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Ratio of Stolarsky means: monotonicity and comparison

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Dedicated to Professor Zoltán Daróczy on the occasion of his 70th birthday

Abstract. The Stolarsky mean ([29], [30]) $S_{a,b}(x,y)$ of the numbers $x, y > 0$ with parameters $a, b \in \mathbb{R}$ is defined by

$$S_{a,b}(x,y) = \left(\frac{b(x^a - y^a)}{a(x^b - y^b)}\right)^{\frac{1}{a-b}} \qquad \text{if } ab(a-b)(x-y) \neq 0,$$

while for $ab(a-b)(x-y) = 0$ the function $S_{a,b}(x,y)$ is extended continuously. We study monotonicity properties of the ratio

$$R_{a,b}(x,y,z) := \frac{S_{a,b}(x,y)}{S_{a,b}(x,z)} \quad (a, b \in \mathbb{R},\ 0 < x < y < z)$$

in the parameters a, b where $0 < x < y < z$ and completely solve the comparison problem

$$R_{a,b}(x,y,z) \leq R_{c,d}(x,y,z) \qquad (a,b,c,d \in \mathbb{R},\ 0 < x < y < z)$$

for this ratio. This generalizes, among others, the results of C. E. M. Pearce and J. Pečarić [27] and F. Qi, Sh.-X. Chen and Ch.-P. Chen [9], [15].

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1. Introduction

Let $a > 0$ then for any $n \in \mathbb{N}$

$$\frac{n}{n+1} < \left(\frac{1}{n} \sum_{k=1}^{n} k^a \Big/ \frac{1}{n+1} \sum_{k=1}^{n+1} k^a \right)^{1/a} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{1}$$

The left hand side inequality is due to Alzer [2], the right hand side inequality is due to Martins [23], the inequality between the far ends is due to Minc and Sathre [24]. There is a rich literature on inequality (1), various extensions, generalizations of it were found in [17], [19], [21], [28], [31], see also [15] and the references there.

In [4], [6], [10], [11] Alzer's inequality is extended to all real a. In [3], [12] it was proved that Martin's inequality is reversed for $a < 0$. A survey of some recent results on these two inequalities has been presented in [1], [5].

In 2004 Ch.-P. Chen and F. Qi [9] proved that for fixed $0 < x < y < z$ the function

$$r \to \frac{L_r(x,y)}{L_r(x,z)} \quad (r \in \mathbb{R}) \tag{2}$$

is strictly decreasing and it has been re-proved in [2007] by F. Qi, Sh.-X. Chen and Ch.-P. Chen ([15], see also the correction in [16] Remark 3, p. 801). Here $L_r(x,y) := S_{r+1,1}(x,y)$ is the generalized logarithmic mean and $S_{a,b}(x,y)$ is the Stolarsky mean of $x, y > 0$ with parameters $a, b \in \mathbb{R}$ defined by

$$S_{a,b}(x,y) = \begin{cases} \left(\dfrac{b(x^a - y^a)}{a(x^b - y^b)} \right)^{\frac{1}{a-b}} & \text{if } ab(a-b)(x-y) \neq 0, \\ \left(\dfrac{x^a - y^a}{a(\ln x - \ln y)} \right)^{\frac{1}{a}} & \text{if } a(x-y) \neq 0,\ b = 0, \\ \left(\dfrac{b(\ln x - \ln y)}{x^b - y^b} \right)^{-\frac{1}{b}} & \text{if } b(x-y) \neq 0,\ a = 0, \\ \exp\left(-\dfrac{1}{a} + \dfrac{x^a \ln x - y^a \ln y}{x^a - y^a} \right) & \text{if } b(x-y) \neq 0,\ a = b, \\ \sqrt{xy} & \text{if } x - y \neq 0,\ a = b = 0, \\ x & \text{if } x - y = 0. \end{cases}$$

$S_{a,b}(x,y)$ is a C^∞ function on the domain $\{(a,b,x,y) \mid a,b \in \mathbb{R},\ x,y > 0\}$.

The monotonicity result of [9], [15] also follows from Theorem 4 of Pearce and Pečarić [27] who proved that the function

$$r \to \frac{L_r(x_1,x_2)}{L_r(y_1,y_2)} \quad (r \in \mathbb{R})$$

is nondecreasing, provided that $x_1, x_2, y_1, y_2 > 0$ with $\max\left(\frac{x_1}{x_2}, \frac{x_2}{x_1}\right) \geq \max\left(\frac{y_1}{y_2}, \frac{y_2}{y_1}\right)$.

Due to the monotonicity of the function (2) and the known inequality $L_0(x,y) < L_r(x,y) < \max\{x,y\}$, (valid for $r > 0$, $x \neq y$), one gets that

$$\frac{y}{z} < \frac{L_r(x,y)}{L_r(x,z)} < \frac{L_0(x,y)}{L_0(x,z)} \quad (r > 0,\ 0 < x < y < z).$$

This is the connection to (1), as the last inequality can be considered as a continuous analogue of (1).

Recently Ch.-P. Chen [7], [8] established a more general result: Let a, b, c, d be fixed positive numbers with $a \neq b$, $c \neq d$ and r, s be real numbers. Then the function

$$(r,s) \to \frac{S_{r,s}(a,b)}{S_{r,s}(c,d)}$$

is strictly $\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}$ with both r and s according as

$$\frac{\min\{a,b\}}{\max\{a,b\}} \lesseqgtr \frac{\min\{c,d\}}{\max\{c,d\}}.$$

Let, for $a, b \in \mathbb{R}$, $0 < x < y < z$

$$R_{a,b}(x,y,z) := \frac{S_{a,b}(x,y)}{S_{a,b}(x,z)}. \tag{3}$$

Here we establish monotonicity properties of $R_{a,b}$ and completely solve the comparison problem

$$R_{a,b}(x,y,z) \leq R_{c,d}(x,y,z) \quad (a,b,c,d \in \mathbb{R},\ 0 < x < y < z)$$

of these ratios.

2. Monotonicity of the ratio of Stolarsky means

The next theorem is a generalization of the monotonicity result of [9], [15].

Theorem 1 (first monotonicity property)**.** *For fixed $a, b \in \mathbb{R}$, $0 < x < y < z$ the functions*

$$f_0(r) := R_{a,r}(x,y,z) \quad (r \in \mathbb{R}) \qquad f_1(r) := R_{r,b}(x,y,z) \quad (r \in \mathbb{R}) \tag{4}$$

are strictly decreasing on $\mathbb{R}$.

PROOF. By the symmetry $R_{a,b}(x,y,z) = R_{b,a}(x,y,z)$ it is enough to prove the statement for f_1. Let

$$g_1(r) := \ln f_1(r) = \begin{cases} \dfrac{l(r) - l(b)}{r - b} & \text{if } r \neq b, \\ l'(b) & \text{if } r = b \end{cases} \tag{5}$$

where

$$l(r) := \begin{cases} \ln|y^r - x^r| - \ln|z^r - x^r| & \text{if } r \neq 0, \\ \ln|\ln y - \ln x| - \ln|\ln z - \ln x| & \text{if } r = 0. \end{cases}$$

The function f_1 is (strictly) decreasing if and only if g_1 is (strictly) decreasing. To prove the decreasingness of g_1 we need some lemmas.

The next lemma is due to L. Galvani [20], see also [25], p. 20, Theorem 1.3.1.

Lemma 2 ([25], p. 20)**.** *Let $f : I \to \mathbb{R}$ be a function defined on an interval I. Then f is convex/concave (respectively strictly convex/concave) if and only if the associated functions s_r defined by*

$$s_r : I \setminus \{r\} \to \mathbb{R}, \qquad s_r(x) = \frac{f(x) - f(r)}{x - r}$$

are increasing/decreasing (respectively strictly increasing/decreasing) on $I \setminus \{r\}$ for all $r \in I$.

We also need some properties of the second derivative of the function l.

Lemma 3. *The function l'' is*

(i) *an even function and* $\lim_{r\to\infty} l''(r) = \lim_{r\to-\infty} l''(r) = 0$,

(ii) *strictly increasing if* $r > 0$, *strictly decreasing if* $r < 0$

(iii) $\dfrac{\ln^2\left(\frac{y}{x}\right) - \ln^2\left(\frac{z}{x}\right)}{12} < l''(r) < 0 \quad (r \in \mathbb{R})$

PROOF. A simple calculation shows that

$$\begin{aligned} l''(r) &= -x^r y^r \left(\frac{\ln x - \ln y}{x^r - y^r}\right)^2 + x^r z^r \left(\frac{\ln x - \ln z}{x^r - z^r}\right)^2 \\ &= -\frac{\ln^2\left(\dfrac{y}{x}\right)}{\left(\dfrac{y}{x}\right)^r - 2 + \left(\dfrac{x}{y}\right)^r} + \frac{\ln^2\left(\dfrac{z}{x}\right)}{\left(\dfrac{z}{x}\right)^r - 2 + \left(\dfrac{x}{z}\right)^r} \end{aligned}$$

$$l'''(r) = x^r y^r (x^r + y^r) \left(\frac{\ln x - \ln y}{x^r - y^r}\right)^3 - x^r z^r (x^r + z^r) \left(\frac{\ln x - \ln z}{x^r - z^r}\right)^3$$

$$= \frac{1}{r^3}\left[H\left(\left(\frac{y}{x}\right)^r\right) - H\left(\left(\frac{z}{x}\right)^r\right)\right] \tag{6}$$

where

$$H(u) := u(1+u)\left(\frac{\ln u}{u-1}\right)^3 \qquad (u \in \mathbb{R},\ u \neq 1).$$

The second form of l'' shows that it is an *even* function and its limit at $r \to \pm\infty$ is zero, proving (i). To justify (ii) we write H' as

$$H'(u) = \left(\frac{\ln u}{(u-1)^2}\right)^2 k(u) \tag{7}$$

with $k(u) = -(u^2+4u+1)\ln u + 3(u^2-1)$ $(u > 0)$. It is easy to check that $k(1) = k'(1) = k''(1) = 0, k'''(u) = \frac{-2(u-1)^2}{u^3}$ therefore by Taylor's theorem

$$k(u) = \frac{-2(\xi-1)^2}{6\xi^3}$$

where $u < \xi < 1$ for $0 < u < 1$ and $1 < \xi < u$ for $1 < u$. This shows that

$$\begin{aligned} k(u) &> 0 \qquad \text{if } 0 < u < 1, \\ k(u) &< 0 \qquad \text{if } 1 < u, \end{aligned}$$

thus by (7) H is strictly increasing for $0 < u < 1$ and strictly decreasing for $1 < u$.

If $r > 0$ then $1^r < \left(\frac{y}{x}\right)^r < \left(\frac{z}{x}\right)^r$ hence by (6) $l'''(r) > 0$ and l'' is strictly increasing, while *for* $r < 0$ we have $1^r > \left(\frac{y}{x}\right)^r > \left(\frac{z}{x}\right)^r$ hence by (6) $l'''(r) < 0$ and l'' is strictly decreasing.

The negativity of l'' follows from (i) and (ii) and due to these properties $l''(r)$ is not less than $\lim_{r\to 0} l''(r)$. To calculate this limit we rewrite l'' as

$$l''(r) = \frac{B^2}{2(\cosh Br - 1)} - \frac{A^2}{2(\cosh Ar - 1)}$$

where $0 < A = \ln\left(\frac{y}{x}\right) < B = \ln\left(\frac{z}{x}\right)$. Applying L'Hospital's rule four times we get that

$$\lim_{r\to 0} l''(r) = \frac{A^2 - B^2}{12}$$

proving (iii). □

Continuation of the proof of Theorem 1. Applying Lemma 1 for the difference ratio (5) and using (iii) we conclude that the function l is strictly concave hence g_1 is strictly decreasing. □

Theorem 4 (second monotonicity property)**.** *For fixed $a \in \mathbb{R}$, $0 < x < y < z$ the function*

$$f_2(r) := R_{r+a,r}(x, y, z) \quad (r \in \mathbb{R}) \tag{8}$$

is strictly decreasing on $\mathbb{R}$.

Proof. Let

$$g_2(r) := \ln f_2(r) = \begin{cases} \dfrac{l(r+a) - l(r)}{a} & \text{if } a \neq 0, \\ l'(r) & \text{if } a = 0. \end{cases} \tag{9}$$

If $a \neq 0$ then by the Lagrange mean value theorem there is a ξ between zero and a such that

$$g_2'(r) = \begin{cases} \dfrac{l'(r+a) - l'(r)}{a} = l''(\xi) & \text{if } a \neq 0, \\ l''(r) & \text{if } a = 0 \end{cases}$$

thus in both cases $g_2'(r) < 0$, proving our theorem. □

The next monotonicity result was proved in the special case $2\alpha = 1$ in [27].

Theorem 5 (third monotonicity property)**.** *For fixed $\alpha \in \mathbb{R}$, $0 < x < y < z$ the function*

$$f_3(r) := R_{r,2\alpha-r}(x, y, z) \quad (r \in \mathbb{R}) \tag{10}$$

(j) *is strictly decreasing on $]-\infty, \alpha]$ and strictly increasing on $[\alpha, \infty[$ if $\alpha > 0$,*

(jj) *is strictly increasing on $]-\infty, \alpha]$ and strictly decreasing on $[\alpha, \infty[$ if $\alpha < 0$,*

(jjj) *is constant ($= \sqrt{\frac{y}{z}}$) on $]-\infty, \infty[$ if $\alpha = 0$.*

Proof. Let

$$g_3(r) := \ln f_3(r) = \begin{cases} \dfrac{l(r) - l(2\alpha - r)}{2(r-\alpha)} & \text{if } r \neq \alpha, \\ l'(\alpha) & \text{if } r = \alpha \end{cases}$$

and

$$h_3(r) := 4(r-\alpha)^2 g_3'(r) \quad (r \in \mathbb{R}),$$

then expressing h_3 and its derivative by l we get that

$$h_3(r) = 2\big[(l'(r) - l'(2\alpha - r))(r - \alpha) - l(r) + l(2\alpha - r)\big]$$

$$h_3'(r) = 2\big[l''(r) - l''(2\alpha - r)\big](r - \alpha).$$

Assume first that $\alpha > 0$ then for $r < \alpha$ we have $r < 2\alpha - r$.

If $r > 0$ then by (ii) $l''(r) - l''(2\alpha - r) < 0$, hence

$$h_3'(r) = 2\big[l''(r) - l''(2\alpha - r)\big](r - \alpha) > 0. \tag{11}$$

If $r < 0$ then $2\alpha - r > 0$, $0 < -r < 2\alpha - r$ and by (i), (ii) $l''(r) - l''(2\alpha - r) = l''(-r) - l''(2\alpha - r) < 0$ and (11) holds again.

By Taylor's formula $h_3(r) = h_3(\alpha) + h_3'(\xi)(r - \alpha) = h_3'(\xi)(r - \alpha)$ where $r < \xi < \alpha$, therefore

$$g_3'(r) = \frac{h_3(r)}{4(r-\alpha)^2} = \frac{h_3'(\xi)}{4(r-\alpha)} < 0$$

as $h_3'(\xi) > 0, r - \alpha < 0$, proving the first statement of (j). The other proposition of (j), and (jj) can be proved similarly, (jjj) is justified by direct calculation. □

Theorem 6 (fourth monotonicity property)**.** *For fixed* $a > 0 > b$, $0 < x < y < z$ *the function*

$$f_4(r) := R_{ar,br}(x,y,z) \quad (r \in \mathbb{R}) \tag{12}$$

(k) *is strictly decreasing on* $\mathbb{R}$ *if* $a + b > 0$,

(kk) *is strictly increasing on* $\mathbb{R}$ *if* $a + b < 0$,

(kkk) *is constant* $(=\sqrt{\frac{y}{z}})$ *on* $\mathbb{R}$ *if* $a + b = 0$.

We postpone its proof after the next theorem.

Theorem 7 (criterion for the comparison)**.** *The comparison inequality*

$$R_{a,b}(x,y,z) \le R_{c,d}(x,y,z) \quad (a,b,c,d \in \mathbb{R},\ 0 < x < y < z). \tag{13}$$

holds if and only if

$$\frac{\dfrac{a}{1-v^a} - \dfrac{b}{1-v^b}}{a-b} \ge \frac{\dfrac{c}{1-v^c} - \dfrac{d}{1-v^d}}{c-d} \quad (a,b,c,d \in \mathbb{R},\ 0 < v < 1) \tag{14}$$

is satisfied (provided that $abcd(a-b)(c-d) \neq 0$, *otherwise the appropriate limits should stand in* (14)*). If the inequality is strict in* (14) *then it is also strict in* (13).

Proof. Rearranging (13) and using the homogeneity of the Stolarsky mean we get

$$\frac{xS_{a,b}\left(1, \dfrac{y}{x}\right)}{xS_{c,d}\left(1, \dfrac{y}{x}\right)} \le \frac{xS_{a,b}\left(1, \dfrac{z}{x}\right)}{xS_{c,d}\left(1, \dfrac{z}{x}\right)} \quad (0 < x < y < z).$$

With $u = \frac{y}{x}$, $v = \frac{z}{x}$ we can rewrite this as

$$\frac{S_{a,b}\,(1,u)}{S_{c,d}\,(1,u)} \leq \frac{S_{a,b}\,(1,v)}{S_{c,d}\,(1,v)} \quad (1 < u < v).$$

This holds if and only if

$$\frac{d}{du}\frac{S_{a,b}\,(1,u)}{S_{c,d}\,(1,u)} \geq 0.$$

A simple calculation shows that this is equivalent to

$$(\ln S_{a,b}(1,u))' \geq (\ln S_{c,d}(1,u))' \quad (1 < u),$$

or (assuming $abcd(a-b)(c-d) \neq 0$) by

$$\frac{\dfrac{au^{a-1}}{u^a-1} - \dfrac{bu^{b-1}}{u^b-1}}{a-b} \geq \frac{\dfrac{cu^{c-1}}{u^c-1} - \dfrac{du^{d-1}}{u^d-1}}{c-d} \quad (1 < u).$$

Multiplying by u and writing $v = 1/u \in \,]0,1[$ we obtain the required inequality (14). As each inequality is this proof was equivalent with the preceding one (14) implies (13) completing the equivalence of these two inequalities. Scrutinizing the proof we easily see that the statement concerning strict inequalities holds. □

Now we can complete the proof of Theorem 6.

Proof of Theorem 6. By the preceding theorem it is enough to show that the function

$$r \longrightarrow \frac{\dfrac{ar}{1-v^{ar}} - \dfrac{br}{1-v^{br}}}{ar-br} \quad (r \in \mathbb{R})$$

(for $r = 0$ defined continuously by its limits $1/2$) is strictly decreasing(increasing) for all fixed $0 < v < 1$ according to $a+b > 0\,(a+b < 0)$. Let $v^r = e^x$, $(x \in \mathbb{R})$ then we need to show that the function

$$U(x) := \frac{1}{a-b}\left(\frac{a}{1-e^{ax}} - \frac{b}{1-e^{bx}}\right) \quad (x \in \mathbb{R})$$

is strictly decreasing(increasing) if $a+b < 0\,(a+b > 0)$. As increasing r will decrease x, the monotonicity is interchanged here. The derivative of U can be written as

$$U'(x) = \frac{a^2b^2e^{(a+b)x}}{(a-b)(1-e^{ax})^2(1-e^{bx})^2}\left[\left(\frac{1-e^{bx}}{be^{\frac{bx}{2}}}\right)^2 - \left(\frac{1-e^{ax}}{ae^{\frac{ax}{2}}}\right)^2\right]$$

$$= \frac{a^2 b^2 e^{(a+b)x}}{(a-b)(1-e^{ax})^2(1-e^{bx})^2} \left[\left(\frac{2\sinh\frac{bx}{2}}{b} \right)^2 - \left(\frac{2\sinh\frac{ax}{2}}{a} \right)^2 \right]$$

and here the first factor (the fraction) is always positive. Using well-known identities and power series development we get

$$\left(\frac{2\sinh\frac{ax}{2}}{a} \right)^2 = \frac{2}{a^2}(\cosh(ax) - 1) = \sum_{n=1}^{\infty} \frac{2a^{2n-2}x^{2n}}{(2n)!}.$$

hence we can write the second factor (the bracket) of U' as

$$\sum_{n=1}^{\infty} \frac{2(b^{2n-2} - a^{2n-2})x^{2n}}{(2n)!}$$
$$= 2(a+b)(a-b) \sum_{n=1}^{\infty} \frac{(b^{2(n-2)} + b^{2(n-3)}a^2 + \cdots + a^{2(n-2)})x^{2n}}{(2n)!}$$

which shows that for $a + b > 0$ the derivative U' is positive, for $a + b < 0$ it is negative, proving (k) and (kk). The third statement (kkk) can be easily checked by direct calculation. □

Figures 1–4 illustrate the four monotonicity properties of $R_{a,b}$. In each of them the *directions of decrease* are indicated by arrows.

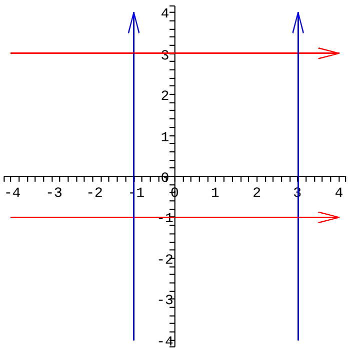

Figure 1. First monotonicity property.

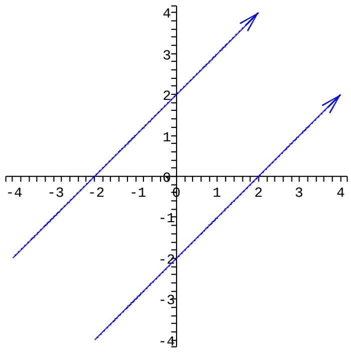

Figure 2. Second monotonicity property.

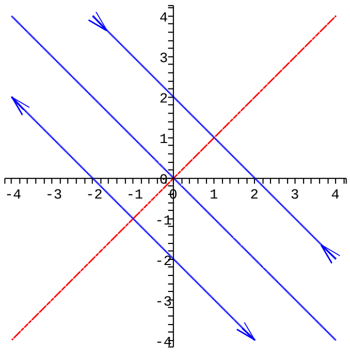

Figure 3. Third monotonicity property.

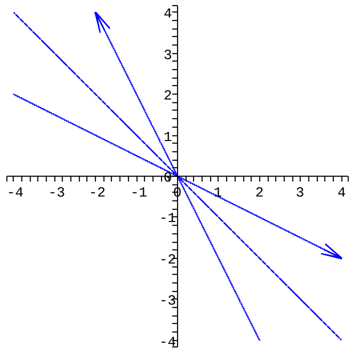

Figure 4. Fourth monotonicity property.

From the monotonicity properties we easily obtain

Theorem 8 (equality of ratios)**.** *The equality*

$$R_{a,b}(x,y,z) = R_{c,d}(x,y,z) \quad (a,b,c,d \in \mathbb{R},\ 0 < x < y < z)$$

holds if and only if either $a+b = c+d = 0$*, or at least one of the sums* $a+b$*,* $c+d$ *is not zero, and* $a = b$*,* $c = d$ *or* $a = d$*,* $b = c$*.*

Proof. In the first case equality is obvious as by (kkk) $R_{a,b}(x,y,z) = \sqrt{\frac{y}{z}} =$ constant if $a+b = 0$. If e.g. $a+b \neq 0$ and $a \geq b$ then (by the above monotonicity properties) moving the point (a,b) to the point (c,d) (different from (a,b)) strictly increases or decreases the $R_{a,b}(x,y,z)$ unless $a+b = c+d$. In the latter case equality can occur only at one point (c,d) with $c \leq d$ and this indeed happens if $a = d$, $b = c$. $\square$

3. Necessary and sufficient conditions for the comparison

First we find some necessary conditions for (13) to hold.

Theorem 9 (necessary conditions for the comparison)**.** *If the comparison inequality*

$$R_{a,b}(x,y,z) \leq R_{c,d}(x,y,z) \quad (a,b,c,d \in \mathbb{R},\ 0 < x < y < z).$$

holds then we have

$$S_{c,d}(x,y) \leq S_{a,b}(x,y) \quad (a,b,c,d \in \mathbb{R},\ x,y > 0). \tag{15}$$

Proof. Taking the limits $x \nearrow y$ in the comparison inequality we obtain that

$$S_{c,d}(y,z) \le S_{a,b}(y,z) \quad (a,b,c,d \in \mathbb{R},\ 0 < y < z)$$

which, by the symmetry $S_{a,b}(y,z) = S_{a,b}(z,y)$ of the Stolarsky means implies (15). □

The solution of the comparison problem (15) is known (cf. [22], [26], [13]). To formulate this solution and some further results we introduce the following notations. For $u, v \in \mathbb{R}$, let

$$\alpha(u,v) := \begin{cases} \dfrac{|u|-|v|}{u-v} & \text{if } u \ne v, \\ \operatorname{sign}(u) & \text{if } u = v, \end{cases}$$

$$\beta(u,v) := \begin{cases} \dfrac{u-v}{\log(u/v)} & \text{if } 0 < uv \text{ and } u \ne v, \\ u & \text{if } 0 < uv \text{ and } u = v, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma(u,v) := \begin{cases} \min\{u,v\} & \text{if } u,v \ge 0, \\ 0 & \text{if } uv < 0, \\ \max\{u,v\} & \text{if } u,v \le 0. \end{cases}$$

For the comparison of Stolarsky means, we recall

Theorem 10 (see [13], Theorem 1)**.** *The comparison inequality*

$$S_{c,d}(x,y) \le S_{a,b}(x,y) \quad (a,b,c,d \in \mathbb{R},\ x,y \in \mathbb{R}_+)$$

of Stolarsky means holds if and only if the parameters $a,b,c,d \in \mathbb{R}$ satisfy the conditions

$$c + d \le a + b, \quad \alpha(c,d) \le \alpha(a,b), \quad \text{and} \quad \beta(c,d) \le \beta(a,b). \tag{16}$$

Our *main result* concerning the comparison is

Theorem 11 (the main result)**.** *The comparison inequality*

$$R_{a,b}(x,y,z) \le R_{c,d}(x,y,z) \quad (a,b,c,d \in \mathbb{R},\ 0 < x < y < z).$$

holds if and only if the parameters $a, b, c, d \in \mathbb{R}$ satisfy the conditions

$$c + d \leq a + b, \quad \alpha(c,d) \leq \alpha(a,b), \quad \gamma(c,d) \leq \gamma(a,b). \tag{17}$$

Suppose now that (17) *holds.*

If $0 \neq c + d \leq a + b$ and $(a,b) \neq (c,d)$, $(a,b) \neq (d,c)$ then strict inequality holds in (13).

If $0 = c + d < a + b$, then again strict inequality holds in the comparison inequality (13).

If $0 = c + d = a + b$, then equality holds in the comparison inequality (13).

Proof. The conditions (17) are *necessary.* Assume the comparison inequality then by Theorem 9 and Theorem 10 the inequalities (16) are satisfied. In Figures 5–11 for seven fixed points (c,d), $c = 5, 4, 3, 2, 1, 0, -1$, $d = c - 4$ we plotted the points (a,b) for which (16) holds. The domains of these (a,b)'s are indicated by 45 degree parallel lines. The points (a,b) for which (17) holds differ from this only, if $c > 0$, $d > 0$ (Figure 5) and if $c < 0$, $d < 0$ (Figure 11).

The set of points (a,b) for which (17) holds are on a thick broken line and to the north-east direction from these broken lines. On Figure 12 these boundaries are plotted together for all the mentioned points (c,d). The set of points (a,b) which satisfy (16) but does not satisfy (17) are two rectangles and two triangles (both are symmetric to the line $y = x$). The rectangle and the triangle (below the line $y = x$!) on Figures 5 and 11 are distinguished by 135 degree parallel lines.

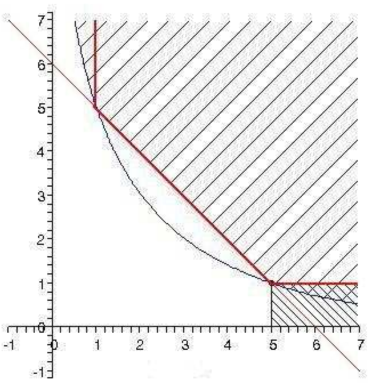

Figure 5. $(c,d) = (5,1)$

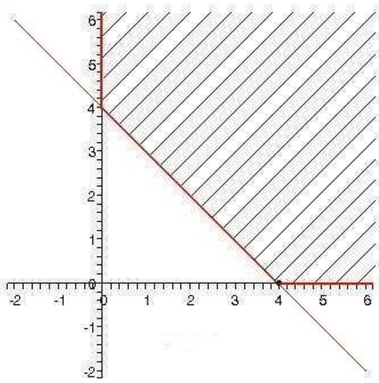

Figure 6. $(c,d) = (4,0)$

By the symmetry $R_{a,b} = R_{b,a}$ we have to show only that if $a \geq b$ and (a, b) is in the distinguished rectangle (Figure 5), or in the triangle (Figure 11), then (13) or the inequality (14) equivalent to it does not hold.

Suppose that (13) holds, $c \geq d > 0$ and $a > c$, $0 < b < d$. By Theorem 7 we have that

$$C(v) := \frac{\dfrac{a}{1-v^a} - \dfrac{b}{1-v^b}}{a-b} - \frac{\dfrac{c}{1-v^c} - \dfrac{d}{1-v^d}}{c-d} \geq 0 \quad (0 < v < 1).$$

C can be extended continuously by defining $C(0) := \lim_{v\to 0} C(v) = 0$. In the expression

$$v^{1-d}C'(v) := \frac{\dfrac{a^2 v^{a-d}}{(1-v^a)^2} - \dfrac{b^2 v^{b-d}}{(1-v^b)^2}}{a-b} - \frac{\dfrac{c^2 v^{c-d}}{(1-v^c)^2} - \dfrac{d^2}{(1-v^d)^2}}{c-d} \geq 0 \quad (0 < v < 1)$$

the exponents of v on the right hand side satisfy $a - d > 0$, $b - d < 0$, $c - d \geq 0$, $a > 0$, $b > 0$, $c > 0$, $d > 0$ thus taking the limit $v \to 0$ we get that

$$\lim_{v\to 0} v^{1-d}C'(v) = -\infty.$$

This implies that for $v > 0$ small enough $C'(v) < 0$ hence by the mean value theorem there exists a ξ such that $0 < \xi < v$ and

$$C(v) = C(0) + C'(\xi)v = C'(\xi)v < 0$$

which is a contradiction.

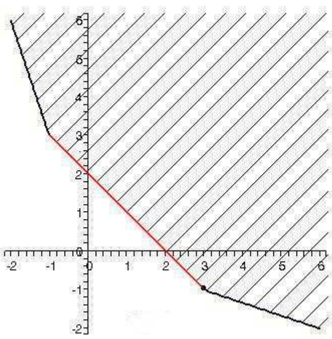

Figure 7. $(c, d) = (3, -1)$

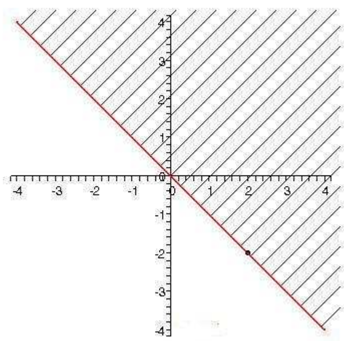

Figure 8. $(c, d) = (2, -2)$

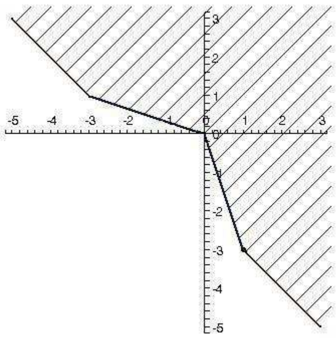

Figure 9. $(c,d) = (1,-4)$

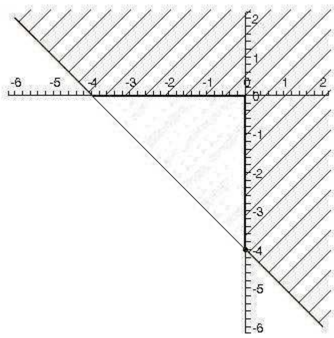

Figure 10. $(c,d) = (0,-4)$

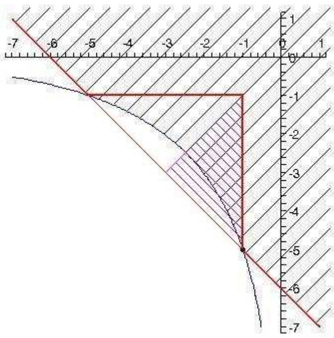

Figure 11. $(c,d) = (-1,-5)$

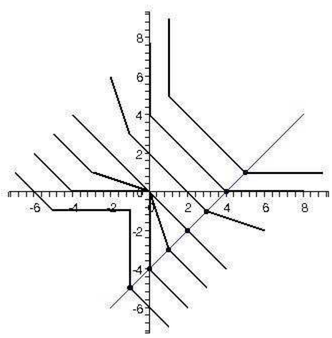

Figure 12. All points (c,d)

If the point (c,d) is in the second or fourth quadrant (including the boundaries) then the points (a,b) satisfying (16) and (17) coincide (this can clearly be seen in Figures 6–10. To complete the proof of necessity we have to show only that if (c,d) is strictly in the third quadrant $c \geq d$ and $a \geq b$ and (a,b) is in the distinguished triangle (Figure 11), then (13) or the inequality (14) equivalent to it does not hold.

Assume that $c + d \leq a + b$, $d < c < 0$, $b < a < 0$, then we show that the conditions

$$b < c, \quad a < c$$

cannot hold. By Theorem 7 $C(v) \geq 0$ for $0 < v < 1$. Again we extend C continuously to $v = 0$ by $C(0) := \lim_{v\to 0} C(v) = 0$. A simple calculation shows that

$$v^{-a-1}C'(v) := \frac{\dfrac{a^2}{(v^{-a}-1)^2} - \dfrac{b^2 v^{a-b}}{(v^{-b}-1)^2}}{a-b} - \frac{\dfrac{c^2 v^{a-c}}{(v^{-c}-1)^2} - \dfrac{d^2 v^{a-d}}{(v^{-d}-1)^2}}{c-d} \geq 0$$

$$(0 < v < 1).$$

Here the denominators $a - b$, $c - d$ are positive and the squares $(v^{-\alpha} - 1)^2$ with $\alpha = a, b, c, d$ tend to 1 as $v \to 0$. The exponents of v in the numerators of the fractions satisfy

$$a - b > 0, a - c < 0, \quad \text{and} \quad a - d > 0$$

(the latter inequality comes from writing $c + d \leq a + b$ as $a - d \geq c - b$ and using the assumption $b < c$ to conclude $0 < c - b \leq a - d$). Taking the limit $v \to 0$ we get that $\lim_{v\to 0} v^{-a-1}C'(v) = -\infty$, therefore for small positive v's $C'(v)$ is negative, and arguing as earlier, we conclude that $C(v) < 0$ for small $v > 0$, which is a contradiction.

Thus the conditions (17) are necessary.

Sufficiency. Using the four monotonicity properties of $R_{a,b}$ one can easily see that (17) is sufficient for the comparison.

The statement on strict inequalities is also a simple consequence of the monotonicity properties. □

4. Closing remarks, open problems

It is worth to mention that the necessary and sufficient conditions (17) *coincide* with the necessary and sufficient condition for the comparison

$$G_{c,d}(x, y) \leq G_{a,b}(x, y) \quad (x, y > 0) \tag{18}$$

of Gini means (see [13], Theorem 4.1), defined by

$$G_{a,b}(x, y) = \begin{cases} \left(\dfrac{x^a + y^a}{x^b + y^b}\right)^{\frac{1}{a-b}} & \text{if } a - b \neq 0, \\ \exp\left(\dfrac{x^a \ln x + y^a \ln y}{x^a + y^a}\right) & \text{if } a - b = 0. \end{cases}$$

Thus we can reformulate the main result as follows:

The comparison inequality (13) *holds if and only if the inequality* (18) *is satisfied.*

Using the identity

$$R_{a,b}(u,v,w) = \big(R_{-a,-b}(u^{-1},v^{-1},w^{-1})\big)^{-1} \quad (u,v,w>0)$$

it is easy to rewrite Theorem 11 to the case when the variables in the comparison inequality have opposite order $x>y>z>0$.

Forming other ratios, like

$$\widetilde{R}_{a,b}(x,y,z) := \frac{S_{a,b}(x,y)}{S_{a,b}(y,z)} \quad (a,b\in\mathbb{R},\ 0<x<y<z)$$

one can easily prove that the modified comparison

$$\widetilde{R}_{a,b}(x,y,z) \le \widetilde{R}_{c,d}(x,y,z) \quad (a,b,c,d\in\mathbb{R},\ 0<x<y<z)$$

holds if and only if $\widetilde{R}_{a,b}(x,y,z) = \widetilde{R}_{c,d}(x,y,z)$ satisfied for all $x,y,z>0$.

Let now $\mathcal{M}_{a,b}$ $(a,b\in\mathbb{R})$ be a two-parameter, symmetric, homogeneous mean defined for positive variables and let us form the ratio

$$\mathcal{R}_{a,b}(x,y,z) := \frac{\mathcal{M}_{a,b}(x,y)}{\mathcal{M}_{a,b}(x,z)} \quad (a,b\in\mathbb{R},\ 0<x<y<z).$$

For what means $\mathcal{M}_{a,b}$ has this ratio simple monotonicity properties?

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