, $M_{(1,1)}$ is the strip bounded by C_1 and $C_{(1,1,1)}$, $M_{(1,1,1)}$ is the open half-plane which consists of all points that lie below the line $C_{(1,1)}$. In this case $A^+ = \{1, (1,1), (1,1,1)\}$ and $A^- = \{-1\}$.

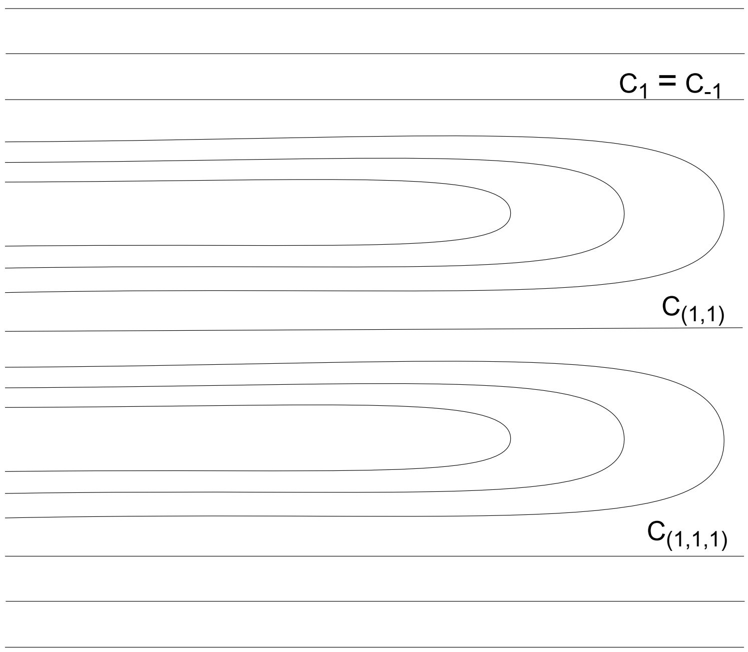

Figure 1. A flow of free mappings with three maximal parallelizable regions

Another possibility of choosing A^+ and A^- is the following: $(A^+)' = \{1, (1,1)\}$, $(A^-)' = \{-1, (-1,-1)\}$. Then taking $(C_1)' = C_{(1,1)}$, $(C_{(1,1)})' = C_{(1,1,1)}$, $(C_{(-1,-1)})' = C_1$, we have $(M_1)' = M_{(1,1)}$, $(M_{(1,1)})' = M_{(1,1,1)}$, $(M_{(-1,-1)})' = M_1$.

Proof of Theorem 2.2. We will show that the construction described in Remark 2.3 gives families $\{C_\alpha : \alpha \in A\}$ and $\{M_\alpha : \alpha \in A\}$ that satisfy all relations in the assertion of Theorem 2.2. Since $C_1 = C_p$, $p \in M_1$ and M_1 is invariant, the inclusion $C_1 \subset M_1$ holds. Moreover, by the fact that $C_{-1} = C_1$ and $M_{-1} = M_1$, we have $C_{-1} \subset M_{-1}$. From the Jordan theorem we obtain that the set $\mathbb{R}^2 \setminus C_p$ has exactly two components (we denote these components by H_1 and H_{-1}). For the set $C_1 \cup H_1$ we will construct an admissible class A^+ of finite sequences of positive integers and a subfamily $\{M_\alpha : \alpha \in A^+\}$ of $\{M_j : j \in J\}$, where $\{M_j : j \in J\}$ is the family occurring in Lemma 2.1, such that $C_1 \cup H_1 \subset \bigcup_{\alpha \in A^+} M_\alpha$. The same procedure can be applied to $C_{-1} \cup H_{-1}$, to obtain an admissible class A^- of finite sequences of negative integers and a subfamily $\{M_\alpha : \alpha \in A^-\}$ of $\{M_j : j \in J\}$ such that $C_{-1} \cup H_{-1} \subset \bigcup_{\alpha \in A^-} M_\alpha$.

Assume that for an $n \geq 1$ we have constructed the set of admissible sequences of positive integers A_n of length n, the set of admissible sequences of negative integers A_{-n} of length n, the orbits C_α and the maximal parallelizable regions M_α for $\alpha \in A_n \cup A_{-n}$ such that $C_\alpha \subset M_\alpha$ and $M_\alpha \in \{M_j : j \in J\}$. Consider the maximal parallelizable region M_α for an $\alpha \in A_n \cup A_{-n}$. From Proposition 1.1, we obtain that $\operatorname{fr} M_\alpha$ is a union of orbits. Moreover, by Propositions 1.2 and 1.3, for all $q_1, q_2 \in \operatorname{fr} M_\alpha \cap H_\alpha$ such that $C_{q_1} \neq C_{q_2}$ the relation $|C_\alpha, C_{q_1}, C_{q_2}|$ holds and the components $\mathbb{R}^2 \setminus C_{q_1}$, $\mathbb{R}^2 \setminus C_{q_2}$ which do not contain M_α are disjoint open sets. Hence $\operatorname{fr} M_\alpha \cap H_\alpha$ consists of at most countable many of orbits, since for each $q \in \operatorname{fr} M_\alpha \cap H_\alpha$ the component of $\mathbb{R}^2 \setminus C_q$ which has no common points with M_α contains a point with rational coordinates.

Let $(C_{\alpha * i})_{i \in I_\alpha}$, where $I_\alpha = \{1, \ldots, k_\alpha\}$ for some $k_1 \in \mathbb{Z}_+$ or $I_\alpha = \mathbb{Z}_+$ if $\alpha \in A_n$ and $I_\alpha = \{-1, \ldots, k_\alpha\}$ for some $k_\alpha \in \mathbb{Z}_-$ or $I_\alpha = \mathbb{Z}_-$ if $\alpha \in A_{-n}$, be a sequence of all orbits contained in $\operatorname{fr} M_\alpha \cap H_\alpha$ (the orbits contained in $\operatorname{fr} M_\alpha \cap H_\alpha$ are indexed bijectively by sequences of the form $\alpha * i$). For each $i \in I_\alpha$ denote by $H_{\alpha * i}$ the component of $\mathbb{R}^2 \setminus C_{\alpha * i}$ which has no common points with M_α. It can happen that $\operatorname{fr} M_\alpha \cap H_\alpha = \emptyset$. In such a case we have $I_\alpha = \emptyset$.

For each $i \in I_\alpha$ we take a point $q_{\alpha * i} \in C_{\alpha * i}$. Then there exists a positive integer m such that $q_{\alpha * i} \in \operatorname{cl} B(p, m)$. Since $C_{\alpha * i} \cap \operatorname{cl} B(p, m)$ is a compact set, there exists a point $p_{\alpha * i} \in C_{\alpha * i} \cap \operatorname{cl} B(p, m)$ such that the Euclidean distance $d(p, p_{\alpha * i})$ between p and $p_{\alpha * i}$ is the minimum of the distances between the point p and those in the orbit $C_{\alpha * i}$. Denote by $m_{\alpha * i}$ the minimum of the set of all positive integers m such that $d(p, p_{\alpha * i}) \leq m$. Let $M_{\alpha * i}$ be an element of the subfamily $\{M_j : j = 1, \ldots, j_{m_{\alpha * i}}\}$ of the family $\{M_j : j \in J\}$ that contains $p_{\alpha * i}$. Then $C_{\alpha * i} \subset M_{\alpha * i}$, since $p_{\alpha * i} \in C_{\alpha * i}$ and $M_{\alpha * i}$ is invariant. Thus we have constructed an admissible class of finite sequences A, a family of orbits $\{C_\alpha : \alpha \in A\}$ and a family of maximal parallelizable regions $\{M_\alpha : \alpha \in A\}$ such that $C_\alpha \subset M_\alpha$ for $\alpha \in A$.

Now we will prove that A is at most contable. To this end we show by induction that for each positive integer n the set $A_n \cup A_{-n}$ is at most contable. Since $A_1 = \{1\}$ and $A_{-1} = \{-1\}$, the statement holds for $n = 1$. Assume that the set $A_n \cup A_{-n}$ is at most contable for some n. Since for each $\alpha \in A_n \cup A_{-n}$ the set I_α is at most contable, so is $A_{n+1} = \{\alpha * i : \alpha \in A_n, i \in I_\alpha\}$ and $A_{-n-1} = \{\alpha * i : \alpha \in A_{-n}, i \in I_\alpha\}$. Thus for each positive integer n the set $A_n \cup A_{-n}$ is at most contable and consequently so is $A = \bigcup_{n \in \mathbb{Z}_+} A_n \cup A_{-n}$.

Fix a sequence $\alpha * i \in A$. We will show that $M_\alpha \cap M_{\alpha * i} \neq \emptyset$. Let $q \in C_{\alpha * i}$. Then there exists a ball $B(q, \varepsilon)$ for some $\varepsilon > 0$ such that $B(q, \varepsilon) \subset M_{\alpha * i}$, since $C_{\alpha * i} \subset M_{\alpha * i}$ and $M_{\alpha * i}$ is an open set. By the fact that $C_{\alpha * i} \subset \operatorname{fr} M_\alpha$ we obtain that there exists a point $r \in B(q, \varepsilon) \cap M_\alpha$. Thus $r \in M_\alpha \cap M_{\alpha * i}$. Fix an $\alpha \in A$.

Since M_α is a maximal parallelizable region, we have $\operatorname{fr} M_\alpha = J(M_\alpha)$. Hence by the fact that $C_{\alpha*i} \subset \operatorname{fr} M_\alpha$, we obtain that $C_{\alpha*i} \subset J(M_\alpha)$.

Fix $\alpha * i_1, \alpha * i_2 \in A$ such that $i_1 \neq i_2$. By Proposition 1.3 $|C_\alpha, C_{\alpha*i_1}, C_{\alpha*i_2}|$, since $C_{\alpha*i_1}$ and $C_{\alpha*i_2}$ are contained in $\operatorname{fr} M_\alpha \cap H_\alpha$. Take a sequence $\alpha * i * l \in A$. Since the orbit $C_{\alpha*i*l}$ is contained in the component of $\mathbb{R}^2 \setminus C_{\alpha*i}$ which does not contain any point from M_α and $C_\alpha \subset M_\alpha$, the relation $C_\alpha | C_{\alpha*i} | C_{\alpha*i*l}$ holds.

Now we shall show that for every point $q \in \mathbb{R}^2$ there exists an $\alpha \in A$ such that $q \in M_\alpha$. Fix a point $q \in \mathbb{R}^2$. Let $q \in H_1$ (for $q \in H_{-1}$ the proof runs in the same way). Suppose, on the contrary, that $q \notin \bigcup_{\alpha \in A^+} M_\alpha$. Then $q \notin M_1$. Hence there exists exactly one $k_1 \in I_1$ such that $q \in H_{1*k_1}$. Since $q \notin M_{1*k_1}$, there exists exactly one $k_2 \in I_{1*k_1}$ such that $q \in H_{1*k_1*k_2}$. If $q \in H_{(1,k_1\ldots,k_{n-1})}$ for some $(1, k_1 \ldots, k_{n-1}) \in A^+$, then there exists exactly one $k_n \in I_{(1,k_1\ldots,k_{n-1})}$ such that $q \in H_{(1,k_1\ldots,k_{n-1},k_n)}$, since otherwise $q \in M_{(1,k_1\ldots,k_{n-1})}$. In such a way we obtain a sequence $(k_n)_{n\in\mathbb{Z}_+}$ such that $q \in H_{(1,k_1\ldots,k_n)}$ for every $n \in \mathbb{Z}_+$. Put $\alpha_1 := 1$ and $\alpha_n := (1, k_1, \ldots, k_{n-1})$ for $n \geq 2$. Thus we get the sequence $(\alpha_n)_{n\in\mathbb{Z}_+}$ of elements of A^+ such that $q \in H_{\alpha_n}$ for $n \in \mathbb{Z}_+$ and the sequence of orbits $(C_{\alpha_n})_{n\in\mathbb{Z}_+}$ such that and $C_{\alpha_n} \mid C_{\alpha_{n+1}} \mid C_{\alpha_{n+2}}$ for $n \in \mathbb{Z}_+$, since $\alpha_{n+1} = \alpha_n * k_n$ and $\alpha_{n+2} = \alpha_n * k_n * k_{n+1}$.

Now we prove that the elements of the sequence $(M_{\alpha_n})_{n\in\mathbb{Z}_+}$ are distinct. Fix an $n \in \mathbb{Z}_+$. Then by the definition of $H_{\alpha_{n+1}}$, we have $M_{\alpha_n} \subset \mathbb{R}^2 \setminus (C_{\alpha_{n+1}} \cup H_{\alpha_{n+1}})$. Let k be a positive integer such that $k > n+1$. Then M_{α_k} contains a point from $H_{\alpha_{n+1}}$, since $C_{\alpha_k} \subset H_{\alpha_{n+1}}$ and $C_{\alpha_k} \subset M_{\alpha_k}$. Hence $M_{\alpha_n} \neq M_{\alpha_k}$. Moreover, $M_{\alpha_n} \neq M_{\alpha_{n+1}}$, since $C_{\alpha_{n+1}} \subset M_{\alpha_{n+1}}$ and $C_{\alpha_{n+1}} \cap M_{\alpha_n} = \emptyset$. If k is a positive integer such that $k < n$, then $M_{\alpha_k} \subset \mathbb{R}^2 \setminus (C_{\alpha_{k+1}} \cup H_{\alpha_{k+1}})$. Hence $M_{\alpha_k} \subset \mathbb{R}^2 \setminus (C_{\alpha_n} \cup H_{\alpha_n})$. Thus $M_{\alpha_n} \neq M_{\alpha_k}$, since $C_{\alpha_n} \subset M_{\alpha_n}$.

Since $q \in H_{\alpha_n}$ for every $n \in \mathbb{Z}_+$, the segment with endpoints p and q has a common point with C_{α_n} for each $n \in \mathbb{Z}_+$. Denote by ρ the distance between p and q and by m_q the smallest positive integer m such that $m \geq \rho$. For each $n \in \mathbb{Z}_+$ consider the point $p_{\alpha_n} \in C_{\alpha_n}$ such that the Euclidean distance $d(p, p_{\alpha_n})$ between p and p_{α_n} is the minimum of the distances between the point p and those in the orbit C_{α_n} and the number m_{α_n} being the minimum of the set of all positive integers m such that $d(p, p_{\alpha_n}) \leq m$.

Fix an $n \in \mathbb{Z}_+$. By the construction the region M_{α_n} is an element of the subfamily $\{M_j : j = 1, \ldots, j_{m_{\alpha_n}}\}$ of the family $\{M_j : j \in J\}$. Since C_{α_n} has a common point with the segment with endpoints p and q, we have $m_{\alpha_n} \leq m_q$. Hence M_{α_n} is an element of the subfamily $\{M_j : j = 1, \ldots, j_{m_q}\}$ of the family $\{M_j : j \in J\}$ for each $n \in \mathbb{Z}_+$. Thus we get a contradiction, since the elements of the sequence $(M_{\alpha_n})_{n\in\mathbb{Z}_+}$ are distinct and the subfamily $\{M_j : j = 1,\dots,j_{m_q}\}$ is finite. $\square$

Corollary 2.5. *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of free mappings. Then there exists a family of connected subsets of the plane $\{U_\alpha : \alpha \in A\}$, where $A = A^+ \cup A^-$ is an admissible class of finite sequences, such that*

$$\bigcup_{\alpha\in A} U_\alpha = \mathbb{R}^2,$$

$U_\alpha = C_\alpha \cup N_\alpha$ for an orbit C_α and a parallelizable region N_α, $\operatorname{fr} U_\alpha = C_\alpha \cup \bigcup_{\alpha * i\in A} C_{\alpha*i}$, $N_{\alpha_1} \cap N_{\alpha_2} = \emptyset$ for distinct $\alpha_1, \alpha_2 \in A$, $C_\alpha \subset J(\mathbb{R}^2)$ for $\alpha \in A \setminus \{-1, 1\}$, $C_1 = C_{-1}$, $C_{\alpha_1} \neq C_{\alpha_2}$ for distinct $\alpha_1, \alpha_2 \in A$ satisfying at least one of the conditions $\alpha_1 \notin \{-1, 1\}$, $\alpha_2 \notin \{-1, 1\}$ and

$$f^t(x) = \varphi_\alpha^{-1}(\varphi_\alpha(x) + (t, 0)), \quad x \in U_\alpha,\ t \in \mathbb{R}$$

for some homeomorphisms

$$\varphi_\alpha : U_\alpha \overset{onto}{\to} \mathbb{R} \times [0, +\infty) \quad \text{for } \alpha \in A^+,$$

$$\varphi_\alpha : U_\alpha \overset{onto}{\to} \mathbb{R} \times (-\infty, 0] \quad \text{for } \alpha \in A^-.$$

Proof. Putting $N_\alpha = M_\alpha \cap H_\alpha$, $U_\alpha = C_\alpha \cup N_\alpha$ and $\varphi_\alpha = \psi_{\alpha|U_\alpha}$ for all $\alpha \in A$, where M_α, H_α, ψ_α and A are those occuring Theorem 2.2, we obtain our assertion directly from Theorem 2.2. $\square$

3. Dependence on common regions

In this section we describe the relations between parallelizing homeomorphisms defined on the common parts of the maximal parallelizable regions of the family constructed in the previous section.

Proposition 3.1. *Let $\{M_\alpha : \alpha \in A^+\}$ and $\{\psi_\alpha : \alpha \in A^+\}$ be the families of maximal parallelizable regions and homeomorphisms, respectively, occurring in Theorem 2.2. Then for each $\alpha * i \in A^+$*

$$\psi_{\alpha*i}(M_\alpha \cap M_{\alpha*i}) = \mathbb{R} \times (c_{\alpha*i}, 0),$$

$$\psi_\alpha(M_\alpha \cap M_{\alpha*i}) = \mathbb{R} \times (c_\alpha, d_\alpha),$$

where $c_\alpha \in \mathbb{R} \cup \{-\infty\}$*,* $d_\alpha \in \mathbb{R} \cup \{+\infty\}$ *and* $c_{\alpha*i} \in [-\infty, 0)$ *are some constants such that* $c_\alpha < d_\alpha$ *and at least one of the constants* c_α*,* d_α *is finite, and there exist a continuous function* $\mu_{\alpha*i} : (c_\alpha, d_\alpha) \to \mathbb{R}$ *and a homeomorphism* $\nu_{\alpha*i} : (c_\alpha, d_\alpha) \to (c_{\alpha*i}, 0)$ *such that the homeomorphism*

$$h_{\alpha*i} : \psi_\alpha(M_\alpha \cap M_{\alpha*i}) \to \psi_{\alpha*i}(M_\alpha \cap M_{\alpha*i})$$

given by the relation $h_{\alpha*i} = \psi_{\alpha*i} \circ (\psi_{\alpha|_{M_\alpha \cap M_{\alpha*i}}})^{-1}$ *has the form*

$$h_{\alpha*i}(t, s) = (\mu_{\alpha*i}(s) + t, \nu_{\alpha*i}(s))$$

for all $t \in \mathbb{R}$ *and* $s \in (c_\alpha, d_\alpha)$*.*

Proof. Fix an $\alpha * i \in A^+$. Then, by Theorem 2.2, $M_\alpha \cap M_{\alpha*i} \neq \emptyset$ and

$$\psi_\alpha^{-1}(\psi_\alpha(x) + (t, 0)) = \psi_{\alpha*i}^{-1}(\psi_{\alpha*i}(x) + (t, 0))$$

for all $x \in M_\alpha \cap M_{\alpha*i}$ and $t \in \mathbb{R}$. This means that for all $t \in \mathbb{R}$

$$\psi_\alpha^{-1} \circ T^t \circ \psi_\alpha = \psi_{\alpha*i}^{-1} \circ T^t \circ \psi_{\alpha*i}$$

on $M_\alpha \cap M_{\alpha*i}$, where T^t is given by

$$T^t(x) = x + (t, 0) \quad \text{for } x \in \mathbb{R}^2.$$

Hence

$$\psi_{\alpha*i} \circ \psi_\alpha^{-1} \circ T^t = T^t \circ \psi_{\alpha*i} \circ \psi_\alpha^{-1} \tag{1}$$

on $\psi_\alpha(M_\alpha \cap M_{\alpha*i})$.

Define a function $h_{\alpha*i} : \psi_\alpha(M_\alpha \cap M_{\alpha*i}) \to \psi_{\alpha*i}(M_\alpha \cap M_{\alpha*i})$ by the relation $h_{\alpha*i} = \psi_{\alpha*i} \circ (\psi_{\alpha|_{M_\alpha \cap M_{\alpha*i}}})^{-1}$. From the proof of Theorem 2.2 we get that $\psi_{\alpha*i}(M_\alpha \cap M_{\alpha*i}) = \mathbb{R} \times (c_{\alpha*i}, 0)$ for some $c_{\alpha*i} \in [-\infty, 0)$ and $\psi_\alpha(M_\alpha \cap M_{\alpha*i}) = \mathbb{R} \times (c_\alpha, d_\alpha)$ for some $c_\alpha \in \mathbb{R} \cup \{-\infty\}$ and $d_\alpha \in \mathbb{R} \cup \{+\infty\}$ such that $c_\alpha < d_\alpha$. Moreover, at least one of the constants c_α, d_α is finite, since $J(C_{\alpha*i}) \cap M_\alpha \neq \emptyset$. Thus $h_{\alpha*i} : \mathbb{R} \times (c_\alpha, d_\alpha) \to \mathbb{R} \times (c_{\alpha*i}, 0)$.

To obtain the form of $h_{\alpha*i}$ we consider the functions $u_{\alpha*i} : \mathbb{R} \times (c_\alpha, d_\alpha) \to \mathbb{R}$ and $v_{\alpha*i} : \mathbb{R} \times (c_\alpha, d_\alpha) \to (c_{\alpha*i}, 0)$ such that $h_{\alpha*i} = (u_{\alpha*i}, v_{\alpha*i})$. From (1) we get

$$(h_{\alpha*i} \circ T^t)(y_1, y_2) = (T^t \circ h_{\alpha*i})(y_1, y_2)$$

for all $(y_1, y_2) \in \mathbb{R} \times (c_\alpha, d_\alpha)$ and $t \in \mathbb{R}$, since $T^t(\mathbb{R} \times (c_\alpha, d_\alpha)) = \mathbb{R} \times (c_\alpha, d_\alpha)$. Hence for all $(y_1, y_2) \in \mathbb{R} \times (c_\alpha, d_\alpha)$ and $t \in \mathbb{R}$

$$h_{\alpha*i}(y_1 + t, y_2) = h_{\alpha*i}(y_1, y_2) + (t, 0),$$

which means that

$$u_{\alpha*i}(y_1+t,y_2)=u_{\alpha*i}(y_1,y_2)+t$$

and

$$v_{\alpha*i}(y_1+t,y_2)=v_{\alpha*i}(y_1,y_2).$$

Putting $y_1=0$ and $y_2=s$ we obtain

$$u_{\alpha*i}(t,s)=u_{\alpha*i}(0,s)+t$$

and

$$v_{\alpha*i}(t,s)=v_{\alpha*i}(0,s)$$

for all $t\in\mathbb{R}$ and $s\in(c_\alpha,d_\alpha)$. Consequently

$$h_{\alpha*i}(t,s)=(\mu_{\alpha*i}(s)+t,\nu_{\alpha*i}(s))$$

for all $t\in\mathbb{R}$ and $s\in(c_\alpha,d_\alpha)$, where $\mu_{\alpha*i}(s)=u_{\alpha*i}(0,s)$ and $\nu_{\alpha*i}(s)=v_{\alpha*i}(0,s)$.

The functions $\mu_{\alpha*i}$ and $\nu_{\alpha*i}$ are continuous, since so are $u_{\alpha*i}$ and $v_{\alpha*i}$. To prove that $\nu_{\alpha*i}$ is one-to-one, let us fix $s_1,s_2\in(c_\alpha,d_\alpha)$ such that $s_1\neq s_2$. Let $t_1,t_2\in\mathbb{R}$. Then the orbits of $C_{(s_1,t_1)}$ and $C_{(s_2,t_2)}$ of (s_1,t_1) and (s_2,t_2), respectively, of the flow $\{T^t_{|\mathbb{R}\times(c_\alpha,d_\alpha)}:t\in\mathbb{R}\}$ are distinct, since each orbit of the flow $\{T^t:t\in\mathbb{R}\}$ is a horizontal line. From (1) and the definition of $h_{\alpha*i}$ we get

$$h_{\alpha*i}(T^t(s,t))=T^t(h_{\alpha*i}(s,t))$$

for all $(s,t)\in\mathbb{R}\times(c_\alpha,d_\alpha)$ and $t\in\mathbb{R}$. Hence

$$h_{\alpha*i}(C_{(s,t)})=C_{h_{\alpha*i}(s,t)}$$

for all $(s,t)\in\mathbb{R}\times(c_\alpha,d_\alpha)$, where $C_{h_{\alpha*i}(s,t)}$ denotes the orbit of the point $h_{\alpha*i}(s,t)$ of the flow $\{T^t_{|\mathbb{R}\times(c_{\alpha*i},0)}:t\in\mathbb{R}\}$. Thus $C_{h_{\alpha*i}(s_1,t_1)}\neq C_{h_{\alpha*i}(s_2,t_2)}$, since $h_{\alpha*i}$ is a homeomorphism. Consequently, the second coordinates of $h_{\alpha*i}(s_1,t_1)$ and $h_{\alpha*i}(s_2,t_2)$ are distinct, which means that $\nu_{\alpha*i}(s_1)\neq\nu_{\alpha*i}(s_2)$. Using the invariance of domain theorem we obtain that $\nu_{\alpha*i}$ is a homeomorphism, since $(c_\alpha,d_\alpha)\subset\mathbb{R}$, $\nu_{\alpha*i}:(c_\alpha,d_\alpha)\to\mathbb{R}$ is injective continuous map and $\nu_{\alpha*i}((c_\alpha,d_\alpha))=(c_{\alpha*i},0)$. $\square$

Remark 3.2. The homeomorphisms $\nu_{\alpha*i}$ occurring in Proposition 3.1 can be either increasing or decreasing. Let us denote by $C^\alpha_{\alpha*i}$ the unique orbit contained in $M_\alpha\cap J(C_{\alpha*i})$ (see Proposition 1.4). From the construction described in Theorem 2.2 we obtain that, in the case where $C_\alpha|C^\alpha_{\alpha*i}|C_{\alpha*i}$ or $C_\alpha=C^\alpha_{\alpha*i}$, the homeomorphism $\nu_{\alpha*i}$ is decreasing and $c_\alpha>0$ or $c_\alpha=0$, respectively. However, in case $|C_\alpha,C^\alpha_{\alpha*i},C_{\alpha*i}|$, the homeomorphism $\nu_{\alpha*i}$ is increasing and $d_\alpha>0$.

References

[1] N. P. Bhatia and G. P. Szegö, Stability Theory of Dynamical Systems, *Springer-Verlag, Berlin, Heidelberg, New York*, 1970.

[2] W. Kaplan, Regular curve-families filling the plane I, *Duke Math. J.* **7** (1940), 154–185.

[3] W. Kaplan, Regular curve-families filling the plane II, *Duke Math. J.* **8** (1941), 11–46.

[4] Z. Leśniak, On an equivalence relation for free mappings embeddeable in a flow, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13**, no. 7 (2003), 1911–1915.

[5] Z. Leśniak, On maximal parallelizable regions of flows of the plane, *Int. J. Pure Appl. Math.* **30**, no. 2 (2006), 151–156.

[6] Z. Leśniak, On parallelizability of flows of free mappings, *Aequationes Math.* **71**, no. 3 (2006), 280–287.

[7] Z. Leśniak, On boundaries of parallelizable regions of flows of free mappings, *Abstr. Appl. Anal.* (2007), 8.

[8] R. C. McCann, Planar dynamical systems without critical points, *Funkcial. Ekvac.* **13** (1970), 67–95.

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