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On certain tensor fields on contact metric manifolds II

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To the memory of Professor András Rapcsák

In Sasakian manifolds, MATSUMOTO and CHŪMAN [3] defined a contact Bochner curvature tensor (see also YANO [6]). This is invariant with respect to a D-homothetic deformation (see TANNO [5] about a D-homothetic deformation). On the other hand, if a contact metric manifold is normal, it is called a Sasakian manifold. In this paper, we define a new tensor field which is invariant with respect to the D-homothetic deformation and call it an EH-tensor. Then we shall establish that a contact metric manifold is a Sasakian manifold if and only if its EH-tensor vanishes.

1. Preliminaries

Let M be a $(2n+1)$-dimensional contact metric manifold with the structure tensors (ϕ, ξ, η, g). Then these satisfy

$$\phi\xi = 0,\ \eta(\xi) = 1,\ \phi^2 = -I + \eta \otimes \xi,\ \eta(X) = g(\xi, X)$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ g(\phi X, Y) = d\eta(X, Y)$$

for any vector fields X and Y on M. Now we define an operator h by $h = -\frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ denotes Lie differentiation. Then the vector field ξ is Killing if and only if h vanishes. It is well known that h and ϕh are symmetric operators, h anti-commutes with ϕ (i.e., $\phi h + h\phi = 0$), $h\xi = 0$, $\eta_0 h = 0$, $\operatorname{Tr} h = 0$ and $\operatorname{Tr} \phi h = 0$, where $\operatorname{Tr} h$ denotes the trace of h. Moreover the following general formulas for a contact metric manifold M were obtained

$$\nabla_X \xi = \phi X + \phi h X \tag{1.1}$$

$$\frac{1}{2}(R(\xi, X)\xi - \phi R(\xi, \phi X)\xi) = h^2 X + \phi^2 X \tag{1.2}$$

$$g(Q\xi, \xi) = 2n - \operatorname{Tr} h^2 \tag{1.3}$$

$$(1.4)\qquad \sum_{i=1}^{2n+1}(\nabla_{E_i}\phi)E_i = -2n\xi \ (\{E_i\} \text{ is an orthonormal frame})$$

$$(1.5)\quad (\nabla_{\phi X}\phi)\phi Y + (\nabla_X\phi)Y = -2g(X,Y)\xi + \eta(Y)(X + hX + \eta(X)\xi)$$

$$((\nabla_X\phi)\xi = -\eta(X)\xi + X + hX)$$

$$(1.6)\qquad \phi(\nabla_\xi h)X = X - \eta(X)\xi - h^2X - R(X,\xi)\xi,$$

where ∇ is the covariant differentiation with respect to g, Q is the Ricci operator of M and R is the curvature tensor field of M (see., [1], [2] and [4]). Moreover, by means of $\phi h\xi = 0$, we get

$$(1.7)\qquad \phi(\nabla_Y h)\xi = -hY - h^2Y.$$

If ξ is Killing in a contact metric manifold M, M is said to be a K-contact Riemannian manifold. If a contact metric manifold M is normal (i.e., $N+2d\eta\otimes\xi = 0$, where N denotes the Nijenhuis tensor formed with ϕ) M is called a Sasakian manifold. A Sasakian manifold is a K-contact Riemannian manifold. In a Sasakian manifold with structure tensors (ϕ,ξ,η,g), we have

$$\nabla_X\xi = \phi X,$$

$$(\nabla_X\phi)Y = R(X,\xi)Y = -g(X,Y)\xi + \eta(Y)X,\ \phi Q = Q\phi,\ Q\xi = 2n\xi$$

(see e.g., [7]).

2. D-homothetic deformations

Let M be an $(m+1)$-dimensional $(m = 2n)$ contact metric manifold. Now we define the tensor field B^{es} in M by

$$\begin{aligned}(2.1)\quad B^{es}(X,Y) &= R(X,Y) + h\phi X\wedge h\phi Y\\ &+ \frac{1}{2(m+4)}(QY\wedge X - (\phi Q\phi Y)\wedge X + \frac{1}{2}(\eta(Y)Q\xi\wedge X\\ &+ \eta(QY)\xi\wedge X) - QX\wedge Y + (\phi Q\phi X)\wedge Y - \frac{1}{2}(\eta(X)Q\xi\wedge Y\\ &+ \eta(QX)\xi\wedge Y) + (Q\phi Y)\wedge\phi X + (\phi QY)\wedge\phi X\\ &- (Q\phi X)\wedge\phi Y - (\phi QX)\wedge\phi Y + 2g(Q\phi X,Y)\phi\\ &+ 2g(\phi QX,Y)\phi + 2g(\phi X,Y)\phi Q + 2g(\phi X,Y)Q\phi\\ &- \eta(X)QY\wedge\xi + \eta(X)(\phi Q\phi Y)\wedge\xi + \eta(Y)QX\wedge\xi\\ &- \eta(Y)(\phi Q\phi X)\wedge\xi) - \frac{k+m}{m+4}(\phi Y\wedge\phi X + 2g(\phi X,Y)\phi)\end{aligned}$$

$$- \frac{k-4}{m+4} Y \wedge X + \frac{k}{m+4} (\eta(Y)\xi \wedge X + \eta(X) Y \wedge \xi)$$
$$- \frac{1}{(m+4)(m+2)} \operatorname{Tr} h^2 (\phi Y \wedge \phi X + 2g(\phi X, Y)\phi$$
$$+ Y \wedge X + \eta(X)\xi \wedge Y - \eta(Y)\xi \wedge X),$$

where $k = \frac{S+m}{m+2}$ (S is the scalar curvature tensor of M) and $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ (cf., [3]). Using (2.1), B^{es} satisfies the following identities in a contact metric manifold M:

$$\begin{aligned} B^{es}(X,Y)Z &= -B^{es}(Y,X)Z \\ B^{es}(X,Y)Z + B^{es}(Y,Z)X + B^{es}(Z,X)Y &= 0 \\ g(B^{es}(X,Y)Z,W) &= -g(Z, B^{es}(X,Y)W), \\ g(B^{es}(X,Y)Z,W) &= g(B^{es}(Z,W)X,Y). \end{aligned} \tag{2.2}$$

If M is a Sasakian manifold, B^{es} coincides with the contact Bochner curvature tensor by Mastumoto and Chūman [3] and the following are satisfied:

$$\begin{aligned} B^{es}(\xi,Y)Z = B^{es}(X,Y)\xi = 0 \quad \text{and} \\ B^{es}(\phi X, \phi Y)Z = B^{es}(X,Y)Z, \end{aligned} \tag{2.3}$$

where we have used $R(\phi X,Y)Z - R(\phi Y,X)Z = g(\phi Z,X)Y - g(\phi Z,Y)X - g(Z,X)\phi Y + g(Z,Y)\phi X$ in a Sasakian manifold.

We consider a D-homothetic deformation $g^* = \alpha g + \alpha(\alpha-1)\eta \otimes \eta$, $\phi^* = \phi$, $\xi^* = \alpha^{-1}\xi$, $\eta^* = \alpha\eta$ on a contact metric manifold M, where α is a positive constant. For a D-homothetic deformation we say that $M(\phi,\xi,\eta,g)$ is D-homothetic to $M(\phi^*,\xi^*,\eta^*,g^*)$. It is well known that if a contact metric manifold $M(\phi,\xi,\eta,g)$ is D-homothetic to M $(\phi^*,\xi^*,\eta^*,g^*)$, then $M(\phi^*,\xi^*,\eta^*,g^*)$ is a contact metric manifold. Moreover if $M(\phi,\xi,\eta,g)$ is a K-contact Riemannian manifold (resp. Sasakian manifold), then $M(\phi^*,\xi^*,\eta^*,g^*)$ is also a K-contact Riemannian manifold (resp. Sasakian manifold) (see [5]). Denoting by ${W^i}_{jk}$ the difference ${}^*{\Gamma^i}_{jk} - {\Gamma^i}_{jk}$ of Christoffel symbols, by (1.1) we have in a contact metric manifold M

$$\begin{aligned} W(X,Y) &= (\alpha-1)(\eta(Y)\phi X + \eta(X)\phi Y) \\ &\qquad + \frac{\alpha-1}{2\alpha} ((\nabla_X \eta)(Y) + (\nabla_Y \eta)(X))\xi \quad \text{(see [5])} \\ &= (\alpha-1)(\eta(Y)\phi X + \eta(X)\phi Y) + \frac{\alpha-1}{\alpha} g(\phi h X, Y)\xi. \end{aligned}$$

Putting this into

$$R^*(X,Y)Z = R(X,Y)Z + (\nabla_X W)(Z,Y) - (\nabla_Y W)(Z,X) + W(W(Z,Y),X) - W(W(Z,X),Y)$$

and using (1.1) we have

$$\begin{aligned} R^*(X,Y)Z &= R(X,Y)Z + (\alpha-1)(2g(\phi X,Y)\phi Z \\ &+ g(\phi Z,Y)\phi X - g(\phi Z,X)\phi Y \\ &+ \eta(Y)(\nabla_X\phi)(Z) + \eta(Z)(\nabla_X\phi)(Y) \\ &- \eta(X)(\nabla_Y\phi)(Z) - \eta(Z)(\nabla_Y\phi)(X)) - (\alpha-1)^2(\eta(Z)\eta(X)Y \\ &- \eta(Z)\eta(Y)X) - \frac{\alpha-1}{\alpha}(g(X,(\nabla_Y\phi)hZ)\xi - g(Y,(\nabla_X\phi)hZ)\xi \\ &+ g(X,\phi(\nabla_Y h)Z)\xi - g(Y,\phi(\nabla_X h)Z)\xi \\ &+ g(X,\phi hZ)\phi hY - g(Y,\phi hZ)\phi hX) \\ &- \frac{(\alpha-1)^2}{\alpha}(\eta(X)g(hZ,Y)\xi - \eta(Y)g(hZ,X)\xi). \end{aligned} \tag{2.4}$$

Choosing a ϕ^*-basis with respect to g^* (if we take a ϕ^*-basis $(\xi^*, X_1, \phi^* X_1, \ldots, X_n, \phi^* X_n)$, a ϕ-basis with respect to g is $(\xi, \sqrt{\alpha}X_1, \sqrt{\alpha}\,\phi X_1, \ldots, \sqrt{\alpha}X_n, \sqrt{\alpha}\,\phi X_n)$) and using (1.4) and (1.6), we get

$$\begin{aligned} \mathrm{Ric}^*(X,Y) &= \mathrm{Ric}(X,Y) + (\alpha-1)(-2g(X,Y) \\ &+ 2(2n+1)\eta(X)\eta(Y)) + 2n(\alpha-1)^2\eta(X)\eta(Y) \\ &- \frac{\alpha-1}{\alpha}(-g(X,Y) + \eta(X)\eta(Y) - 2g(hX,Y) \\ &+ g(hX,hY) + g(R(X,\xi)\xi,Y)), \end{aligned} \tag{2.5}$$

where Ric is the Ricci curvature of M. From (2.5), we find

$$\begin{aligned} Q^*X = \frac{1}{\alpha}QX &+ \frac{\alpha-1}{\alpha}(-2X + 2(2n+1)\eta(X)\xi) \\ &- \frac{\alpha-1}{\alpha^2}g(X,Q\xi)\xi - 2n\left(\frac{\alpha-1}{\alpha}\right)^2\eta(X)\xi \\ &- \frac{\alpha-1}{\alpha^2}(-X + \eta(X)\xi - 2hX + h^2X + R(X,\xi)\xi), \end{aligned} \tag{2.6}$$

where we have used $Q^*\xi = \frac{1}{\alpha}Q\xi - \frac{\alpha-1}{\alpha^2}g(Q\xi,\xi)\xi + 2n\frac{\alpha^2-1}{\alpha^2}\xi$.

By virture of (1.3) we have

$$S^* = \frac{1}{\alpha}S - 2n\frac{\alpha-1}{\alpha} + \frac{\alpha-1}{\alpha^2}\,\mathrm{Tr}\,h^2. \tag{2.7}$$

Moreover, using the definition of h, we have

$$h^* = \frac{1}{\alpha} h, \tag{2.8}$$

from which we get

$$\operatorname{Tr} h^{*2} = \frac{1}{\alpha^2} \operatorname{Tr} h^2. \tag{2.9}$$

By means of (1.2), (2.1), (2.4), (2.5), (2.6), (2.7), (2.8) and (2.9), after some lengthy computation, we obtain

$$\begin{aligned}
{}^*B^{es}(X,Y)Z &= B^{es}(X,Y)Z + (\alpha-1)(\eta(Y)g(X,Z)\xi \\
&- \eta(X)g(Y,Z)\xi + 2\eta(X)\eta(Z)Y - 2\eta(Y)\eta(Z)X \\
&+ \eta(Y)(\nabla_X\phi)Z + \eta(Z)(\nabla_X\phi)Y - \eta(X)(\nabla_Y\phi)Z \\
&- \eta(Z)(\nabla_Y\phi)X) - \frac{\alpha-1}{\alpha}(g(X,(\nabla_Y\phi)hZ)\xi \\
&- g(Y,(\nabla_X\phi)hZ)\xi + g(X,\phi(\nabla_Y h)Z)\xi - g(Y,\phi(\nabla_X h)Z)\xi) \\
&- \frac{(\alpha-1)^2}{\alpha}(\eta(X)g(hZ,Y)\xi - \eta(Y)g(hZ,X)\xi) \\
&+ \frac{1}{2(2n+4)}\bigg\{\frac{3(\alpha-1)}{2\alpha}(\eta(X)\eta(Z)g(Y,Q\xi)\xi \\
&- \eta(Y)\eta(Z)g(X,Q\xi)\xi + g(Y,Z)g(X,Q\xi)\xi \\
&- g(X,Z)g(Y,Q\xi)\xi) + \frac{1}{2}\frac{\alpha-1}{\alpha}(g(Y,Z)\eta(X) \\
&- g(X,Z)\eta(Y))g(Q\xi,\xi)\xi \\
&+ \frac{\alpha-1}{\alpha}(-g(\phi X,Z)g(\phi Y,Q\xi)\xi + g(\phi Y,Z)g(\phi X,Q\xi)\xi \\
&- 2g(\phi X,Y)g(\phi Z,Q\xi)\xi \\
&+ 4ng(X,Z)\eta(Y)\xi - 4ng(Y,Z)\eta(X)\xi)\bigg\}.
\end{aligned} \tag{2.10}$$

Now we introduce in M an EH-tensor by

$$\begin{aligned}
EH(X,Y)Z &= B^{es}(X,Y)Z - \eta(X)B^{es}(\xi,Y)Z \\
&- \eta(Y)B^{es}(X,\xi)Z - \eta(Z)B^{es}(X,Y)\xi \\
&- \eta(B^{es}(X,Y)Z)\xi - B^{es}(\phi X,\phi Y)Z \\
&+ \eta(Z)B^{es}(\phi X,\phi Y)\xi + \eta(B^{es}(\phi X,\phi Y)Z)\xi \\
&+ \eta(X)\eta(B^{es}(\xi,Y)Z)\xi + \eta(Y)\eta(B^{es}(X,\xi)Z)\xi
\end{aligned} \tag{2.11}$$

$$+\eta(Y)\eta(Z)\phi B^{es}(\phi X,\xi)\xi+\eta(X)\eta(Z)\phi B^{es}(\xi,\phi Y)\xi.$$

In particular, if M is Sasakian, then $EH=0$ from (2.2) and (2.3).

Theorem 2.1. *The EH-tensor is invariant with respect to the D-homothetic deformation $M(\phi,\xi,\eta,g)\to M(\phi^*,\xi^*,\eta^*,g^*)$ on a contact metric manifold M.*

Proof. Using (1.5), (1.6), (1.7) and (2.10), we find

$$\begin{aligned}
(2.12)\quad & -\eta^*(X)\overset{*}{B}{}^{es}(\xi^*,Y)Z=-\eta(X)\overset{*}{B}{}^{es}(\xi,Y)Z \\
&=-\eta(X)B^{es}(\xi,Y)Z+(\alpha-1)(-\eta(X)\eta(Z)Y \\
&+\eta(X)\eta(Z)hY+\eta(X)(\nabla_Y\phi)Z) \\
&+\frac{\alpha-1}{\alpha}(\eta(X)\eta(Y)\eta(Z)\xi+\eta(X)g(Y,R(Z,\xi)\xi)\xi) \\
&+\frac{(\alpha-1)^2}{\alpha}\eta(X)g(Y,Z)\xi+\frac{(\alpha-1)(\alpha-2)}{\alpha}\eta(X)g(Y,hZ)\xi \\
&+\frac{1}{2(2n+4)}\left\{\frac{3(\alpha-1)}{2\alpha}(-g(Y,Z)\eta(X)g(Q\xi,\xi)\xi\right. \\
&+\eta(X)\eta(Y)\eta(Z)g(Q\xi,\xi)\xi) \\
&+\frac{1}{2}\frac{\alpha-1}{\alpha}(-\eta(X)g(Y,Z)g(Q\xi,\xi)\xi \\
&+\eta(X)\eta(Y)\eta(Z)g(Q\xi,\xi)\xi) \\
&\left.+\frac{\alpha-1}{\alpha}(4ng(Y,Z)\eta(X)\xi-4n\eta(X)\eta(Y)\eta(Z)\xi)\right\}
\end{aligned}$$

$$\begin{aligned}
(2.13)\quad & -\eta^*(Y)\overset{*}{B}{}^{es}(X,\xi^*)Z \\
&=-\eta(Y)B^{es}(X,\xi)Z+(\alpha-1)(\eta(Y)\eta(Z)X \\
&-\eta(Y)\eta(Z)hX-\eta(Y)(\nabla_X\phi)Z)-\frac{\alpha-1}{\alpha}(\eta(X)\eta(Y)\eta(Z)\xi \\
&+\eta(Y)g(X,R(Z,\xi)\xi)\xi)-\frac{(\alpha-1)^2}{\alpha}g(X,Z)\eta(Y)\xi \\
&-\frac{(\alpha-1)(\alpha-2)}{\alpha}g(hX,Z)\eta(Y)\xi \\
&+\frac{1}{2(2n+4)}\left\{\frac{3(\alpha-1)}{2\alpha}(g(X,Z)\eta(Y)g(Q\xi,\xi)\xi\right. \\
&-\eta(X)\eta(Y)\eta(Z)g(Q\xi,\xi)\xi
\end{aligned}$$

$$+\frac{1}{2}\frac{\alpha-1}{\alpha}(g(X,Z)\eta(Y)g(Q\xi,\xi)\xi$$
$$-\eta(X)\eta(Y)\eta(Z)g(Q\xi,\xi)\xi)$$
$$+\frac{\alpha-1}{\alpha}(-4ng(X,Z)\eta(Y)\xi+4n\eta(X)\eta(Y)\eta(Z)\xi)\Big\}$$

(2.14)
$$-\eta^*(Z)\overset{*}{B}{}^{es}(X,Y)\xi^* = -\eta(Z)B^{es}(X,Y)\xi$$
$$+(\alpha-1)(-\eta(X)\eta(Z)Y+\eta(Y)\eta(Z)X-\eta(Y)\eta(Z)hX$$
$$+\eta(X)\eta(Z)hY-\eta(Z)(\nabla_X\phi)Y+\eta(Z)(\nabla_Y\phi)X)$$

(2.15)
$$-\eta^*(\overset{*}{B}{}^{es}(X,Y)Z)\xi^* = -\eta(B^{es}(X,Y)Z)\xi$$
$$-\frac{\alpha-1}{\alpha}(\eta(X)g(hY,Z)\xi-\eta(Y)g(hX,Z)\xi$$
$$-g(X,(\nabla_Y\phi)hZ)\xi+g(Y,(\nabla_X\phi)hZ)\xi-g(X,\phi(\nabla_Y h)Z)\xi$$
$$+g(Y,\phi(\nabla_X h)Z)\xi)+\frac{1}{2(2n+4)}\Big\{\frac{3(\alpha-1)}{2\alpha}$$
$$(-g(Y,Z)g(X,Q\xi)\xi+g(X,Z)g(Y,Q\xi)\xi$$
$$-\eta(X)\eta(Z)g(Y,Q\xi)\xi+\eta(Y)\eta(Z)g(X,Q\xi)\xi)$$
$$+\frac{1}{2}\frac{\alpha-1}{\alpha}(-\eta(X)g(Y,Z)g(Q\xi,\xi)\xi$$
$$+\eta(Y)g(X,Z)g(Q\xi,\xi)\xi)+\frac{\alpha-1}{\alpha}(g(\phi X,Z)g(\phi Y,Q\xi)\xi$$
$$-g(\phi Y,Z)g(\phi X,Q\xi)\xi+2g(\phi X,Y)g(\phi Z,Q\xi)\xi$$
$$-4ng(X,Z)\eta(Y)\xi+4ng(Y,Z)\eta(X)\xi)\Big\}.$$

Using (1.3), (1.5), (1.6) and (2.15), we get

(2.16)
$$\eta^*(X)\eta^*(\overset{*}{B}{}^{es}(\xi^*,Y)Z)\xi^*+\eta^*(Y)\eta^*(\overset{*}{B}{}^{es}(X,\xi^*)Z)\xi^*$$
$$=\frac{\alpha-1}{\alpha}(2g(Y,hZ)\eta(X)\xi-2g(X,hZ)\eta(Y)\xi$$
$$+g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi$$
$$-g(Y,R(Z,\xi)\xi)\eta(X)\xi+g(X,R(Z,\xi)\xi)\eta(Y)\xi)$$
$$+\frac{1}{2n+4}\frac{\alpha-1}{\alpha}(g(X,Z)\eta(Y)-g(Y,Z)\eta(X))\operatorname{Tr}h^2$$

From (1.5) and (2.14) we have

$$\begin{aligned}(2.17)\qquad &\eta^*(Y)\eta^*(Z)\phi^*\overset{*}{B}{}^{es}(\phi^*X,\xi^*)\xi^*\\ &=\eta(Y)\eta(Z)\phi B^{es}(\phi X,\xi)\xi+2(\alpha-1)\eta(Y)\eta(Z)hX\\ &\eta^*(X)\eta^*(Z)\phi^*\overset{*}{B}{}^{es}(\xi^*,\phi^*Y)\xi^*\\ &=-\eta(X)\eta(Z)\phi B^{es}(\xi,\phi Y)\xi-2(\alpha-1)\eta(X)\eta(Z)hY.\end{aligned}$$

Therefore, using (1.3), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), (2.16) and (2.17), we get

$$\begin{aligned}(2.18)\quad &\overset{*}{B}{}^{es}(X,Y)Z-\eta^*(X)\overset{*}{B}{}^{es}(\xi^*,Y)Z\\ &-\eta^*(Y)\overset{*}{B}{}^{es}(X,\xi^*)Z-\eta^*(Z)\overset{*}{B}{}^{es}(X,Y)\xi^*\\ &-\eta^*(\overset{*}{B}{}^{es}(X,Y)Z)\xi^*+\eta^*(X)\eta^*(\overset{*}{B}{}^{es}(\xi^*,Y)Z)\xi^*\\ &+\eta^*(Y)\eta^*(\overset{*}{B}{}^{es}(X,\xi^*)Z)\xi^*+\eta^*(Y)\eta^*(Z)\phi^*\overset{*}{B}{}^{es}(\phi^*X,\xi^*)\xi^*\\ &+\eta^*(X)\eta^*(Z)\phi^*\overset{*}{B}{}^{es}(\xi^*,\phi^*Y)\xi^*\\ &=B^{es}(X,Y)Z-\eta(X)B^{es}(\xi,Y)Z-\eta(Y)B^{es}(X,\xi)Z\\ &-\eta(Z)B^{es}(X,Y)\xi-\eta(B^{es}(X,Y)Z)\xi\\ &+\eta(X)\eta(B^{es}(\xi,Y)Z)\xi+\eta(Y)\eta(B^{es}(X,\xi)Z)\xi\\ &+\eta(Y)\eta(Z)\phi B^{es}(\phi X,\xi)\xi+\eta(X)\eta(Z)\phi B^{es}(\xi,\phi Y)\xi.\end{aligned}$$

Here, substituting ϕ^*X for X and ϕ^*Y for Y in (2.18), we obtain our result.

3. Contact metric manifolds with vanishing EH-tensor

We define

$$(3.1)\qquad s^{\#}=\sum_{i,j=1}^{2n+1}g(R(E_i,E_j)\phi E_j,\phi E_i),$$

where $\{E_i\}$ is an orthonormal frame.

Lemma 3.1 ([4]). *For any $(2n+1)$-dimensional contact metric manifold M, we have*

$$s^{\#}-S+4n^2=\operatorname{Tr}h^2+\frac{1}{2}\{\|\nabla\phi\|^2-4n\}\geq 0.$$

Moreover M is Sasakian if and only if $\|\nabla\phi\|^2 - 4n = 0$ or equivalently

$$s^{\#} - S + 4n^2 = 0.$$

Theorem 3.1. *Let M be a contact metric manifold. M is a Sasakian manifold if and only if the EH-tensor vanishes.*

Proof. Since the EH-tensor of M vanishes, we have

$$\begin{aligned}(3.2)\qquad & g(B^{es}(X,Y)Z,W) - g(B^{es}(\phi X,\phi Y)Z,W)\\ &= \eta(X)g(B^{es}(\xi,Y)Z,W) + \eta(Y)g(B^{es}(X,\xi)Z,W)\\ &+ \eta(Z)g(B^{es}(X,Y)\xi,W) + \eta(W)\eta(B^{es}(X,Y)Z)\\ &- \eta(Z)g(B^{es}(\phi X,\phi Y)\xi,W) - \eta(W)\eta(B^{es}(\phi X,\phi Y)Z)\\ &- \eta(X)\eta(W)\eta(B^{es}(\xi,Y)Z) - \eta(Y)\eta(W)\eta(B^{es}(X,\xi)Z)\\ &- \eta(Y)\eta(Z)g(\phi B^{es}(\phi X,\xi)\xi,W)\\ &- \eta(X)\eta(Z)g(\phi B^{es}(\xi,\phi Y)\xi,W).\end{aligned}$$

Taking $X = E_i$, $Y = E_j$, $Z = \phi E_j$, $W = \phi E_i$ ($\{E_i\}$ is a ϕ-basis) in each member of (3.2) and summing over i and j, we obtain

$$\sum_{i,j=1}^{2n+1} g(B^{es}(E_i,E_j)\phi E_j,\phi E_i) = s^{\#} - S + 4n^2 - 2\operatorname{Tr} h^2 + (\operatorname{Tr} h^2)^2$$

$$\sum_{i,j=1}^{2n+1} g(B^{es}(\phi E_i,\phi E_j)\phi E_j,\phi E_i) = S - 2g(Q\xi,\xi) + \frac{2(n+1)}{n+2}(g(Q\xi,\xi) - S)$$

$$+ \frac{3n(k+2n)}{n+2} + \frac{n(2n-1)(k-4)}{n+2} - \frac{2}{n+2}\operatorname{Tr} h^2.$$

Thus, from (1.3) and $k = \frac{S+2n}{2n+2}$, we get

$$s^{\#} - S + 4n^2 - 2\operatorname{Tr} h^2 + (\operatorname{Tr} h^2)^2 = 0.$$

On the other hand, from (3.2) we find

$$\sum_{i=1}^{2n+1} g(B^{es}(E_i,\xi)\xi,E_i) = -\sum_{i=1}^{2n+1} g(B^{es}(\phi E_i,\xi)\xi,\phi E_i).$$

Hence we have $g(Q\xi,\xi) = 2n$.

By Lemma 3.1 we have our result.

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