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The identifying problem related to linear functional operators with linear arguments

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Dedicated to Professor Zoltán Daróczy on his 70th birthday

Abstract. In the work we consider a general linear functional operator $\mathcal{P}$ (see below) arising when studying different problems of analysis, geometry and even gas dynamics. The class of these operators includes such popular operators as Cauchy and Jensen operators, the quadratical operator and many others. Because of linearity $\mathcal{P}$, the homogeneous equation $\mathcal{P}F \approx 0$ plays a very special role. In particular, it relates to the description of approximate solutions of this equation, including different definitions of this notion. Some example of such description is provided by the well-known Hyers–Ulam theorem (by no means connected with any stability). In the present work we introduce the notion Identifying problem. The essence of this problem consists of searching some subspaces $\mathcal{K}$, $C_{\langle\tau\rangle}$ and submanifold Γ, guaranteeing the validity of a specific á priory estimate of the operator $\mathcal{P}$. In the presence of this estimate a description of approximate solutions to the equation $\mathcal{P}F \approx 0$ is derived automatically. In this work the described procedure: searching above-mentioned subspaces and manifold, proving the needed á priori estimate and, finally, describing the required approximate solutions is derived by using some sufficiently general class of functional equations.

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1. Introduction

In this work we formulate a new identifying problem for a general linear functional operator of the form

$$\mathcal{P}F(x) := \sum_{j=1}^{N} c_j(x) F \circ a_j(x), \quad x \in D \subset \mathbb{R}^p.$$

Here $F \in C(I, B)$ (the space of all B-valued continuous functions on I) with $I = (-1, 1)$, B a Banach space with norm $|\cdot|_B$, *coefficients* c_j and *arguments* a_j of $\mathcal{P}$ are continuous functions $D \to \mathbb{R}$ and $D \to I$, respectively, D is a domain with compact closure. The interest in this class of operators is motivated by the fact that many processes and phenomenons in nature or in society are determined by relations connecting the values of a function F at different points (but not values of its derivatives or integrals) (see, for example, [P4]). The importance of this class is emphasized by the fact that it contains such popular operators as the Cauchy operator $\mathfrak{C}$, the Jensen operator $\mathfrak{J}$, the quadratical operator $\mathfrak{Q}$ and many others.

We deal with the general problem of approximate solvability of linear functional equations $\mathcal{P}F = H$. The case $H = 0$ plays here the exceptional role for the linearity reasons. But even an exact formulation of the problem in question admits different interpretations when dealing with the equation $\mathcal{P}F \approx 0$.

Until now the general theory of the operators $\mathcal{P}$ has not developed, even for $x \in \mathbb{R}$. Some particular results are related to isolated operators introduced in the XIX century. One such result turned out to be the well-known Hyers–Ulam theorem (see below) undoubtedly characterizing the localization of some approximate solution to the Cauchy equation $(\mathfrak{C}F)(x, y) \approx 0$, although an unexplainable tradition connects it with a notion "stability".

In the present work we describe a new approach to the approximate solvability of the equation $\mathcal{P}F \approx 0$. This approach looks significantly more flexible than all previous ones, as it is based on considerably smaller à priory information about $\mathcal{P}F$ and allows considerably wider class of target subspaces localizing this F.

To this end we introduce in Section 2 the new notion *Identifying problem* for $\mathcal{P}$ and discuss shortly its connection with searching approximate solutions to the equation $\mathcal{P}F \approx 0$. As a main result of the work we formulate in Section 3 the (generic) conditions of solvability of this problem for the operator $\mathcal{P}$ with linear arguments a_j and constant coefficients c_j. The proof of the corresponding assertion is given in Section 4. For the generalized Cauchy and Jensen operators the analogous results may be found in the author's papers [P1] and [P2].

It would be extremely interesting to find some applied problems in any field whose study is closely connected to the results and methods discussed in this work.

2. Some notions and notations. Formulation of the problem

Definition 1. A term $c_k F \circ a_k$ is called *leading* term of $\mathcal{P}$ if the function a_k maps D onto I and $c_k \neq 0$.

Let Γ be an arbitrary C – curve in D, i.e.

$$\Gamma = \{x \in D \mid x = \zeta(t),\ t \in I\},\ \zeta \in C(I, D),$$

with ζ a one-to-one map: $I \to \Gamma$ (parametrization of Γ).

Definition 2. A curve $\Gamma \subset D$ is called $\mathcal{P}$ – *admissible*, if for a leading term $c_k F \circ a_k$, the function a_k maps Γ one-to-one onto I.

Introduce the operator

$$\mathcal{P}_\Gamma : F(t) \to \sum_{j=1}^{N} c_j(x) F \circ a_j(x) \,|_{x=\zeta(t)}, \quad t \in I,$$

which can be interpreted as the restriction of $\mathcal{P}$ to Γ.

In the following we systematically use the spaces $C_{\langle\tau\rangle} = C_{\langle\tau\rangle}(I, B)$, $\tau \in \mathbb{R}^+ = \{\tau \in \mathbb{R},\ \tau > 0\}$. By definition, $F(t) \in C_{\langle\tau\rangle}(I, B)$ if

$$F(t) = \sum_{j=0}^{\tau} t^j b_j + t^\tau f(t), \quad t \in I, \tag{1}$$

where $b_j \in B$, $f \in C(I, B)$ and $f(0) = 0$, as $\tau \in \mathbb{N}$, and

$$F(t) = \sum_{j=1}^{[\tau]} t^j b_j + t_+^\tau \varphi_+(t) + t_-^\tau \varphi_-(t), \quad t \in I, \tag{2}$$

with $[\tau]$ being the integral part of τ, where $\varphi_\pm \in C(I, B)$, and $\varphi_\pm(t) = \text{const}$ for $t \lessgtr 0$, $t_+ = \max\{t, 0\}$, $t_- = \max\{-t, 0\}$, as $\tau \notin \mathbb{N}$. The topology in $C_{\langle\tau\rangle}$ is defined by the norms

$$|F|_{\langle\tau\rangle} = \sum_{j=1}^{\tau} |b_j|_B + \sup_t |f(t)|_B, \quad \text{if } \tau \in \mathbb{N},$$

and

$$|F|_{\langle\tau\rangle} = \sum_{j=1}^{[\tau]} |b_j|_B + \sup_t |\varphi_+(t)|_B + \sup_t |\varphi_-(t)|_B,$$

otherwise. Now everything is ready to formulate the new
Identifying problem for $\mathcal{P}$. *Given an operator $\mathcal{P}$, find a finite-dimensional subspace $\mathcal{K} \subset C(I, B)$, a smooth submanifold $\Gamma \subset D$ of a positive codimension and a subspace $C_{\langle\tau\rangle}(I, B) \subset C(I, B)$ such that, the à priori estimate*

$$\inf_{\varphi \in \mathcal{K}} |F - \varphi|_{\langle\tau\rangle} < c|\mathcal{P}_\Gamma F|_{\langle\tau\rangle}, \quad F \in C_{\langle\tau\rangle}(I, B), \tag{3}$$

holds with a constant c not depending on F. If such a triple $(\mathcal{K}, \Gamma, C_{\langle\tau\rangle})$ is found, we say that the identifying problem for the operator $\mathcal{P}$ is $(\mathcal{K}, \Gamma)$ – solvable in the space $C_{\langle\tau\rangle}$. In this case, as it follows from (3), given an arbitrary $\varepsilon > 0$, if $|\mathcal{P}_\Gamma F|_{\langle\tau\rangle} < \varepsilon/c$ for a function $F \in C_{\langle\tau\rangle}(I, B)$, then $F = \varphi + h_\varepsilon$ with φ a function from $\mathcal{K}$ and $|h_\varepsilon|_{\langle\tau\rangle} < \varepsilon$. Thus, φ turns out to be an approximate solution to the equation $\mathcal{P}F \approx 0$, localizing close to $\mathcal{K}$.

Note that in the simplest situation of the Cauchy operator $\mathfrak{C}$ the identifying problem has been considered in some sense by Ulam and solved by Hyers. Their result states that *for an arbitrary $\varepsilon > 0$ and for each function $F \in C(I, B)$ satisfying inequality $|\mathcal{P}F|_B < \varepsilon$, there is a (linear) function $\varphi \in \ker \mathfrak{C}$ such that $|F - \varphi|_C < c\varepsilon$ with c a constant not depending on F or ε.* If to interpret $|\mathcal{P}F|_B$ as $|\mathcal{P}_{\mathbb{R}^2} F|_B$, by analogy with $|\mathcal{P}_\Gamma F|_B$, and denote $\ker \mathfrak{C} = \mathcal{K}$, then this assertion can be reformulated in the following way:
the identifying problem for the Cauchy operator $\mathfrak{C}$ is $(\ker \mathfrak{C}, \mathbb{R}^2)$ - solvable in the space C.

The essential difference between the two problems is that a starting point in searching an approximate solution F in the identifying problem is a smallness of $\mathcal{P}F$ on some (unknown) submanifold $\Gamma \subset D$, whereas in the Ulam problems this smallness is required on the full domain D.

The other significant difference between both problems in question is that the Ulam problem deals with an approximate solution to the equation $\mathcal{P}F \approx 0$ lying in the subspace $\ker \mathcal{P}$ only, whereas identifying problem admits such solution from an arbitrary fixed finite-dimensional subspace $\mathcal{K}$.

In the present work we discuss the identifying problem for the operator $\mathcal{P}$ under condition of linearity of all arguments a_j (just in this sense the title of the paper should be understood). In this case the operator $\mathcal{P}$ takes the form

$$\mathcal{P}F(x) := \sum_{k=0}^{m} \alpha_k F(\overline{a}_k \cdot x) - \sum_{j=1}^{n} \beta_j F(\overline{b}_j \cdot x), \quad x \in D, \tag{4}$$

with positive real coefficients $\alpha_k,\ \beta_j,$ real linear form $\overline{a}_k \cdot x,\ \overline{b}_j \cdot x$ on $\mathbb{R}^p$ and canonical domain

$$D = \{x \in \mathbb{R}^p \mid |\overline{a}_k \cdot x| \le 1;\ |\overline{b}_j \cdot x| \le 1,\ 0 \le k \le m,\ 1 \le j \le n\}.$$

3. Formulation of the results

Denote by $\mathcal{P}_+ F$ and $\mathcal{P}_- F$ both sums in (4), respectively, and suppose that

$$m \le n, \quad m + 1 \le p. \tag{5}$$

Definition 3. We say that the operator $\mathcal{P}_-$ *subordinates* the operator $\mathcal{P}_+$ if

$$\sum_{k=0}^{m} \alpha_k < \sum_{j=1}^{n} \beta_j \tag{6}$$

and, for an arbitrary natural value k, $1 \le k \le m$, there is an integer j_k, $1 \le j_k \le n$, such that

$$\beta_{j_k} > \alpha_k. \tag{7}$$

We associate with the operator $\mathcal{P} = \mathcal{P}_+ - \mathcal{P}_-$ the $m \times p$ – matrix

$$\Pi = \begin{pmatrix} \overline{a}_1 - \overline{b}_{j_1} \\ \dots \\ \overline{a}_m - \overline{b}_{j_m} \end{pmatrix}$$

and the $(m+1) \times p$ – matrices

$$\Pi_{\overline{a}_0} = \begin{pmatrix} \overline{a}_0 \\ \overline{a}_1 - \overline{b}_{j_1} \\ \dots \\ \overline{a}_m - \overline{b}_{j_m} \end{pmatrix}, \qquad \Pi_{\overline{b}_k} = \begin{pmatrix} \overline{b}_k \\ \overline{a}_1 - \overline{b}_{j_1} \\ \dots \\ \overline{a}_m - \overline{b}_{j_m} \end{pmatrix}, \quad k = 1, \dots, n.$$

Definition 4. Given a real-valued $m \times p$ – matrix $\mathcal{M}$, $m \le p$, and m – tuple $\sigma = (\sigma_1, \dots, \sigma_m)$ with natural $\sigma_1 < \dots < \sigma_m \le p$ we define $\mathcal{M}^\sigma$ as a square matrix whose j – column coincides with σ_j – column of $\mathcal{M}$, $j = 1, \dots, m$.

Now everything is ready to formulate the main result of the present work.

Theorem 1. *Given an operator $\mathcal{P}$ of the type* (4) *satisfying the conditions:*

(i) *the operator $\mathcal{P}_-$ subordinates $\mathcal{P}_+$;*

(i) *there is a multi-index* $\sigma = (\sigma_1, \ldots, \sigma_{m+1})$ *such that all numbers* $\det \Pi^\sigma_{\overline{b}_k}$, $k = 1, \ldots, n$, *separate the values* 0 *and* $\det \Pi^\sigma_{\overline{a}_0}$.

Then the identifying problem for $\mathcal{P}$ *is* $(\mathcal{K}, \Gamma)$ *– solvable in any space* $C_{\langle\tau\rangle}(I, B)$ *with* $\tau > \rho$, *where* ρ *is a unique* z *– root of the equation*

$$\sum_{k=1}^{m} (\beta_{j_k} - \alpha_k)(\det \Pi^\sigma_{\overline{b}_k})^z + \sum_{\substack{k=1 \\ k \neq \sigma_q}}^{n} \beta_k (\det \Pi^\sigma_{\overline{b}_k})^z = (\det \Pi^\sigma_{\overline{a}_0})^z.$$

Here $\mathcal{K} = \{t^\rho \zeta\}_{\zeta \in B}$ *or* $\mathcal{K} = \{0\}$ *if* $\rho \in \mathbb{N}$ *or* $\rho \notin \mathbb{N}$, *respectively, and* $\Gamma = \{x \in D \mid x = \overline{\lambda} t,\ t \in I\}$ *with* $\overline{\lambda}$ *a solution of the equation* $\Pi_{\overline{a}_0} \overline{\lambda} = (1, 0, \ldots, 0)$ *such that* $\lambda_q = 0$ *for all* $q \neq \sigma_1, \ldots, \sigma_{m+1}$.

4. Proof

We will prove the theorem by determining successively a curve Γ, a space $C_{\langle\tau\rangle}(I, B)$ and a finite-dimensional subspace $\mathcal{K} \subset C_{\langle\tau\rangle}$ figuring in the formulation of the identifying problem. Without loss of generality we assume that $\alpha_0 = 1$. We note first of all that $F(\overline{a}_0 \cdot x)$ is a *leading* term of the operator $\mathcal{P}$ in D. By convexity and central symmetry of the domain D it follows, that the curve $\Gamma = \{x \mid x = t\overline{\lambda},\ t \in I\}$ with $\overline{\lambda} = (\lambda_1, \ldots, \lambda_p)$ a solution of the equation $\Pi_{\overline{a}_0} \overline{\lambda} = (1, 0, \ldots, 0)$ is $\mathcal{P}$-admissible. It is not difficult to determine explicitly a solution $\overline{\lambda}$ of the above equation under condition (ii). Indeed, by (ii), $\det \Pi^\sigma_{\overline{a}_0} \neq 0$ and, therefore, there is a unique solution $\overline{\mu} = (\mu_1, \ldots, \mu_{m+1})$ of the equation $\Pi^\sigma_{\overline{a}_0} \overline{\mu} = (1, 0, \ldots, 0)$. It follows that the vector $\overline{\lambda} = (\lambda_1, \ldots, \lambda_p)$ with $\lambda_{\sigma_k} = (-1)^{k+1} \mu_k / \det \Pi^\sigma_{\overline{a}_0}$, $k = 1, \ldots, m+1$, and $\lambda_q = 0$ if $q \neq \sigma_k$ for any k, satisfies the equation in question. Moreover, as is easily seen, each value $\mu_k,\ k = 1, \ldots, m+1$, is nothing but the $m \times m$ - minor of the matrix Π^σ obtained by removing from Π^σ the k-th column. It is clear that for an arbitrary vector $\overline{b}_k = (b^1_k, \ldots b^p_k)$

$$\overline{b}_k \cdot \overline{\lambda} = \sum_{j=1}^{p} b^j_k \lambda_j = \sum_{j=1}^{m+1} b^{\sigma_j}_k \lambda_{\sigma_j} = \det \Pi^\sigma_{\overline{b}_k} / \det \Pi^\sigma_{\overline{a}_0}.$$

Thus, the operator $\mathcal{P}_\Gamma$ takes the form

$$(\mathcal{P}_\Gamma F)(t) := F(t) - \sum_{k=1}^{m} (\beta_{j_k} - \alpha_k) F\Big((\det \Pi^\sigma_{\overline{b}_{j_k}} / \det \Pi^\sigma_{\overline{a}_0}) t \Big)$$
$$- \sum_{\substack{l=1 \\ l \neq j_k}}^{n} \beta_l F\Big((\det \Pi^\sigma_{\overline{b}_l} / \det \Pi^\sigma_{\overline{a}_0}) t \Big), \quad t \in I,$$

or shortly

$$(\mathcal{P}_\Gamma F)(t) := F(t) - \sum_{k=1}^{n} \gamma_k F(\delta_k t), \quad t \in I, \tag{8}$$

where

$$\gamma_k = \begin{cases} \beta_{j_k} - \alpha_k, & \text{if } k \le m, \\ \beta_{r(k)}, & m < k \le n, \end{cases} \qquad \delta_k = \begin{cases} \det \Pi^{\sigma}_{\overline{b}_{j_k}} / \det \Pi^{\sigma}_{\overline{a}_0}, & 1 \le k \le m, \\ \det \Pi^{\sigma}_{\overline{b}_{r(k)}} / \det \Pi^{\sigma}_{\overline{a}_0}, & m < k \le n, \end{cases}$$

with $r(k)$ a $(k-m)$ – th natural number in the ordered set $\{1, \dots, n\} \setminus \{j_1, \dots, j_m\}$, if $k > m$. We note that for all k, $1 \le k \le n$,

$$\gamma_k > 0 \quad \text{and} \quad 0 < \delta_k < 1, \tag{9}$$

by (7) and (ii), respectively, and, in addition, $\sum_{k=1}^{n} \gamma_k > 1$, by (6).

Our next step is describing the kernel $\ker \mathcal{P}_\Gamma$ of the operator $\mathcal{P}_\Gamma$. To this end introduce the number $\rho > 0$ as a unique z-root of the equation

$$\Phi(z) := \sum_{k=1}^{n} \gamma_k (\delta_k)^z = 1.$$

The existence of such number follows from continuity and decreasing of the function Φ_z, $z \ge 0$, and from relations

$$\Phi(0) = \sum_{k=1}^{n} \gamma_k > 1, \quad \Phi(\infty) = 0$$

following from (9). Introduce the space $\mathcal{K}_\rho = \ker \mathcal{P}_\Gamma \cap C_{\langle \rho \rangle}(I, B)$ and describe it. We consider separately two cases: $\rho \in \mathbb{N}$ and $\rho \notin \mathbb{N}$.

Let $\rho \in \mathbb{N}$ and $F \in C_{\langle \rho \rangle}$. Then, by (1), the relation $\mathcal{P}_\Gamma F = 0$ in a detailed form looks as follows:

$$\sum_{j=0}^{\rho} b_j t^j \left(1 - \sum_{k=1}^{\rho} \gamma_k \, \delta_k^j \right) + t^\rho \left[f(t) - \sum_{k=1}^{n} \gamma_k \, \delta_k^\rho \, f(\delta_k t) \right] = 0, \quad t \in I. \tag{10}$$

Let us compare the behavior of both parts of this relation as $t \to 0$. From the monotonicity of the function $\Phi(z)$ and by the choice of the value ρ it follows that all the factors $1 - \sum_{k=1}^{\rho} \gamma_k \, \delta_k^j$, $0 \le j \le \rho - 1$, do not vanish. This is possible only if $b_j = 0$ for the same j. Therefore the first sum in (10) is equal to zero on I and

$$f(t) - \sum_{k=1}^{n} \gamma_k \, \delta_k^\rho \, f(\delta_k t) = 0, \quad t \in I. \tag{11}$$

Let us show that $f=0$. Denote $\mathcal{M}=\max_I f$ and let $f(\widehat{t})=\mathcal{M}$. Then, by the relation $\sum_{k=1}^n \gamma_k\,\delta_k^\rho=1$, we find that $f(\delta_k\widehat{t})=\mathcal{M}$ for all k, $1\le k\le n$. Repeating this argument leads to the relation $f(\delta_k^q\widehat{t})=\mathcal{M}$ for all natural q, which in turn, by (9), results in $f(0)=\mathcal{M}$. Applying the same arguments to the minimal value m of f on I we arrive at the relation $f(0)=m$ and consequently

$$f=\text{const}\ \text{ on } I.$$

It remains to remember that, by definition, $f(0)=0$. Thus, we have proved that

$$\mathcal{K}_\rho=\{t^\rho\sigma\}_{\sigma\in B}. \tag{12}$$

Let $\rho\notin\mathbb{N}$ and $\mathcal{P}_\Gamma F=0$ for a function $F\in C_{\langle\rho\rangle}(I,B)$. Then, by (2),

$$\sum_{j=0}^{[\rho]} t^j c_j\left(1-\sum_{k=1}^n \gamma_k\,\delta_k^j\right)+t_+^\rho\left[\varphi_+(t)-\sum_{k=1}^n\gamma_k\,\delta_k^\rho\,\varphi_+(\delta_k t)\right]$$
$$+\,t_-^\rho\left[\varphi_-(t)-\sum_{k=1}^n\gamma_k\,\delta_k^\rho\,\varphi_-(\delta_k t)\right]=0 \tag{13}$$

It is immediate, as above, that all $c_j=0$. On the other hand, as the supports of the functions φ_+ and φ_- are disjoint, the relations

$$\varphi_\pm(t)-\sum_{k=1}^n\gamma_k\,\delta_k^\rho\,\varphi_\pm(\delta_k t)=0,\quad t\gtrless 0,$$

hold. Applying to $\varphi_\pm$ the above approach with max and min results in the relations

$$\varphi_\pm(t)=\text{const}\ \text{ for } t\gtrless 0.$$

It follows that

$$\mathcal{K}_\rho=\{t_+^\rho\sigma_++t_-^\rho\sigma_-\}_{\sigma_+,\,\sigma_-\in B} \tag{14}$$

Assume now that estimate (3) is true with $\mathcal{K}=\ker\mathcal{P}_\Gamma\cap C_{\langle\tau\rangle}$ and $\tau>\rho$. It follows that $\mathcal{K}\subset\mathcal{K}_{\langle\rho\rangle}$. If $\rho\in\mathbb{N}$, then $\mathcal{K}_{\langle\rho\rangle}\subset\mathcal{K}$ because $\mathcal{K}_\rho=\{t^\rho\sigma\}_{\sigma\in B}$ and $t^\rho\in C^\infty(I)$. Therefore, $\mathcal{K}=\mathcal{K}_{\langle\rho\rangle}$. If $\rho\notin\mathbb{N}$, then by (14), any function $\psi\in\mathcal{K}$ can be represented in the form

$$\psi=t_+^\rho\sigma_++t_-^\rho\sigma_-$$

with $\sigma_\pm$ from B. On the other hand, being an element of $\mathcal{K}$, ψ has a form $\psi=t^\tau\zeta(t)$ with $\zeta\in C(I,B)$. However, the relation

$$t^\tau\zeta(t)=t_+^\rho\sigma_++t_-^\rho\sigma_-,\quad t\in I,$$

may be true, under condition $\tau > \rho$, only if $\sigma_+ = \sigma_- = 0$. Therefore, $\mathcal{K} = \{0\}$. Thus, to complete the proof of the theorem it remains to obtain à priori estimate (3) with the subspace $\mathcal{K}$ in question. It will be done on the basis of Proposition 1 in [P1]. We recall this proposition for completeness.

Let $L : B_1 \to B_2$ be a closed linear operator between Banach spaces and $\mathcal{K} = \ker L$. If the range $\mathcal{R}(L)$ is closed, then there is a positive constant c such that the à priori estimate

$$\inf_{\varphi \in \mathcal{K}} |F - \varphi|_{B_1} < c|LF|_{B_2} \tag{15}$$

holds for all elements $F \notin \mathcal{K}$. In our case we choose as B_1 and B_2 the space $C_{\langle\tau\rangle}(I, B)$ with $\tau > \rho$, as L the operator $\mathcal{P}_\Gamma$ from (8). As the closedness of the operator $\mathcal{P}_\Gamma$ follows from its continuity, to obtain estimate (15) it suffices, by the above Proposition, to prove the solvability of the equation

$$F(t) - \sum_{k=1}^{n} \gamma_k F(\delta_k t) = H(t) \tag{16}$$

in the space $C_{\langle\tau\rangle}(I, B)$ for an arbitrary function H from some closed subspace of $C_{\langle\tau\rangle}(I, B)$. To this end we compare the asymptotic expansions of the left-hand side in (16) (see (8) and (13)) with the similar expansions

$$\sum_{j=1}^{[\tau]} t^j h_j + t^\tau h(t) \quad \text{and} \quad \sum_{j=0}^{[\tau]} t^j h_j + t_+^\tau h_+(t) + t_-^\tau h_-(t)$$

of the function H (with $h(t)$ and $h_\pm(t)$ continuous functions) corresponding to the cases $\rho \in \mathbb{N}$ and $\rho \notin \mathbb{N}$, respectively. It is easily seen that if $\rho \notin \mathbb{N}$, then

$$b_j = \left(1 - \sum_{k=1}^{n} \gamma_k \, \delta_k^j\right)^{-1} h_j, \quad 0 \le j \le [\tau], \tag{17}$$

and if $\rho \in \mathbb{N}$, then this is true for $j \neq \rho$, and

$$h_\rho = \left(1 - \sum_{k=1}^{n} \gamma_k \, \delta_k^\rho\right) b_\rho = 0. \tag{18}$$

In the first case we find by this all elements b_j in representation (2). In the second case, by (18), the range of the operator $\mathcal{P}_\Gamma$ coincides with the closed subspace of functions $H \in C_{\langle\tau\rangle}(I, B)$ with $H^{(\rho)}(0) = 0$ (however, this does not obstruct using the above proposition from [P1]). All the above elements b_j in (1) are defined by (17) with the exception of b_ρ, which can be taken arbitrarily. It remains to determine the functions f, φ_+ and φ_- in (1) and (2), which satisfy the functional equations

$$f(t) - \sum_{k=1}^{n} \gamma_k \, \delta_k^\tau \, f(\delta_k t) = h(t), \quad t \in I,$$

and

$$\varphi_\pm(t) - \sum_{k=1}^{n} \gamma_k \, \delta_k^\tau \, \varphi_\pm(\delta_k t) = h_\pm(t), \quad t \in I_\pm,$$

respectively, with f, $h \in C(I,B)$, $\varphi_\pm$ and $h_\pm \in C(I_\pm, B)$, $I_+ = \{t \in I,\, t \geq 0\}$, $I_- = \{t \in I,\, t \leq 0\}$. All these equations have the same operator form $g - \mathcal{A}g = h$ where

$$\mathcal{A} : g(t) \to \sum_{k=1}^{n} \gamma_k \, \delta_k^\tau \, g(\delta_k t)$$

is a linear operator on $C(I,B)$ with the norm

$$\|\mathcal{A}\| \leq \sum_{k=1}^{n} \gamma_k \, \delta_k^\tau < 1.$$

The latter relation follows by the choice of ρ, relation $\tau > \rho$ and (9).

Applying the classical result in functional analysis (the invertibility of the operator $E - \mathcal{A}$, where E stands for the identical operator) proves the unique solvability of equation in question for an arbitrary function $h \in C(I,B)$. Thus, we have proved solvability in the space $C_{\langle\tau\rangle}(I,B)$ of equation (16) for all elements H from a close subspace in $C_{\langle\tau\rangle}(I,B)$ and hence the validity of à priory estimate (3).

5. Concluding remarks

1. It is worth noting that the Theorem guaranties the solvability of the identifying problem for the operator $\mathcal{P}$ in any space $C_{\langle\tau\rangle}$ with $\tau > \rho$, if Γ is a $C_{\langle\tau\rangle}$ - curve. Such conclusions play a crucial role in some applications of the results obtained in partial differential equations where the corresponding results are called "increasing in smoothness". We note also that the analogous problem of increasing in smoothness for the Ulam problem have never been studied.

2. The solvability of the identifying problem for several important classes of the operator $\mathcal{P}$ (Cauchy type, Jensen type and others) has been established under a different name in the works [P1], [P2], [P3].

3. The solvability of the Identifying problem for the operator $\mathcal{P}$ in the space $C(I,B)$ remains to be very interesting unsolved problem.

References

[P1] B. Paneah, A new approach to the stability of linear functional operators, *Aequationes Math.* (*to appear*).

[P2] B. Paneah, On the stability of the linear functional operators structurally associated with the Jensen operator, *Grazer Math. Bericht* **351** (2007), 129–138.

[P3] B. Paneah, Some remarks on stability and solvability of linear functional equations, *Banach J. Math. Anal.* **1**, no. 1 (2007), 56–65.

[P4] B. Paneah, On the general theory of the Cauchy type functional equations with applications in analysis, *Aeq. Math.* **74** (2007), 119–157.

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