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On selections of general linear inclusions

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Dedicated to Professor Zoltán Daróczy on his 70^{th} *birthday*

Abstract. In this paper we prove that a set-valued map with closed, convex and uniformly bounded values in a Banach space, which satisfy a general linear inclusion, admits a selection that satisfies a general linear equation.

1. Introduction

The main notions of set-valued analysis as linearity, convexity, subadditivity, superadditivity, affinity are defined by functional inclusions. An important problem in set-valued analysis is to find selections of set-valued maps satisfying some conditions as continuity, measurability, integrability, etc (see e.g. [3]).

In the theory of functional equations one of the main topics is Hyers–Ulam stability (cf. e.g. [5], [8], [2] and the references therein). A first result on this topic was given by D. H. Hyers [7] who obtained the following result for the Cauchy functional equation:

Let X be a linear normed space, Y a Banach space and $\varepsilon > 0$. Then for every $f : X \to Y$ satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon, \quad x, y \in X, \tag{1.1}$$

there exists a unique additive function $g : X \to Y$ such that

$$\|f(x) - g(x)\| \le \varepsilon, \quad x \in X. \tag{1.2}$$

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An interesting connection between the stability of the Cauchy equation and subadditive set-valued maps was established by W. Smajdor [14] and by R. Ger and Z. Gajda [6]. They observed that if f is a solution of (1.1), then the set-valued map $F : X \to \mathcal{P}_0(Y)$ ($\mathcal{P}_0(X)$ denotes the collection of all nonempty subsets of Y) defined by the relation

$$F(x) = f(x) + B(0, \varepsilon), \quad x \in X, \tag{1.3}$$

where $B(0, \varepsilon)$ is the closed ball of center 0 and radius ε in Y, is subadditive and the function g from (1.2) is an additive selection of F, i.e. $g(x) \in F(x)$ for every $x \in X$.

Now one may ask under what conditions a subadditive set-valued map admits an additive selection. An answer to this question is given in [6]. Furthermore this result was generalized by D. Popa [12], [13], who considered multifunctions satisfying a general linear inclusion instead a subadditive set-valued map.

The transposition of the general linear equation, considered, among others, by J. Aczél, Z. Daróczy and L. Losonczi (cf. [1] and the references therein), for set-valued maps leads to the study of the following two linear inclusions:

$$F(\alpha x + \beta y + c) \subseteq \gamma F(x) + \delta F(y) + C \tag{1.4}$$

$$\alpha F(x) + \beta F(y) \subseteq F(\gamma x + \delta y + c) + C \tag{1.5}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers $F : X \to \mathcal{P}_0(Y)$, X, Y are real vector spaces, $c \in X$ and $C \in \mathcal{P}_0(Y)$.

Subadditive and superadditive set-valued functions, defined by particular cases of the linear inclusions (1.4) and (1.5) were studied by W. Smajdor [14], [15] and A. Smajdor [16].

D. Popa proved that a set valued map satisfying the general linear inclusion (1.4), in appropriate conditions, admits a selection f satisfying the general linear equation

$$f(\alpha x + \beta y + c) = \gamma f(x) + \delta f(y), \quad x, y \in X. \tag{1.6}$$

The goal of this paper is to obtain an analogous result for the general linear inclusion (1.5). This result is in connection with the results of convex analysis for set-valued maps. Indeed if in (1.5) one take $\alpha = \gamma = 1 - t$, $\beta = \delta = t$, where $t \in [0, 1]$ is fixed, $c = 0_X$, and C is a convex cone, then F is called (t, C)-convex set-valued map (if $C = \{0_Y\}$ then F is called t-convex set valued map) (see [9] and the references therein). If $\alpha = \gamma$, $\beta = \delta$ are positive fixed numbers, $c = 0_X$, and $C = \{0_Y\}$ then F is a generalized convex process [3].

2. Main result

Throughout this section we denote by X a real linear space and by Y a real Banach space with the zero vectors denoted by 0_X and 0_Y, the collection of all nonempty subsets of Y is denoted by $\mathcal{P}_0(Y)$ and $ccl(Y)$ denotes the family of all nonempty convex and closed subsets of Y. For $A, B \in \mathcal{P}_0(Y)$ and $\lambda, \mu \in \mathbb{R}$ we define the sets $A + B$ and λA by

$$\begin{aligned} A + B &= \{x \mid x = a + b,\ a \in A,\ b \in B\} \\ \lambda A &= \{x \mid x = \lambda a,\ a \in A\}. \end{aligned} \tag{2.1}$$

The next properties will be used often in the sequel

$$\begin{aligned} \lambda(A + B) &= \lambda A + \lambda B \\ (\lambda + \mu)A &\subseteq \lambda A + \mu A \end{aligned} \tag{2.2}$$

If A is a convex set and $\lambda\mu \geq 0$, then we have

$$(\lambda + \mu)A = \lambda A + \mu A. \tag{2.3}$$

For a set $A \in \mathcal{P}_0(Y)$ we denote by $d(A)$ the *diameter* of A, i.e.

$$d(A) = \sup\{\|x - y\| : x, y \in A\}. \tag{2.4}$$

A *selection* of a set-valued map $F : X \to \mathcal{P}_0(Y)$ is a single valued map $f : X \to Y$ with the property $f(x) \in F(x)$ for all $x \in X$. Given a point $z \in X$, $z \neq 0$, denote by L_z the half-line with the origin 0_X containing z, i.e. $L_z = \{tz : t \geq 0\}$.

The main result of this paper is contained in the next theorem.

Theorem 2.1. *Let K be a convex cone in X containing 0_X, α, β, γ, δ be positive numbers $\alpha + \beta \neq 1$.*

i) *If $\alpha + \beta < 1$, then for every solution $F : K \to ccl(Y)$ of the linear inclusion*

$$\alpha F(x) + \beta F(y) \subseteq F(\gamma x + \delta y), \quad x, y \in K, \tag{2.5}$$

satisfying $\sup\{d(F(x)) : x \in L_z\} < \infty$, for every $z \in K$, there exists a unique selection $f : K \to Y$ of F satisfying the general linear equation

$$\alpha f(x) + \beta f(y) = f(\gamma x + \delta y), \quad x, y \in K. \tag{2.6}$$

ii) *If* $\alpha+\beta>1$, *then every solution* $F: K\to\mathcal{P}_0(Y)$ *of the linear inclusion* (2.5), *satisfying* $\sup\{d(F(x)):\ x\in L_z\}<\infty$, *for every* $z\in K$, *is single-valued.*

Proof. i) *Existence.* Suppose that $\alpha+\beta<1$ and (2.5) is satisfied. Following the method used by Gajda and Ger in [6], we put $y=x$ in (2.5) and taking account that F has convex values we get

$$(\alpha+\beta)F(x)\subseteq F((\gamma+\delta)x),\quad x\in K. \tag{2.7}$$

Replacing x by $\frac{x}{(\gamma+\delta)^{n+1}}$, in (2.7) and multiplying by $(\alpha+\beta)^n$, $n\in\mathbb{N}$, we get

$$(\alpha+\beta)^{n+1}F\left(\frac{x}{(\gamma+\delta)^{n+1}}\right)\subseteq(\alpha+\beta)^nF\left(\frac{x}{(\gamma+\delta)^{n}}\right). \tag{2.8}$$

Fix $x\in K$ and denote

$$F_n(x)=(\alpha+\beta)^nF\left(\frac{x}{(\gamma+\delta)^{n}}\right),\quad n\geq 0. \tag{2.9}$$

By (2.8) follows that $(F_n(x))_{n\geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y and

$$\lim_{n\to\infty}d(F_n(x))=\lim_{n\to\infty}(\alpha+\beta)^nd\left(F\left(\frac{x}{(\gamma+\delta)^{n}}\right)\right)=0, \tag{2.10}$$

in view of the uniform boundedness of the values of F on the half-line L_x.

Hence the $\bigcap_{n\geq 0}F_n(x)$ is a singleton and we denote

$$f(x)=\bigcap_{n\geq 0}F_n(x),\quad x\in K. \tag{2.11}$$

Thus we have obtained a single-valued mapping $f: K\to Y$ satisfying the condition $f(x)\in F_0(x)=F(x)$, i.e. a selection of F.

Let $x,y\in K$ be fixed. By (2.5) and (2.9) we have

$$\begin{aligned}\alpha F_n(x)+\beta F_n(y)&=(\alpha+\beta)^n\left(\alpha F\left(\frac{x}{(\gamma+\delta)^{n}}\right)+\beta F\left(\frac{y}{(\gamma+\delta)^{n}}\right)\right)\\&\subseteq(\alpha+\beta)^nF\left(\frac{\gamma x+\delta y}{(\gamma+\delta)^{n}}\right)=F_n(\gamma x+\delta y),\ n\geq 0,\end{aligned}$$

and, taking account that $(F_n(x))_{n\geq 0}$ is decreasing, it follows

$$\begin{aligned}\alpha f(x)+\beta f(y)&=\alpha\bigcap_{n\geq 0}F_n(x)+\beta\bigcap_{n\geq 0}F_n(y)\subseteq\bigcap_{n\geq 0}(\alpha F_n(x)+\beta F_n(y))\\&\subseteq\bigcap_{n\geq 0}F_n(\gamma x+\delta y)=f(\gamma x+\delta y).\end{aligned}$$

Therefore

$$\alpha f(x) + \beta f(y) = f(\gamma x + \delta y), \quad x, y \in K. \tag{2.12}$$

The existence is proved.

Uniqueness. Suppose that there exist two selections $f_1, f_2 : K \to Y$ of F satisfying the equation (2.6). The following relations hold

$$(\alpha + \beta)^n f_k(x) = f_k((\gamma + \delta)^n x), \quad k \in \{1, 2\}, \tag{2.13}$$

for every positive integer n and all $x \in K$, in view of the relation (2.6). We have

$$\begin{aligned} \frac{1}{(\alpha + \beta)^n} \|f_1(x) - f_2(x)\| &= \left\| f_1\left(\frac{x}{(\gamma + \delta)^n}\right) - f_2\left(\frac{x}{(\gamma + \delta)^n}\right) \right\| \\ &\le d\left(F\left(\frac{x}{(\gamma + \delta)^n}\right)\right) \end{aligned} \tag{2.14}$$

for every $x \in K$ and every $n \geq 0$. The uniform boundedness of F on the half-line L_x leads to $f_1(x) = f_2(x)$ for every $x \in K$. The uniqueness is proved.

ii) Suppose that $\alpha + \beta > 1$ and F satisfies (2.5). Then (2.7) holds and replacing x in (2.7) by $(\gamma + \delta)^n x$, dividing by $(\alpha + \beta)^{n+1}$, $n \in \mathbb{N}$, we get

$$\frac{F((\gamma + \delta)^n x)}{(\alpha + \beta)^n} \subseteq \frac{F((\gamma + \delta)^{n+1} x)}{(\alpha + \beta)^{n+1}}, \quad x \in K. \tag{2.15}$$

Let $x \in K$ be fixed. Taking account of (2.15) it follows that the sequence of sets $(F'_n(x))_{n \geq 0}$ given by

$$F'_n(x) = \frac{F((\gamma + \delta)^n x)}{(\alpha + \beta)^n}, \quad n \geq 0,$$

is increasing, hence the sequence of real numbers $(d(F'_n(x)))_{n \geq 0}$ is increasing too. But

$$\lim_{n \to \infty} d(F'_n(x)) = \lim_{n \to \infty} \frac{1}{(\alpha + \beta)^n} d(F((\gamma + \delta)_n x) = 0, \tag{2.16}$$

in view of the uniform boundedness of F on the half-line L_x.

Thus $d(F'_n(x)) = 0$ for every $n \in \mathbb{N}$, hence F is single valued and satisfies the equation

$$\alpha F(x) + \beta F(y) = F(\gamma x + \delta y), \quad x, y \in K. \tag{2.17}$$

The theorem is proved. □

The following result is a simple consequence of Theorem 2.1.

Corollary 2.1. *Let K be a convex cone in X containing 0_X, C a nonempty compact and convex subset of Y, $\alpha, \beta, \gamma, \delta > 0$, $\alpha + \beta < 1$, $\gamma + \delta \neq 1$, $c \in K$ and $x_0 = \frac{c}{1-\gamma-\delta}$. Suppose that $F : K + x_0 \to ccl(Y)$ satisfies the general linear inclusion*

$$\alpha F(x) + \beta F(y) \subseteq F(\gamma x + \delta y + c) + C, \quad x, y \in K + x_0 \tag{2.18}$$

and $\sup\{d(F(x)) : \ x \in L_z + x_0\} < \infty$ *for every $z \in K$. Then there exists a unique single valued mapping $f : K + x_0 \to Y$ satisfying the equation*

$$\alpha f(x) + \beta f(y) = f(\gamma x + \delta y + c), \quad x, y \in K + x_0 \tag{2.19}$$

and

$$f(x) \in F(x) + \frac{1}{1-\alpha-\beta}C, \quad x \in K + x_0. \tag{2.20}$$

Proof. Let $G : K \to ccl(Y)$ be defined by the relation

$$G(x) = F(x + x_0) + \frac{1}{1-\alpha-\beta}C, \quad x \in K. \tag{2.21}$$

The definition of G is correct since the sum of a closed set and a compact set is closed and the sum of two convex sets is a convex set. We will prove that G satisfies the following relation

$$\alpha G(x) + \beta G(y) \subseteq G(\gamma x + \delta y), \quad x, y \in K. \tag{2.22}$$

Indeed,

$$\begin{aligned}
\alpha G(x) + \beta G(y) &= \alpha F(x + x_0) + \beta F(y + x_0) + \frac{\alpha+\beta}{1-\alpha-\beta}C \\
&\subseteq F(\gamma(x + x_0) + \delta(y + x_0) + c) + C + \frac{\alpha+\beta}{1-\alpha-\beta}C \\
&= F(\gamma x + \delta y + x_0) + \frac{1}{1-\alpha-\beta}C \\
&= G(\gamma x + \delta y), \ x, y \in K.
\end{aligned}$$

Taking account of Theorem 2.1 it follows that there exists a unique selection g of G satisfying

$$\alpha g(x) + \beta g(y) = g(\gamma x + \delta y), \quad x, y \in K. \tag{2.23}$$

Now let $f : K + x_0 \to Y$ be defined by the relation

$$f(x) = g(x - x_0), \quad x \in K + x_0. \tag{2.24}$$

Then

$$\alpha f(x)+\beta f(y)=f(\gamma x+\delta y+c),\quad x,y\in K+x_0 \tag{2.25}$$

and

$$f(x)\in F(x)+\frac{1}{1-\alpha-\beta}C,\quad x\in K+x_0. \tag{2.26}$$

□

Corollary 2.1 leads to the following result on the stability of the general linear equation.

Corollary 2.2. *Let K be a convex cone in X containing 0_X, α, β, γ, δ, $\varepsilon>0$, $\alpha+\beta<1$, $\gamma+\delta\neq 1$, $c\in K$, $k\in Y$ and $x_0=\frac{c}{1-\gamma-\delta}$. Suppose that $f:K+x_0\to Y$ satisfies the following relation*

$$\|f(\gamma x+\delta y+c)-\alpha f(x)-\beta f(y)-k\|\leq\varepsilon,\quad x,y\in K+x_0. \tag{2.27}$$

Then there exists a unique function $g:K+x_0\to Y$ satisfying:

$$g(\gamma x+\delta y+c)=\alpha g(x)+\beta g(y)+k,\quad x,y\in K+x_0 \tag{2.28}$$

and

$$\|f(x)-g(x)\|\leq\frac{\varepsilon}{1-\alpha-\beta},\quad x\in K+x_0. \tag{2.29}$$

Proof. Define the function $h:K+x_0\to Y$ by the relation

$$h(x)=f(x)+\frac{k}{\alpha+\beta-1},\quad x\in K+x_0. \tag{2.30}$$

Then h satisfies the inequality

$$\|h(\gamma x+\delta y+c)-\alpha h(x)-\beta h(y)\|\leq\varepsilon,\quad x,y\in K+x_0. \tag{2.31}$$

Now we consider the set-valued map $F:K+x_0\to ccl(Y)$ given by

$$F(x)=h(x)+\frac{1}{1-\alpha-\beta}B(0,\varepsilon),\quad x\in K+x_0. \tag{2.32}$$

We have

$$\begin{aligned}\alpha F(x)+\beta F(y)&=\alpha h(x)+\frac{\alpha}{1-\alpha-\beta}B(0,\varepsilon)+\beta h(y)+\frac{\beta}{1-\alpha-\beta}B(0,\varepsilon)\\&=\alpha h(x)+\beta h(y)+\frac{\alpha+\beta}{1-\alpha-\beta}B(0,\varepsilon)\end{aligned}$$

$$\begin{aligned}&\subseteq h(\gamma x+\delta y+c)+B(0,\varepsilon)+\frac{\alpha+\beta}{1-\alpha-\beta}B(0,\varepsilon)\\&=h(\gamma x+\delta y+c)+\frac{1}{1-\alpha-\beta}B(0,\varepsilon)\\&=F(\gamma x+\delta y+c),\quad x,y\in K+x_0.\end{aligned}$$

Then, in view of Corollary 2.1, there exists a unique function $g_1 : K + x_0 \to Y$

$$g_1(\gamma x+\delta y+c)=\alpha g_1(x)+\beta g_1(y),\quad x,y\in K+x_0,$$

with the property

$$g_1(x)\in F(x)=h(x)+\frac{1}{1-\alpha-\beta}B(0,\varepsilon),\quad x\in K+x_0.$$

Finally it follows that the function $g : K + x_0 \to Y$, given by

$$g(x)=g_1(x)+\frac{k}{1-\alpha-\beta},\quad x\in K+x_0, \tag{2.33}$$

satisfies the relations (2.28) and (2.29). The corollary is proved. □

Remark 2.1. The result obtained in Corollary 2.2 is a particular case of a general result obtained by Z. Páles for the stability of the Cauchy functional equation on square-symmetric grupoids [11]. It also crosses with the result on stability of general linear equation on restricted domain obtained recently by J. Brzdęk and A. Pietrzyk [4].

Finally, we consider the selection problem for set-valued maps satisfying (1.5) with $\alpha+\beta=1$. In the special case $\alpha=\beta=\gamma=\delta=1/2$ the next theorem gives conditions under which midconvex set-valued maps have Jensen selections. For $\alpha=\gamma$ and $\beta=\delta=1-\alpha$ we get a result on affine selections of convex set-valued maps. Under other assumptions results of this type were obtained by K. Nikodem [10] and by A. Smajdor and W. Smajdor [17].

Theorem 2.2. *Let $\alpha\in(0,1)$, $\gamma,\delta>0$, C a nonempty compact and convex subset of Y containing 0_Y and K a convex cone in X containing 0_X. Suppose that $F : K\to ccl(Y)$ satisfies*

$$(1-\alpha)F(x)+\alpha F(y)\subseteq F(\gamma x+\delta y)+C,\quad x,y\in K \tag{2.34}$$

and $\sup\{d(F(x)) :\ x\in K\}<\infty$. Then there exists a function $f : K\to Y$ satisfying

$$(1-\alpha)f(x)+\alpha f(y)=f(\gamma x+\delta y),\quad x,y\in K \tag{2.35}$$

and

$$f(x)\in F(x)+\frac{1}{\alpha}C,\quad x\in K. \tag{2.36}$$

Proof. Take $p \in F(0_X)$ and consider the set-valued map $G : K \to ccl(Y)$ given by

$$G(x) = F(x) - p, \quad x \in K. \tag{2.37}$$

Then

$$(1-\alpha)G(x) + \alpha G(y) \subseteq G(\gamma x + \delta y) + C, \quad x, y \in K, \tag{2.38}$$

and $0_Y \in G(0_X)$. Put $y = 0_X$ in (2.38) to get

$$(1-\alpha)G(x) + \alpha G(0_X) \subseteq G(\gamma x) + C, \quad x \in K. \tag{2.39}$$

Replacing x by $\frac{x}{\gamma^{n+1}}$ and multiplying (2.39) by $(1-\alpha)^n$ it follows

$$(1-\alpha)^{n+1} G\left(\frac{x}{\gamma^{n+1}}\right) + \alpha(1-\alpha)^n G(0_X) \subseteq (1-\alpha)^n G\left(\frac{x}{\gamma^n}\right) + (1-\alpha)^n C \tag{2.40}$$

and adding $\frac{(1-\alpha)^{n+1}}{\alpha} C$ to both sides of (2.40) one gets

$$\begin{aligned}(1-\alpha)^{n+1} G\left(\frac{x}{\gamma^{n+1}}\right) &+ \frac{(1-\alpha)^{n+1}}{\alpha} C + \alpha(1-\alpha)^n G(0_X) \\ &\subseteq (1-\alpha)^n G\left(\frac{x}{\gamma^n}\right) + \frac{(1-\alpha)^n}{\alpha} C.\end{aligned} \tag{2.41}$$

Since $0_Y \in G(0_X)$ the following relation holds

$$\begin{aligned}&(1-\alpha)^{n+1} G\left(\frac{x}{\gamma^{n+1}}\right) + \frac{(1-\alpha)^{n+1}}{\alpha} C \\ &\subseteq (1-\alpha)^{n+1} G\left(\frac{x}{\gamma^{n+1}}\right) + \frac{(1-\alpha)^{n+1}}{\alpha} C + \alpha(1-\alpha)^n G(0_X).\end{aligned} \tag{2.42}$$

Now from (2.42) and (2.41) it follows that $(G_n(x))_{n\geq 0}$ defined by

$$G_n(x) = (1-\alpha)^n G\left(\frac{x}{\gamma^n}\right) + \frac{(1-\alpha)^n}{\alpha} C \tag{2.43}$$

is a decreasing sequence of closed sets with

$$\lim_{n\to\infty} d(G_n(x)) = 0. \tag{2.44}$$

Then $\bigcap_{n\geq 0} G_n(x)$ is a singleton. Put

$$g(x) = \bigcap_{n\geq 0} G_n(x), \quad x \in K. \tag{2.45}$$

Replacing x by $\frac{x}{\gamma^n}$, y by $\frac{y}{\delta^n}$, adding $\frac{1}{\alpha}C$ to both sides of (2.38) and multiplying by $(1-\alpha)^n$ one gets

$$(1-\alpha)G_n(x) + \alpha G_n(y) \subseteq G_n(\gamma x + \delta y) + (1-\alpha)^n C. \tag{2.46}$$

Now observe that $((1-\alpha)^n C)_{n\geq 0}$ is a decreasing sequence of compact sets. Indeed, taking account of $0_Y \in C$ it follows that for every $c \in C$

$$(1-\alpha)c = (1-\alpha)c + \alpha \cdot 0_Y \in C, \tag{2.47}$$

thus $(1-\alpha)C \subseteq C$ and forward $(1-\alpha)^{n+1}C \subseteq (1-\alpha)^n C$ for every positive integer n. It is known that if $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ are decreasing sequences of closed sets in a topological vector space and B_1 is compact then

$$\bigcap_{n\geq 0}(A_n + B_n) = \bigcap_{n\geq 0} A_n + \bigcap_{n\geq 0} B_n \tag{2.48}$$

(see Lemma 5.3 from [9]). Using this result we get

$$\begin{aligned}(1-\alpha)g(x) + \alpha g(y) &= (1-\alpha)\bigcap_{n\geq 0} G_n(x) + \alpha \bigcap_{n\geq 0} G_n(y)\\ &\subseteq \bigcap_{n\geq 0}((1-\alpha)G_n(x) + \alpha G_n(y)) \subseteq \bigcap_{n\geq 0}(G_n(\gamma x+\delta y) + (1-\alpha)^n C)\\ &= \bigcap_{n\geq 0} G_n(\gamma x + \delta y) + \bigcap_{n\geq 0}(1-\alpha)^n C = g(\gamma x + \delta y),\ x, y \in K.\end{aligned} \tag{2.49}$$

The function $f : K \to Y$, $f(x) = g(x) + p$, $x \in K$, satisfies the relation

$$(1-\alpha)f(x) + \alpha f(y) = f(\gamma x + \delta y), \quad x \in K. \tag{2.50}$$

On the other hand $g(x) \in G_0(x) = G(x) + \frac{1}{\alpha}C$. It follows

$$f(x) \in F(x) + \frac{1}{\alpha}C, \quad x \in K. \tag{2.51}$$

The theorem is proved. □

Remark 2.2. The selection f from Theorem 2.2 is not uniquely determined. For instance, the set valued map $F : K \to ccl(Y)$, $F(x) = C$, $x \in K$, is a solution of the linear inclusion (2.34) and the function $f : K \to Y$, $f(x) = c$, where $c \in C$ is an arbitrary element, satisfies (2.35) and (2.36).

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