

Iterative roots of mappings with a unique set-value point

By LIN LI (Sichuan), JUSTYNA JARCZYK (Zielona Góra),
WITOLD JARCZYK (Zielona Góra) and WEINIAN ZHANG (Chengdu)

Dedicated to Professor Zoltán Daróczy on the occasion of his 70th birthday

Abstract. We continue the study of the existence of square iterative roots of multifunctions with exactly one set-value point, reported by the third and fourth authors in [7]. First we present nonexistence results in some cases not considered there. Then we give a full description of square roots of strictly monotone, upper semicontinuous multifunctions. The last extends the construction presented by M. KUCZMA [9].

1. Introduction

Iterative roots, or *fractional iterates*, of a given mapping $f : X \rightarrow X$ are functions $g : X \rightarrow X$ satisfying the functional equation

$$g^n = \underbrace{g \circ \cdots \circ g}_{n \text{ times}} = f$$

for a positive integer n . The existence of iterative roots has been intensively studied for almost 200 years, starting from CH. BABBAGE [1]. For the history, results, open questions, and vast literature see the books [16] by GY. TARGONSKI, [10] by M. KUCZMA, and [11] by M. KUCZMA, B. CHOCZEWSKI and R. GER, as well as the survey papers [3] and [2]. It is symptomatic that the existence of iterative roots is rather an exceptional property. And this concerns both the

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purely set-theoretic case (cf., for instance, [17] by G. ZIMMERMAN, but also [15] by R. E. RICE, B. SCHWEIZER and A. SKLAR) and the case of regular mappings, no matter we have in mind the continuity (see [5] by P. D. HUMKE and M. LACZKOVICH), or the holomorphicity (cf. [4] by S. BOGATYI). To avoid this inconvenience there were many ideas how to extend the notion of iterative root. One of them, the so-called phantom root, was introduced by GY. TARGONSKI (cf., among others, [16]). Another one, involving set-valued functions, was introduced by T. POWIERZA [12]–[14] (cf. also [6]).

In [7] the third and fourth authors proposed yet another approach. As the class of continuous functions having a root is small, one can enlarge it a little bit by admitting functions with some discontinuities. At the very beginning one can consider mappings with only isolated points of discontinuity, for instance with exactly one such a point. A function with a unique jump can be identified with a multifunction with a unique set-value point, where the jump of the function is replaced by the set-value of the multifunction.

This is a continuation of paper [7]. In Section 2 we complement [7] by new nonexistence results concerning the purely set-theoretical situation. On the other hand, in Section 3, we deal with the existence of strictly monotonic and upper semicontinuous square roots, extending a classical procedure described for one-valued functions by M. KUCZMA in [9].

2. Nonexistence

Let X be a set and fix its element c . In what follows $\mathcal{F}_c(X)$ stands for the set of all multifunctions $f : X \rightarrow 2^X$ such that $\#f(c) > 1$ and $f(x)$ is a singleton whenever $x \in X \setminus \{c\}$. In the whole paper the following two complementary hypotheses imposed on $f \in \mathcal{F}_c(X)$ will play an important role:

- (i) $\{c\}$ is a value of f ;
- (ii) $\{c\}$ is not a value of f .

We say that a multifunction $f : X \rightarrow 2^X$ is *one-to-one on a set* $A \subset X$ if $f(x_1) \neq f(x_2)$ for all different $x_1, x_2 \in A$; f is called *one-to-one* if it is one-to-one on X .

The first two results will be useful in the next section. However, they seem to be of interest on their own.

Proposition 1. *Let $f \in \mathcal{F}_c(X)$. Then f has no square roots belonging to $\mathcal{F}_c(X)$ and taking the value $\{c\}$.*

PROOF. Suppose on the contrary that $g \in \mathcal{F}_c(X)$ is a square root of f and $g(x_0) = \{c\}$ for an $x_0 \in X$. Then $f(x_0) = g(c)$, whence $x_0 = c$ as $f, g \in \mathcal{F}_c(X)$. But then $g(c) = \{c\}$, a contradiction. \square

Proposition 2. *Let $f \in \mathcal{F}_c(X)$. If the multifunction f is one-to-one, then every square iterative root of f belongs to $\mathcal{F}_c(X)$.*

PROOF. Let $x_0 \in X \setminus \{c\}$. If $u, v \in g(x_0)$ then

$$g(u), g(v) \in g(g(x_0)) = f(x_0),$$

so $g(u) = g(v)$ as $f(x_0)$ is a singleton, and thus $f(u) = f(v)$, i.e. $u = v$ by assumption. Consequently, $\#g(x_0) = 1$. Suppose that $g(c)$ is a singleton, say $g(c) = \{p\}$ with some $p \in X$. If $p = c$ then $f(c) = g(g(c)) = g(c) = \{c\}$. If $p \neq c$ then what we have proved yields $\#g(p) = 1$, and thus $f(c) = g(g(c)) = g(p)$ is a singleton. A contradiction obtained in both cases shows that $\#g(c) > 1$ and completes the proof. \square

Theorem 1. *Let $f \in \mathcal{F}_c(X)$ satisfy hypothesis (i). Then the multifunction f has no square iterative roots, one-to-one on the set $f(c)$. If, in addition, f is one-to-one on $f(c)$, then f has no square iterative roots at all.*

PROOF. Let f have a square root $g : X \rightarrow 2^X$. It follows from the Proposition in [7] that $g(c) = \{u\}$ for a $u \in X$. Then $g(u) = g^2(c) = f(c)$, whence $\#g(u) > 1$, and thus $u \neq c$. Moreover, as f and g commute, we have

$$g(f(c)) = f(g(c)) = f(u).$$

Since $c \neq u$, then $f(u)$ is a singleton. On the other hand $\#f(c) > 1$, which shows that g is not one-to-one on $f(c)$. This completes the first assertion. The second one follows immediately from the obvious fact that if a multifunction $f : X \rightarrow 2^X$ is one-to-one on a set $A \subset X$, then any root of f is also one-to-one on A . \square

Theorem 2. *Let $f \in \mathcal{F}_c(X)$. If $f(c) = \{c, x_0\}$ with some $x_0 \in X$ satisfying $f(x_0) \neq \{x_0\}$, then f has no square iterative roots.*

At first we prove the following fact.

Lemma 1. *Let $f \in \mathcal{F}_c(X)$ satisfy $c \in f(c)$. If $g : X \rightarrow 2^X$ is a square iterative root of f , then $\#g(c) > 1$.*

PROOF. Suppose, on the contrary, that $\#g(c) \leq 1$. As $\#f(c) > 1$ we have $\#g(c) = 1$, and thus $g(c) = \{p\}$ for some $p \in X, p \neq c$. Since $c \in f(c)$ and $\#f(c) > 1$, there is an $x_0 \in X$ such that $x_0 \neq c$ and $\{c, x_0\} \subset f(c)$. Then

$$\{c, x_0\} \subset f(c) = g(g(c)) = g(p),$$

whence

$$\{g(c), g(x_0)\} \subset g(g(p)) = f(p).$$

Now $p \neq c$ implies that $f(p)$ is a singleton, and thus $g(c) = g(x_0)$, and also $f(c) = f(x_0)$, which is impossible as $\#f(c) > 1 = \#f(x_0)$. \square

PROOF OF THEOREM 2. Suppose that f has a square root $g : X \rightarrow 2^X$. By Lemma 1 we get $\#g(c) \geq 2$. Since $c \in f(c) = g(g(c))$, we have $c \in g(p)$ for some $p \in g(c)$. Then

$$g(c) \subset g(g(p)) = f(p),$$

and thus $p = c$. It follows that $c \in g(c)$, whence $g(c) \subset f(c)$. Consequently, $g(c) = f(c) = \{c, x_0\}$ and

$$\{c, x_0\} = f(c) = g(g(c)) = g(c) \cup g(x_0),$$

whence $g(x_0) \subset \{c, x_0\}$. If $c \in g(x_0)$ then $g(c) \subset f(x_0)$, which is impossible as $\#f(x_0) = 1$ because of $x_0 \neq c$. Therefore $g(x_0) = \{x_0\}$, and thus

$$f(x_0) = g(g(x_0)) = g(x_0) = \{x_0\}$$

contrary to the assumption. \square

Some nonexistence results were presented also in [7]. The three examples below show that Theorems 1 and 2 above can be applied in some situations when Theorems 1 and 2 from [7] do not work.

Example 1. As follows from Theorem 1 the multifunction $f : [0, 1] \rightarrow 2^{[0,1]}$, given by

$$f(x) = \begin{cases} \frac{7}{6}x, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ \left\{\frac{7}{12}, \frac{2}{3}\right\}, & \text{if } x = \frac{1}{2}, \\ \frac{2}{3}, & \text{if } x \in \left(\frac{1}{2}, \frac{6}{7}\right], \\ \frac{7}{3}x - \frac{4}{3}, & \text{if } x \in \left(\frac{6}{7}, 1\right], \end{cases}$$

has no square iterative roots, one-to-one on $\left\{\frac{7}{12}, \frac{2}{3}\right\}$.

Example 2. By the same theorem the multifunction $f : [0, 1] \rightarrow 2^{[0,1]}$, defined by

$$f(x) = \begin{cases} \frac{3}{2}x, & \text{if } x \in \left[0, \frac{2}{5}\right), \\ \left\{\frac{1}{4}, \frac{3}{5}\right\}, & \text{if } x = \frac{2}{5}, \\ \frac{5}{4}x - \frac{1}{4}, & \text{if } x \in \left(\frac{2}{5}, 1\right], \end{cases}$$

has no square roots at all.

Example 3. In the case of $f : [0, 1] \rightarrow 2^{[0,1]}$, given by

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{5}, & \text{if } x \in \left[0, \frac{2}{5}\right), \\ \left\{\frac{2}{5}, \frac{3}{5}\right\}, & \text{if } x = \frac{2}{5}, \\ \frac{1}{2}x + \frac{2}{5}, & \text{if } x \in \left(\frac{2}{5}, 1\right], \end{cases}$$

not only the results of [7] but also Theorem 1 is not applicable. However, f has no square roots by Theorem 2.

3. Roots of strictly monotonic upper semicontinuous multifunctions

In this section, assuming that X is a real interval, we study square iterative roots of functions belonging to $\mathcal{F}_c(X)$, strictly monotonic and upper semicontinuous.

Given topological spaces X and Y we say that a multifunction $f : X \rightarrow 2^Y$ is *upper semicontinuous at a point* $x_0 \in X$ if for every open set $V \subset Y$ with $f(x_0) \subset V$ there exists a neighbourhood $U \subset X$ of x_0 such that $f(U) \subset V$. f is called *upper semicontinuous on a set* $A \subset X$ if it is upper semicontinuous at every point of A .

Let X, Y be sets of reals and let $f : X \rightarrow 2^Y$. We say that f is *strictly increasing* [*strictly decreasing*] if $\sup f(x_1) < \inf f(x_2)$ [$\inf f(x_1) > \sup f(x_2)$] whenever $x_1, x_2 \in X$ and $x_1 < x_2$. Multifunctions which are either strictly increasing, or strictly decreasing are said *strictly monotonic*. Clearly every strictly monotonic multifunction is one-to-one.

Given a real number c for every set $A \subset \mathbb{R}$ put

$$A_- = A \cap (-\infty, c) \quad \text{and} \quad A_+ = A \cap (c, \infty).$$

Lemma 2. *Assume that X is an interval and let $f \in \mathcal{F}_c(X)$. If the multifunction f is strictly increasing [strictly decreasing], then it is upper semicontinuous if and only if the functions $f|_{X_-}$ and $f|_{X_+}$ are continuous, $f(c-)$ is the smallest [the greatest] element of $f(c)$ in the case when $X_- \neq \emptyset$, and $f(c+)$ is the greatest [the smallest] element of $f(c)$ in the case when $X_+ \neq \emptyset$.*

PROOF. Consider, for instance, the case of strictly increasing f . Assume that f is upper semicontinuous and, for example, $X_- \neq \emptyset$. Clearly $f|_{X_-}$ and $f|_{X_+}$ are continuous single-valued functions. Moreover, $f(x) < \inf f(c)$ for every $x \in X_-$, whence $f(c-) \leq \inf f(c)$. To show that $f(c-)$ is the smallest element of $f(c)$ it is enough to check that $f(c-) \in f(c)$. Suppose this is not the case. Then $V := (f(c-), \infty) \cap X$ is an open subset of X containing $f(c)$. By the upper semicontinuity of f there is a neighbourhood $U \subset X$ of c such that $f(U) \subset V$. Take any $x \in U_-$. Then $f(x) < f(c-)$. On the other hand, $f(x) \in f(U) \subset V$, that is $f(x) > f(c-)$, which is a contradiction.

Now assume that $f|_{X_-}$ and $f|_{X_+}$ are continuous, $f(c-)$ is the smallest element of $f(c)$ if $X_- \neq \emptyset$, and $f(c+)$ is the greatest element of $f(c)$ if $X_+ \neq \emptyset$. Clearly f is upper semicontinuous at any point of $X \setminus \{c\}$. Now let $V \subset X$ be any open set containing $f(c)$. If $X_- \neq \emptyset$ then $f(c-) \in V$, and thus there is a positive δ_- such that $f(x) \in V$ for every $x \in (c - \delta_-, c) \cap X$. Similarly, if $X_+ \neq \emptyset$ then there is a positive δ_+ such that $f(x) \in V$ whenever $x \in (c, c + \delta_+) \cap X$. Finally, $f((c - \delta, c + \delta) \cap X) \subset V$ for some positive δ . This completes the proof of the upper semicontinuity of the multifunction f . \square

It turns out that, as in the case of single-valued functions (see [9], [10, Chap. XV], [11, Sec. 11.2]), strictly decreasing multifunctions belonging to $\mathcal{F}_c(X)$ have no upper semicontinuous square iterative roots.

Theorem 3. *Assume that X is an interval and let $f \in \mathcal{F}_c(X)$ be strictly decreasing. If the multifunction f is upper semicontinuous on the set $X \setminus \{c\}$, then f has no square iterative roots upper semicontinuous on $X \setminus \{c\}$.*

At first we prove the following auxiliary result.

Lemma 3. *Assume that X is an interval and let $f \in \mathcal{F}_c(X)$. If $\inf f(X_+) < c < \sup f(X_-)$, then the multifunction f has no square iterative roots belonging to $f \in \mathcal{F}_c(X)$ and upper semicontinuous on $X \setminus \{c\}$.*

PROOF. Suppose f has a square root $g \in \mathcal{F}_c(X)$, upper semicontinuous on $X \setminus \{c\}$. If $g(x) = \{c\}$ for an $x \in X \setminus \{c\}$, then $f(x) = g(g(x)) = g(c)$, whence we infer that $g(c)$ is a singleton; a contradiction. Therefore the single-valued continuous functions $g|_{X_-}$ and $g|_{X_+}$ do not take the value c , and thus either $g|_{X_-}, g|_{X_+} < c$, or $g|_{X_-}, g|_{X_+} > c$, or $g|_{X_-} < c < g|_{X_+}$, or $g|_{X_+} < c < g|_{X_-}$. However, by the assumption $\inf f(X_+) < c < \sup f(X_-)$, none of these cases holds. \square

PROOF OF THEOREM 3. Suppose f has a square root $g : X \rightarrow 2^X$, upper semicontinuous on $X \setminus \{c\}$. Then, by Proposition 2, $g \in \mathcal{F}_c(X)$. It follows from Theorem 1 that $\{c\}$ is not a value of f . Therefore, as f is strictly decreasing and the functions $f|_{X_-}$ and $f|_{X_+}$ are continuous, Lemma 3 implies that either $f(X) \subset X_-$, or $f(X) \subset X_+$. Consider, for instance, the latter case. Since $f(c+) > c$ and $f(\sup X-) < \sup X$, there is an $x_0 \in (c, \sup X)$ with $f(x_0) = \{x_0\}$. Actually x_0 is the unique element of the set X with that property. So, as $f(g(x_0)) = g(f(x_0)) = g(x_0)$, we have $g(x_0) = \{x_0\}$. By the continuity of the function $g|_{X_+}$ we can find a positive δ such that $(x_0 - \delta, x_0 + \delta) \subset X_+$ and $g((x_0 - \delta, x_0 + \delta)) \subset X_+$. Since the multifunction f is one-to-one, so is g . Therefore, $g|_{X_+}$ being a continuous single-valued function, is strictly monotonic. Thus it follows from the relation

$$f(x) = g|_{X_+}(g|_{X_+}(x)) \quad \text{for } x \in (x_0 - \delta, x_0 + \delta)$$

that the function $f|_{(x_0 - \delta, x_0 + \delta)}$ is strictly increasing contrary to the assumption. \square

Example 4. According to Theorem 3 the multifunction $f : [0, 1] \rightarrow 2^{[0, 1]}$, defined by

$$f(x) = \begin{cases} -\frac{4}{5}x + \frac{9}{10}, & \text{if } x \in \left[0, \frac{1}{4}\right), \\ \left[\frac{1}{2}, \frac{7}{10}\right], & \text{if } x = \frac{1}{4}, \\ -\frac{4}{15}x + \frac{17}{30}, & \text{if } x \in \left(\frac{1}{4}, 1\right], \end{cases}$$

has no square iterative roots upper semicontinuous on $[0, 1] \setminus \{\frac{1}{4}\}$. On the other hand neither the results of [7], nor Theorems 1 and 2 are applicable.

The assumption of the upper semicontinuity of the multifunction f on $X \setminus \{c\}$ in Theorem 3 can be replaced by the condition $c \in f(c)$.

Theorem 4. *Assume that X is an interval and let $f \in \mathcal{F}_c(X)$. If the multifunction f is strictly decreasing and $c \in f(c)$, then f has no square iterative roots upper semicontinuous on $X \setminus \{c\}$.*

PROOF. As $c \in f(c)$ and f is strictly decreasing, we have $f(X_-) \subset X_+$ and $f(X_+) \subset X_-$, and thus $f(X \setminus \{c\}) \subset X \setminus \{c\}$. Therefore, the existence of a square root of f , upper semicontinuous on $X \setminus \{c\}$, would imply the upper semicontinuity of f on $X \setminus \{c\}$, which contradicts Theorem 3. \square

Now we pass to the question on strictly monotonic square iterative roots of strictly increasing upper semicontinuous multifunctions belonging to $\mathcal{F}_c(X)$. First of all observe that, according to Theorem 1, condition (ii) is necessary for the existence of square roots of such multifunctions. So it is natural to assume it in Theorems 5 and 6 below. As follows from these results the construction of strictly monotonic square roots of multifunctions studied here strictly depends on the set of square roots in the case of single-valued functions. The full description of this set was given by M. KUCZMA in [9] (see also [10, Chap. XV], [11, Sec. 11.2]).

Theorem 5. *Assume that X is an interval and let $f \in \mathcal{F}_c(X)$ be strictly increasing and upper semicontinuous multifunction satisfying condition (ii).*

Let $g : X \rightarrow 2^X$ be a strictly increasing [strictly decreasing] and upper semicontinuous square iterative root of f . If $\sup f(c) < c$ then $X_- \neq \emptyset$, there exists a strictly increasing [strictly decreasing] continuous function $\gamma_- : X_- \rightarrow X_-$ such that $\gamma_-^2 = f|_{X_-}$, $f(X) \subset \gamma_-(X_-)$, and

$$g(x) = \begin{cases} \gamma_-(x), & \text{if } x \in X_-, \\ \gamma_-^{-1}(f(x)), & \text{if } x \in X_+ \cup \{c\}. \end{cases} \quad (3.1)$$

If $c < \inf f(c)$ then $X_+ \neq \emptyset$, there exist a strictly increasing [strictly decreasing] continuous function $\gamma_+ : X_+ \rightarrow X_+$ such that $\gamma_+^2 = f|_{X_+}$, $f(X) \subset \gamma_+(X_+)$, and

$$g(x) = \begin{cases} \gamma_+^{-1}(f(x)), & \text{if } x \in X_- \cup \{c\}, \\ \gamma_+(x), & \text{if } x \in X_+. \end{cases} \quad (3.2)$$

Conversely, the above procedures yield strictly monotonic and upper semicontinuous square iterative roots of f in the cases $\sup f(c) < c$ and $c < \inf f(c)$.

PROOF. Let g be a strictly increasing and upper semicontinuous square root of f . By Propositions 2 and 1 we deduce that the multifunction g does not take the value $\{c\}$.

Assume, for instance, that $\sup f(c) < c$. Then $X_- \neq \emptyset$. By the monotonicity and continuity of $\sup f|_{X_+}$, it follows from condition (ii) that $f(X) \subset X_-$. Therefore, as $g(x) \neq \{c\}$ for every $x \in X$, we infer that $g(X) \subset X_-$. The function $\gamma_- := g|_{X_-}$ maps X_- into itself, is strictly increasing and continuous, and $\gamma_-^2 = f|_{X_-}$. Moreover, $f(X) = g(g(X)) \subset g(X_-) = \gamma_-(X_-)$. For every $x \in X$ we have $f(x) = g(g(x)) = \gamma_-(g(x))$, i.e. $g(x) = \gamma_-^{-1}(f(x))$, which gives the desired representation of the multifunction g . In the case when g is strictly decreasing, the proof is similar.

Now pass to proving the converse. One can easily check that the strict monotonicity of f as well as γ_- and γ_+ gives the strict monotonicity of the multifunction g given by formulas (3.1) and (3.2), respectively. Taking into consideration the upper semicontinuity of f and the continuity of γ_- and γ_+ , and making use of Lemma 2, we deduce that g is upper semicontinuous. Clearly, formulas (3.1) and (3.2) define square iterative roots of f . \square

Remark 1. In the next result the description of strictly decreasing square roots of a strictly increasing and upper semicontinuous multifunction $f \in \mathcal{F}_c(X)$ strongly depends on strictly decreasing continuous functions $\gamma : X_- \rightarrow X_+$ conjugating $f|_{X_-}$ and $f|_{X_+}$, i.e. solutions of the equation

$$\gamma \circ f|_{X_-} = f|_{X_+} \circ \gamma.$$

A construction of all such functions can be deduced from [8] (cf. also [9, p. 171]); there are many of them and, roughly speaking, they depend on an arbitrary function.

Theorem 6. *Assume that X is an interval and let $f \in \mathcal{F}_c(X)$ be strictly increasing and upper semicontinuous multifunction satisfying condition (ii).*

Assume that

$$\inf f(c) \leq c \leq \sup f(c) \tag{3.3}$$

and let $g : X \rightarrow 2^X$ be an upper semicontinuous square iterative root of f .

If g is strictly increasing, then there exist strictly increasing continuous functions $\gamma_- : X_- \rightarrow X_-$ and $\gamma_+ : X_+ \rightarrow X_+$ satisfying $\gamma_-^2 = f|_{X_-}$, $\gamma_+^2 = f|_{X_+}$, and a set $M \subset X$ such that

$$\inf M = \gamma_-(c-) \quad \text{and} \quad \sup M = \gamma_+(c+), \tag{3.4}$$

if $c \in f(c)$ then

$$c \in M, \quad f(c)_- = \gamma_-(M_-) \cup M_- \quad \text{and} \quad f(c)_+ = \gamma_+(M_+) \cup M_+, \tag{3.5}$$

if $c \notin f(c)$ then

$$c \notin M, \quad f(c)_- = \gamma_-(M_-) \quad \text{and} \quad f(c)_+ = \gamma_+(M_+), \quad (3.6)$$

and

$$g(x) = \begin{cases} \gamma_-(x), & \text{if } x \in X_-, \\ M, & \text{if } x = c, \\ \gamma_+(x), & \text{if } x \in X_+. \end{cases} \quad (3.7)$$

If g is strictly decreasing, then there exist a strictly decreasing continuous function $\gamma : X_- \rightarrow X_+$ satisfying the condition

$$f(X_+) \subset \gamma(X_-) \quad (3.8)$$

and the equation

$$f(\gamma(x)) = \gamma(f(x)) \quad (3.9)$$

and a set $M \subset X$ such that

$$\inf M = \gamma^{-1}(f(c_+)) \quad \text{and} \quad \sup M = \gamma(c_-), \quad (3.10)$$

if $c \in f(c)$ then

$$c \in M, \quad f(c)_- = \gamma^{-1}(f(M_+)) \cup M_- \quad \text{and} \quad f(c)_+ = \gamma(M_-) \cup M_+, \quad (3.11)$$

if $c \notin f(c)$ then

$$c \notin M, \quad f(c)_- = \gamma^{-1}(f(M_+)) \quad \text{and} \quad f(c)_+ = \gamma(M_-), \quad (3.12)$$

and

$$g(x) = \begin{cases} \gamma(x), & \text{if } x \in X_-, \\ M, & \text{if } x = c, \\ \gamma^{-1}(f(x)), & \text{if } x \in X_+. \end{cases} \quad (3.13)$$

Conversely, the above procedures yield strictly monotonic and upper semi-continuous square iterative roots of f in the case when (3.3) holds.

PROOF. According to Propositions 2 and 1 the multifunction g does not take the value $\{c\}$.

At first assume that g is strictly increasing. We prove that $g(X_-) \subset X_-$. If this was not the case, then, by the continuity of the function $g|_{X_-}$, we would have $g(X_-) \subset X_+$. Then, as g is strictly increasing, $g(X) \subset X_+$, whence also $f(X) \subset X_+$. This, the monotonicity of f and (3.3) would give $X_- = \emptyset$ and,

consequently, $g(X_-) \subset X_-$. Similarly one can show that $g(X_+) \subset X_+$. Now put $\gamma_- := g|_{X_-}$, $\gamma_+ := g|_{X_+}$ and $M := g(c)$. Clearly, $\gamma_- : X_- \rightarrow X_-$ and $\gamma_+ : X_+ \rightarrow X_+$ are strictly increasing and continuous, $\gamma_-^2 = f|_{X_-}$, $\gamma_+^2 = f|_{X_+}$, and, by Lemma 2, $\gamma_-(c-) = \inf M$ and $\gamma_+(c+) = \sup M$ when $X_- \neq \emptyset$ and $X_+ \neq \emptyset$, respectively. Consider the case $c \in f(c)$. Then $c \in g(x)$ for an $x \in X$. Since $g \in \mathcal{F}_c(X)$ and $\{c\}$ is not a value of g it follows that $x = c$, i.e. $c \in g(c) = M$. Moreover,

$$\begin{aligned} f(c) &= g(g(c)) = g(M) = g(M_-) \cup g(c) \cup g(M_+) \\ &= \gamma_-(M_-) \cup M \cup \gamma_+(M_+) \cup \{c\}, \end{aligned}$$

and thus

$$f(c)_- = \gamma_-(M_-) \cup M_- \quad \text{and} \quad f(c)_+ = \gamma_+(M_+) \cup M_+.$$

Now assume that $c \neq f(c)$. Then $c \notin M$; otherwise $c \in g(c) \subset g(g(c)) = f(c)$. Therefore

$$f(c) = g(g(c)) = g(M) = g(M_-) \cup g(M_+) = \gamma_-(M_-) \cup \gamma_+(M_+),$$

whence

$$f(c)_- = \gamma_-(M_-) \quad \text{and} \quad f(c)_+ = \gamma_+(M_+).$$

Now assume that g is strictly decreasing. We prove that $g(X_-) \subset X_+$ and $g(X_+) \subset X_-$. Suppose the opposite. Then, as g does not take the value $\{c\}$ and is upper semicontinuous and strictly decreasing, we have either $g(X) \subset X_-$, or $g(X) \subset X_+$, and thus either $f(X) \subset X_-$, or $f(X) \subset X_+$, contrary to assumption (3.3). In particular, we get $X_- \neq \emptyset$ and $X_+ \neq \emptyset$ because of the condition $X \setminus \{c\} \neq \emptyset$. The function $\gamma := g|_{X_-}$ maps X_- into X_+ and is strictly decreasing and continuous. Moreover, $f(X_+) = g(g(X_+)) \subset g(X_-) = \gamma(X_-)$. By (i) we have also $f(X_-) \subset X_-$ and $f(X_+) \subset X_+$, whence

$$f(\gamma(x)) = f(g(x)) = g(f(x)) = \gamma(f(x)), \quad x \in X_-,$$

i.e. (3.9) holds, and $f(x) = g(g(x)) = \gamma(g(x))$, i.e.

$$g(x) = \gamma^{-1}(f(x)), \quad x \in X_+.$$

Put $M := g(c)$. Then we have (3.13) and, by Lemma 2, $\inf M = g(c+) = \gamma^{-1}(f(c+))$ and $\sup M = g(c-) = \gamma(c-)$. If $c \in f(c)$ then, as in the preceding case, $c \in M$, and thus

$$f(c) = g(g(c)) = g(M) = g(M_-) \cup g(c) \cup g(M_+) = \gamma(M_-) \cup M \cup \gamma^{-1}(f(M_+)),$$

whence we get

$$f(c)_- = \gamma^{-1}(f(M_+)) \cup M_- \quad \text{and} \quad f(c)_+ = \gamma(M_-) \cup M_+.$$

Assuming that $c \notin f(c)$, as previously we obtain $c \notin M$ and

$$f(c) = g(g(c)) = g(M) = g(M_-) \cup g(M_+) = \gamma(M_-) \cup \gamma^{-1}(f(M_+)),$$

which gives

$$f(c)_- = \gamma^{-1}(f(M_+)) \quad \text{and} \quad f(c)_+ = \gamma(M_-).$$

For a proof of the converse first observe that formulas (3.7) and (3.13) define a strictly increasing and a strictly decreasing multifunction, respectively, which, according to Lemma 2 and conditions (3.4) and (3.10), are upper semicontinuous. To prove that they are square roots of f consider, for instance, the multifunction g given by (3.13). Take any $x \in X$. If $x = c$ then, by the assumed representations (3.11) and (3.12) of $f(c)_-$ and $f(c)_+$ depending on whether c belongs to $f(c)$ or not, we come to the equality $g(g(c)) = f(c)$. If $x \in X_-$ then $g(x) = \gamma(x) \in X_+$, and, by (3.9),

$$g(g(x)) = g(\gamma(x)) = \gamma^{-1}(f(\gamma(x))) = \gamma^{-1}(\gamma(f(x))) = f(x).$$

Finally, if $x \in X_+$ then $f(x) \in X_+$, $g(x) = \gamma^{-1}(f(x)) \in X_-$, and

$$g(g(x)) = g(\gamma^{-1}(f(x))) = \gamma(\gamma^{-1}(f(x))) = f(x). \quad \square$$

The next two results give conditions which are necessary and sufficient for the existence of strictly monotonic upper semicontinuous square roots of strictly increasing and upper semicontinuous multifunctions belonging to $\mathcal{F}_c(X)$.

Theorem 7. *Assume that X is an interval and let $f \in \mathcal{F}_c(X)$ be strictly increasing and upper semicontinuous.*

The multifunction f has a strictly increasing and upper semicontinuous square iterative root if and only if either $c \notin f(X)$, or $c \in f(c)$ and there exist strictly increasing continuous functions $\gamma_- : X_- \rightarrow X_-$ and $\gamma_+ : X_+ \rightarrow X_+$ satisfying $\gamma_-^2 = f|_{X_-}$, $\gamma_+^2 = f|_{X_+}$ and a set $M \subset X$ such that condition (3.5) holds.

PROOF. If f has a strictly increasing and upper semicontinuous square root and $c \in f(X)$, then, by Theorem 1, condition (ii) holds, $c \in f(c)$, and thus the assertion follows from Theorem 6.

Now assume that $c \notin f(X)$. This implies that condition (ii) holds. At first consider the case $\sup f(c) < c$. Then, by the upper semicontinuity of f at c and the continuity of $f|_{X_+}$, we have $f(X) \subset X_-$. Thus, as follows from the general construction of continuous strictly increasing roots of a continuous strictly increasing function (see [9, Case A, pp. 163-165]), there exists a continuous strictly increasing square root $\gamma_- : X_- \rightarrow X_-$ of $f|_{X_-}$ such that $f(X) \subset \gamma_-(X_-)$. Then (cf. Theorem 5) formula (3.1) defines a strictly increasing and upper semicontinuous square iterative root of f . In the case when $c < \inf f(c)$ we proceed similarly. Finally consider case (3.3) and let $\gamma_- : X_- \rightarrow X_-$ and $\gamma_+ : X_+ \rightarrow X_+$, be continuous strictly increasing square roots of $f|_{X_-}$ and $f|_{X_+}$, respectively. As $c \notin f(X)$, we have $c \notin f(c)$. Thus, putting

$$M = \gamma_-^{-1}(f(c)_-) \cup \gamma_+^{-1}(f(c)_+),$$

we see that conditions (3.4) and (3.6) hold, and formula (3.7) (cf. Theorem 6) defines a strictly increasing and upper semicontinuous square iterative root of f .

If $c \in f(c)$ and there exist strictly increasing square roots $\gamma_- : X_- \rightarrow X_-$ and $\gamma_+ : X_+ \rightarrow X_+$ of $f|_{X_-}$ and $f|_{X_+}$, respectively, and a set $M \subset X$ satisfying condition (3.5), then, in particular, (ii) holds, whence the assertion follows from Theorem 6. □

In some cases condition (ii) turns out to be sufficient for the existence of strictly decreasing and upper semicontinuous square iterative roots of f .

Corollary 1. *Assume that X is an interval and let $f \in \mathcal{F}_c(X)$ be strictly increasing and upper semicontinuous. Assume that $c \notin f(c)$ or $f(c)$ is an interval. Then f has a strictly increasing and upper semicontinuous iterative root if and only if condition (ii) holds.*

PROOF. By Theorem 1 condition (ii) is necessary for the existence of the desired root of f . Now we prove the sufficiency of (ii) under the assumption of the corollary.

If $c \notin f(c)$ then, by (ii), $c \notin f(X)$ and we are done by Theorem 7. So assume that $c \in f(c)$ and $f(c)$ is an interval. Take any strictly increasing continuous functions $\gamma_- : X_- \rightarrow X_-$ and $\gamma_+ : X_+ \rightarrow X_+$ satisfying the equations $\gamma_-^2 = f|_{X_-}$ and $\gamma_+^2 = f|_{X_+}$, respectively, and put

$$M = [\gamma_-(c_-), \gamma_+(c_+)].$$

Then conditions (3.4) and (3.6) hold and the assertion follows from Theorem 6. □

Before formulating the next result we recall the notion of a regular fixed point (cf. [9, Case C], [10, Chap. XV, Sec. 5]).

Assume that I is an interval and let $\gamma : I \rightarrow I$ be a continuous strictly increasing function. As γ is monotonic it can be continued onto the closure $\text{cl } I$ of I (in the space $\bar{\mathbb{R}}$ of extended reals). Given a fixed point $\xi \in I$ of γ we put

$$A_\xi = \{x \in \text{cl } I : \gamma(x) = x \leq \xi\} \quad \text{and} \quad B_\xi = \{x \in \text{cl } I : \gamma(x) = x \geq \xi\}.$$

We say that ξ is a *regular* fixed point of γ if

- (a) $\min A_\xi = \inf I$ if and only if $\max B_\xi = \sup I$;¹
- (b) there exists a strictly decreasing function α mapping A_ξ onto B_ξ ;
- (c) for every component (a, b) of the set $\{x \in \text{cl } I : \gamma(x) \neq x\}$, where $a \in A_\xi \cup \{\inf I\}$, $b \in A_\xi$, and $a < b$, the graphs of $\gamma|_{(a,b)}$ and $\gamma|_{(\alpha(a), \alpha(b))}$ lie on the opposite sides of the diagonal.

Theorem 8. *Assume that X is an interval and let $f \in \mathcal{F}_c(X)$ be strictly increasing and upper semicontinuous multifunction satisfying condition (ii).*

The multifunction f has a strictly decreasing and upper semicontinuous square iterative root if and only if either

$$\sup f(c) < c \text{ and } X_- \text{ contains a regular fixed point of } f,$$

or

$$c < \inf f(c) \text{ and } X_+ \text{ contains a regular fixed point of } f,$$

or

$$\inf f(c) \leq c \leq \sup f(c) \quad \text{and} \quad c \notin f(c),$$

or

$c \in f(c)$ and there exist strictly decreasing continuous function $\gamma : X_- \rightarrow X_+$, satisfying condition (3.8) and equation (3.9), and a set $M \subset X$ such that conditions (3.10) and (3.11) hold.

PROOF. Assume that f has a strictly decreasing and upper semicontinuous square root. If $\sup f(c) < c$ then Theorem 5 provides a strictly decreasing continuous square root of the function $f|_{X_-}$, and thus, by [9, the first paragraph on p. 171], X_- contains a regular fixed point of f . In the case when $c < \inf f(c)$ we agree similarly. If $c \in f(c)$ then it is enough to make use of Theorem 6.

Now assume that $\sup f(c) < c$ and X_- contains a regular fixed point of f . Then it follows from condition (a) of the definition of a regular point and from (ii)

¹The definition given in [9] in fact does not require condition (a) explicitly, but as follows from the proofs presented there it should be assumed.

that $\inf X$ is not a fixed point of f (extended, if necessary, to $\text{cl } X$). Therefore, in view of [9, Theorem III], $f|_{X_-}$ has a strictly decreasing continuous square root $\gamma_- : X_- \rightarrow X_-$. Moreover, as follows from the proof of Theorem III [9, pp. 171–173] we can also assume that $f(X) \subset \gamma_-(X_-)$, and thus Theorem 5 provides a strictly decreasing continuous square root of f . In the case when $c < \inf f(c)$ and X_+ contains a regular fixed point of f we proceed analogously.

Finally consider case (3.3). If $c \notin f(c)$ then let $\gamma : X_- \rightarrow X_+$ be any strictly decreasing continuous function satisfying condition (3.8) and equation (3.9) (see [8], also [9, pp. 171–173]) and put $M = M_- \cup M_+$, where

$$M_- = \gamma^{-1}(f(c)_+) \quad \text{and} \quad M_+ = f^{-1}(\gamma(f(c)_-)),$$

and use Theorem 6. The same theorem gives the assertion when $c \in f(c)$ and there exist strictly decreasing continuous function satisfying condition (3.8) and equation (3.9), and a set $M \subset X$ such that conditions (3.10) and (3.11) hold. \square

Corollary 2. *Assume that X is an interval and let $f \in \mathcal{F}_c(X)$ be strictly increasing and upper semicontinuous multifunction satisfying condition (ii). If either*

$$X_- \neq \emptyset \quad \text{and} \quad \inf f(c) = c,$$

or

$$X_+ \neq \emptyset \quad \text{and} \quad \sup f(c) = c,$$

or

$$c < \inf f(c) \quad \text{and} \quad f(\sup X_-) = \sup X,$$

or

$$\sup f(c) < c \quad \text{and} \quad f(\inf X_+) = \inf X,$$

then f has no strictly decreasing and upper semicontinuous square iterative root.

PROOF. Assume that $X_- \neq \emptyset$ and $\inf f(c) = c$. Suppose that f has a strictly decreasing and upper semicontinuous square root. According to Theorem 6 we may find a strictly decreasing continuous function $\gamma : X_- \rightarrow X_+$ satisfying condition (3.8) and equation (3.9), and a set $M \subset X$ such that conditions (3.10) and (3.11) hold. Then, as

$$\gamma^{-1}(f(M_+)) \cup M_- = f(c)_- = \emptyset,$$

we see that $M_- = \emptyset$ and, by (3.8), also $M_+ = \emptyset$. Therefore $M = \{c\}$, which jointly with (3.13) shows that g and, consequently, f are single-valued; a contradiction.

If $c < \inf f(c)$ and $f(\sup X_-) = \sup X$, then the interval X_+ does not contain a regular fixed point of $f|_{X_+}$, and thus (cf. [9, the first paragraph on p. 171]) there is no strictly decreasing continuous square root γ_+ of $f|_{X_+}$. Now the assertion

follows from Theorem 5.

In the remaining two cases we proceed analogously. \square

Example 5. Theorem 7 guaranties that $f : [0, 1] \rightarrow 2^{[0,1]}$, given by

$$f(x) = \begin{cases} \frac{1}{4}x + \frac{3}{16}, & \text{if } x \in \left[0, \frac{3}{4}\right), \\ \left[\frac{3}{8}, \frac{17}{40}\right], & \text{if } x = \frac{3}{4}, \\ \frac{1}{5}x + \frac{11}{40}, & \text{if } x \in \left(\frac{3}{4}, 1\right], \end{cases}$$

has strictly increasing and upper semicontinuous square roots. By Theorem 5, one of them is defined by the formula

$$g(x) = \begin{cases} \frac{1}{2}x + \frac{1}{8}, & \text{if } x \in \left[0, \frac{3}{4}\right), \\ \left[\frac{1}{2}, \frac{3}{5}\right], & \text{if } x = \frac{3}{4}, \\ \frac{2}{5}x + \frac{3}{10}, & \text{if } x \in \left(\frac{3}{4}, 1\right]. \end{cases}$$

Example 6. Define $f : [0, 1] \rightarrow 2^{[0,1]}$ by

$$f(x) = \begin{cases} \frac{1}{4}x + \frac{1}{8}, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ \left[\frac{1}{4}, \frac{3}{4}\right], & \text{if } x = \frac{1}{2}, \\ \frac{1}{4}x + \frac{5}{8}, & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

and $\gamma : [0, \frac{1}{2}) \rightarrow (\frac{1}{2}, 1]$ by $\gamma(x) = -x + 1$. Clearly $f \in \mathcal{F}_{\frac{1}{2}}([0, 1])$ is strictly increasing and upper semicontinuous, and γ is strictly decreasing continuous. Moreover, $f\left(\left(\frac{1}{2}, 1\right]\right) \subset \gamma\left(\left[0, \frac{1}{2}\right)\right)$ and γ satisfies equation (3.9). Taking $M = \left[\frac{1}{4}, \frac{1}{2}\right]$ we see that also conditions (3.10) and (3.11) hold. Therefore, according to Theorem 8, there are strictly decreasing and upper semicontinuous square roots of f . By Theorem 6 one of them is given by

$$g(x) = \begin{cases} -x + 1, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ \left[\frac{1}{4}, \frac{1}{2}\right], & \text{if } x = \frac{1}{2}, \\ -\frac{1}{4}x + \frac{3}{8}, & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

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LIN LI
YANGTZE CENTER OF MATHEMATICS
AND DEPARTMENT OF MATHEMATICS
SICHUAN UNIVERSITY
CHENGDU, SICHUAN 610064
P. R. CHINA

JUSTYNA JARCZYK
FACULTY OF MATHEMATICS
COMPUTER SCIENCE AND ECONOMETRICS
UNIVERSITY OF ZIELONA GÓRA
SZAFRANA 4A
PL-65-516 ZIELONA GÓRA
POLAND

E-mail: j.jarczyk@wmie.uz.zgora.pl

WITOLD JARCZYK
FACULTY OF MATHEMATICS
COMPUTER SCIENCE AND ECONOMETRICS
UNIVERSITY OF ZIELONA GÓRA
SZAFRANA 4A
PL-65-516 ZIELONA GÓRA
POLAND

WEINIAN ZHANG
YANGTZE CENTER OF MATHEMATICS
AND DEPARTMENT OF MATHEMATICS
SICHUAN UNIVERSITY
CHENGDU, SICHUAN 610064
P. R. CHINA

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