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The voice transform on the Blaschke group III

By MARGIT PAP (Pécs) and FERENC SCHIPP (Budapest)

Dedicated to Professor Zoltán Daróczy on his 70th birthday

Abstract. In this paper we present results connected to the voice transform of the Blaschke group generated by a representation of the group on the Bergmann space $H^2(\mathbb{D})$. Sections 1 and 2 contain the basic notations, definitions and results. In Section 3 the matrix elements of the representation on $H^2(\mathbb{D})$ are computed and the properties of the matrix elements are studied. Using these properties it is given a direct proof for the analogue of the Plancherer formula and also it is proved the irreducibility of the representation. It is introduced an orthogonal rational wavelet system and it is showed that the Bergman projection can be expressed with the voice transform and the rational orthogonal wavelet system. As a consequence it is obtained that the matrix elements of the representation form an orthogonal system in $L^2(\mathbb{B})$. It is proved that every element from $H^\infty(\mathbb{D})$ is admissible. Sections 4 contains the proofs.

1. The voice transform

In signal processing and image reconstruction the wavelet and Gábor transforms play an important role. There exists a common generalization of these transforms, the so-called *Voice-transform.* In this section we summarize the basic notations and notions used in the definition of Voice-transform, we also present the definition and the most important properties of this transform.

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In construction of voice transform the starting point will be a locally compact topological group $(G,\cdot)$. It is known that every locally compact topological group has nontrivial left- and right-translation invariant Borel measures, called left invariant and right invariant Haar measures. Let m be a left-invariant Haar measure of G. Let $f : G \to \mathbb{C}$ be a Borel- measurable function which is integrable with respect to the left invariant Haar measure m, the integral of f will be denoted by $\int_G f\,dm = \int_G f(x)\,dm(x)$. Because of left-translation invariance of the measure m it follows that

$$\int_G f(x)\,dm(x) = \int_G f(a^{-1}\cdot x)\,dm(x) \quad (a \in G).$$

There exist groups whose left invariant Haar measure is not right invariant. If the left invariant Haar measure of G is at the same time right invariant then we say that G is *unimodular.* Such measure will be called Haar measure of G. On a given group, Haar measure is unique only up to constant multiples. It is trivial that the commutative groups are unimodular. Furthermore it can be proved that if the left Haar measure is invariant under the inverse transformation $G \ni x \to x^{-1} \in G$, then G is also unimodular.

In the definition of voice transform a *unitary representation of the group* $(G,\cdot)$ is used. Let us consider a Hilbert-space $(H, \langle\cdot,\cdot\rangle)$ and let $\mathcal{U}$ denote the set of unitary bijections $U : H \to H$. Namely, the elements of $\mathcal{U}$ are bounded linear operators which satisfy $\langle Uf, Ug\rangle = \langle f, g\rangle$ $(f, g \in H)$. The set $\mathcal{U}$ with the composition operation $(U \circ V)f := U(Vf)$ $(f \in H)$ is a group the neutral element of which is I, the identity operator on H and the inverse element of $U \in \mathcal{U}$ is the operator U^{-1}, which is equal to the adjoint of $U : U^{-1} = U^*$. The homomorphism U of the group $(G,\cdot)$ on the group $(\mathcal{U},\circ)$ satisfying

$$\text{i)} \quad U_{x\cdot y} = U_x \circ U_y \quad (x, y \in G),$$

$$\text{ii)} \qquad G \ni x \to U_x f \in H \text{ is continuous for all } f \in H \tag{1.1}$$

is called unitary representation of $(G,\cdot)$ on H.

The *voice transform* of $f \in H$ generated by the representation U and by the parameter $\rho \in H$ is the (complex-valued) function on G defined by

$$(V_\rho f)(x) := \langle f, U_x\rho\rangle \quad (x \in G,\ f, \rho \in H). \tag{1.2}$$

For any representation $U : G \to \mathcal{U}$ and for each $f, \rho \in H$ the voice transform $V_\rho f$ is a continuous and bounded function on G.

The set of continuous bounded functions defined on the group G with the supremum norm form a Banach space and $V_\rho : H \to C(G)$ is a bounded linear operator. From the unitarity of $U_x : H \to H$ follows that, for all $x \in G$

$$|(V_\rho f)(x)| = |\langle f, U_x\rho\rangle| \le \|f\|\,\|U_x\rho\| = \|f\|\,\|\rho\|,$$

consequently $\|V_\rho\| \le \|\rho\|$.

The invertibility of V_ρ it is connected to the irreducibility of the representation U which generate the voice transform.

A representation U is called *irreducible* if the only closed invariant subspaces of H, i.e., closed subspaces H_0 which satisfy $U_x H_0 \subset H_0$, are $\{0\}$ and H. Since the closure of the linear span of the set

$$\{U_x\rho : x \in G\} \tag{1.3}$$

is always a closed invariant subspace of H, it follows that U is irreducible if and only if the collection (1.3) is a closed system for any $\rho \in H$, $\rho \ne 0$.

The property of irreducibility gives a simple criterion for deciding when a voice transform is 1 to 1:

Theorem 1 ([6], [12]). *A voice transform V_ρ generated by an unitary representation U is 1 to 1 for all $\rho \in H$, $\rho \ne 0$ if and only if U is irreducible.*

The function $V_\rho f$ is continuous on G but in general is not square integrable. If there exist $\rho \in H, \rho \ne 0$ such that $V_\rho\rho \in L^2_m(G)$, then the representation U is *square integrable* and the function ρ is called admissible for U. For a fixed square integrable U the collection of admissible elements of H will be denoted by H^*. Choosing a convenient $\rho \in H^*$ the voice transform $V_\rho : H \to L^2_m(G)$ will be unitary. This is a consequence of the following theorem:

Theorem 2 ([6], [12]). *Let be $U_x \in \mathcal{U}$ $(x \in G)$ an irreducible square integrable representation of G in H. Then the collection of admissible elements H^* is a linear subspace of H and for every $\rho \in H^*$ the voice transform of the function f is square integrable on G namely $V_\rho f \in L^2_m(G)$, if $f \in H$. Moreover there is a symmetric, positive bilinear map $B : H^* \times H^* \to \mathbb{R}$ such that*

$$[V_{\rho_1} f, V_{\rho_2} g] = B(\rho_1, \rho_2)\langle f, g\rangle \quad (\rho_1, \rho_2 \in H^*,\ f, g \in H),$$

for all $f, g \in H$ and $\rho_1, \rho_2 \in H^*$, where the inner product $[\cdot,\cdot]$ is the usual inner product in $L^2_m(G)$. If the group G is unimodular then $B(\rho,\rho) = c\|\rho\|^2$ $(\rho \in H^*)$, where $c > 0$ is a constant. In this case if we choose ρ so that $\langle \rho, \rho\rangle = 1/c$ then

$$[V_\rho f, V_\rho g] = \langle f, g\rangle \quad (f, g \in H)$$

In the next sections we will construct a voice transform using so called multiplier representations generated by a collection of multiplier functions defined in the following way: $F_a : G \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ $(a \in G)$ is a *collection of multiplier functions* if

$$F_e = 1, \quad F_{a_1 \cdot a_2}(x) = F_{a_1}(a_2 \cdot x) F_{a_2}(x) \quad (a_1, a_2, x \in G),$$

where e is the neutral element of G. It can be proved that

$$(U_a f)(x) := F_{a^{-1}}(x) f(a^{-1} \cdot x) \quad (a, x \in G)$$

satisfies

$$U_{a_1} \circ U_{a_2} = U_{a_1 \cdot a_2} \quad (a_1, a_2 \in G),$$

so is a representation of G on the space of all complex valued functions on G. If F_a is continuous and bounded for every $a \in G$, then $L^2_m(G)$ is an invariant subspace and $(U_a)_{a \in G}$ is a representation on $L^2_m(G)$. The representations obtained as bellow are named *multiplier representations* (see [13]).

Taking as starting point (not necessarily commutative) locally compact groups we can construct in this way important transformations in signal processing and control theory. For example the affine wavelet transform and the Gábor transform are all special voice transforms (see [6], [12]).

2. The voice transform of the Blaschke group on $H^2(\mathbb{D})$

The affine wavelet transform is a voice transform of the affine group which is a subgroup of the Möbius group (i.e. the group of linear fractional transformations with the composition operation). In this section we will study the voice transform of another subgroup of the Möbius group, namely the voice transform of the Blaschke group.

2.1. The Blaschke group. Let us denote by

$$B_a(z) := \epsilon \frac{z - b}{1 - \bar{b}z} \quad (z \in \mathbb{C},\ a = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T},\ \bar{b}z \neq 1) \tag{2.1}$$

the so called *Blaschke functions*, where

$$\mathbb{D}_+ := \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\},$$
$$\mathbb{D}_- := \{z \in \mathbb{C} : |z| > 1\}. \tag{2.2}$$

If $a \in \mathbb{B}$, then B_a is an 1 to 1 map on $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{D}_-$, respectively. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z))$ form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way: $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$. The group $(\mathbb{B}, \circ)$ will be isomorphic with the group $(\{B_a, a \in \mathbb{B}\}, \circ)$. If we use the notations $a_j := (b_j, \epsilon_j)$, $j \in \{1, 2\}$ and $a := (b, \epsilon) =: a_1 \circ a_2$, then

$$b = \frac{b_1\bar{\epsilon}_2 + b_2}{1 + b_1\bar{b}_2\bar{\epsilon}_2} = B_{(-b_2,1)}(b_1\bar{\epsilon}_2), \quad \epsilon = \epsilon_1 \frac{\epsilon_2 + b_1\bar{b}_2}{1 + \epsilon_2\bar{b}_1 b_2} = B_{(-b_1\bar{b}_2,\epsilon_1)}(\epsilon_2). \tag{2.3}$$

The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0, 1) \in \mathbb{B}$ and the inverse element of $a = (b, \epsilon) \in \mathbb{B}$ is $a^{-1} = (-b\epsilon, \bar{\epsilon})$.

Because of $B_a : \mathbb{T} \to \mathbb{T}$ is bijection it follows the existence of a function $\beta_a : \mathbb{R} \to \mathbb{R}$ such that $B_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R})$, where β_a can be expressed in an explicit form. Namely, let us introduce the function

$$\gamma_r(t) := \int_0^t \frac{1 - r^2}{1 - 2r\cos s + r^2}\, ds \quad (t \in \mathbb{R}, 0 \leqq r \leq 1). \tag{2.4}$$

Then

$$\beta_a(t) := \theta + \varphi + \gamma_r(t - \varphi), \quad (a = (re^{i\varphi}, e^{i\theta}) \in \mathbb{B},\ t \in \mathbb{R}, \theta, \varphi \in \mathbb{I} := [-\pi, \pi)). \tag{2.5}$$

The integral of the function $f : \mathbb{B} \to \mathbb{C}$, with respect to the left invariant Haar-measure m of the group $(\mathbb{B}, \circ)$, is given by

$$\int_{\mathbb{B}} f(a)\, dm(a) = \frac{1}{2\pi} \int_{\mathbb{I}} \int_{\mathbb{D}} \frac{f(b, e^{it})}{(1 - |b|^2)^2}\, db_1 db_2 dt, \tag{2.6}$$

where $a = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{D} \times \mathbb{T}$.

It can be shown that this integral is invariant with respect to the left translation $a \to a_0 \circ a$ and under the inverse transformation $a \to a^{-1}$, so this group is unimodular.

We will study the voice transform of the Blaschke group. In the construction it will be used a class of unitary representations of the Blaschke group on the Hilbert space $H = H^2(\mathbb{D})$.

2.2. The Hilbert space $H^2(\mathbb{D})$. We will introduce the voice transform using a class of unitary representations of the Blaschke group on the so called *Bergman space*. In this section we will summarize the basic results connected to the Bergman space (see [3], [5]).

Let denote by $\mathcal{A}$ the set of analytic functions $f : \mathbb{D} \to \mathbb{C}$. Let consider the following subset of analytic functions:

$$H^2(\mathbb{D}) := \left\{ f \in \mathcal{A} : \int_{\mathbb{D}} |f(z)|^2 \, d\sigma(z) < \infty \right\}$$
$$\left(d\sigma(z) := \frac{1}{\pi} dx\, dy, \ z = x + iy \in \mathbb{D} \right). \tag{2.7}$$

The set $H^2(\mathbb{D})$ with the scalar product

$$\langle f, g \rangle := \int_{\mathbb{D}} f(z)\overline{g(z)} \, d\sigma(z) \tag{2.8}$$

is a Hilbert space. The space $H^2(\mathbb{D})$ is the so called *Bergman space*. It can be proved that the function

$$f(z) := \sum_{n=0}^{\infty} c_n z^n \quad (z \in \mathbb{D})$$

from $\mathcal{A}$ belongs to the set $H^2(\mathbb{D})$ if and only if the coefficients satisfies

$$\sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1} < \infty.$$

The Bergman space $H^2(\mathbb{D})$ is a closed subspace of $L^2_\sigma(\mathbb{D})$.

For each $z \in \mathbb{D}$ the point-evaluation map

$$\tau_z : H^2(\mathbb{D}) \to \mathbb{C}, \quad \tau_z(f) = f(z)$$

is a bounded linear functional on $H^2(\mathbb{D})$. Each function $f \in H^2(\mathbb{D})$ has the property

$$|f(z)| \leqq \pi^{-1/2} \delta(z)^{-1} \|f\|_{H^2(\mathbb{D})} \quad (z \in \mathbb{D}),$$

where $\delta(z) = dist(z, \mathbb{T})$. From this it follows that the norm convergence in $H^2(\mathbb{D})$ implies the locally uniform convergence on $\mathbb{D}$. Therefore, by the Riesz Representation Theorem there is a unique element in $H^2(\mathbb{D})$, denoted by $K(., z)$, such that

$$f(z) = \tau_z(f) = \langle f, K(., z) \rangle = \int_{\mathbb{D}} f(\xi)\overline{K(\xi, z)} d\sigma(\xi)$$
$$(f \in H^2(\mathbb{D}), \ z \in \mathbb{D}).$$

The function

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \quad \text{with} \quad K(.,z) \in H^2(\mathbb{D})$$

is called the Bergman kernel for $\mathbb{D}$. Taking $f(\xi) = K(\xi, w)$ for some $w \in \mathbb{D}$ we conclude that the kernel function has the following symmetry property:

$$K(z,w) = \int_{\mathbb{D}} K(\xi,w)\overline{K(\xi,z)}d\sigma(\xi) = \overline{K(w,z)}.$$

For any orthonormal basis $(\varphi_j, j \in \mathbb{N})$ for $H^2(\mathbb{D})$ one has the representation

$$K(\xi,z) = \sum_{j=1}^{\infty} \varphi_j(\xi)\overline{\varphi_j(z)} \quad ((\xi,z) \in \mathbb{D} \times \mathbb{D}) \tag{2.9}$$

with uniform convergence on compact subsets of $\mathbb{D} \times \mathbb{D}$. The set of functions

$$\varphi_j(z) = \sqrt{(j+1)}z^j \quad (z \in \mathbb{D},\ j \in \mathbb{N})$$

form an orthonormal basis in $H^2(\mathbb{D})$, consequently

$$K(\xi,z) = \frac{1}{(1-\overline{z}\xi)^2}. \tag{2.10}$$

The explicit formulae for the kernel function shows that

$$f(z) = \int_{\mathbb{D}} f(\xi)\frac{1}{(1-\bar{\xi}z)^2}d\sigma(\xi) \quad (z \in \mathbb{D},\ f \in H^2(\mathbb{D})). \tag{2.11}$$

Applying this formulae in particular for $f(z) = (1-\xi\overline{z})^{-2}$ for fixed z in the disc we obtain that

$$\|K(z,.)\|_2^2 = \int_{\mathbb{D}} \frac{d\sigma(z)}{|1-\bar{\xi}z|^4} = \frac{1}{(1-|z|^2)^2} = K(z,z) > 0.$$

Since $H^2(\mathbb{D})$ is closed subspace of $L^2_\sigma(\mathbb{D})$ there is an orthogonal projection operator

$$P : L^2_\sigma(\mathbb{D}) \to H^2(\mathbb{D}).$$

The operator P is bounded, Hermitian, of the norm 1 and satisfies $Pf = f$ for $f \in H^2(\mathbb{D})$. It is called *Bergman projection.* Thus the Bergman projection can be expressed by integration with respect to the Bergman kernel in the following way:

$$(Pf)(z) = \langle f, K(.,z)\rangle = \int_{\mathbb{D}} f(\xi)\frac{1}{(1-\bar{\xi}z)^2}d\sigma(\xi) \quad (f \in L^2_\sigma(\mathbb{D}),\ z \in \mathbb{D}).$$

The projection operator is well-defined linear operator on $L^1_\sigma(\mathbb{D})$ mapping each $f \in L^1(\mathbb{D})$ to a function analytic in $\mathbb{D}$ and mapping each

$$f \in H^1(\mathbb{D}) = \left\{ f \in \mathcal{A}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|\, d\sigma(z) < \infty \right\}$$

to itself. It can be proved that for $1 < p < \infty$

$$P : L^p_\sigma(\mathbb{D}) \to H^p(\mathbb{D})$$

is also a bounded map, where

$$H^p(\mathbb{D}) = \left\{ f \in \mathcal{A}(\mathbb{D}) : \left(\int_{\mathbb{D}} |f(z)|^p \, d\sigma(z) \right)^{1/p} < \infty \right\}.$$

3. New results

3.1. The representation of the Blaschke group on $H^2(\mathbb{D})$. Let consider the following set of functions

$$F_a(z) := \frac{\sqrt{\epsilon(1 - |b|^2)}}{1 - \bar{b}z} \quad (a = (b, \epsilon) \in \mathbb{B},\ z \in \overline{\mathbb{D}}). \tag{3.1}$$

F_a induce a unitary representation of Blaschke group on the space $H^2(\mathbb{D})$. Namely, let define

$$U_a f := [F_{a^{-1}}]^2 (f \circ B_a^{-1}) \quad (a \in \mathbb{B}, f \in H^2(\mathbb{D})). \tag{3.2}$$

It can be proved the following theorem:

Theorem 3. *$U_a(a \in \mathbb{B})$ is an unitary representation of the group $\mathbb{B}$ on the Hilbert space $H^2(\mathbb{D})$.*

3.2. The properties of the voice transform induced by the representation U_a. In what follows we will compute the matrix elements of the representation and using the results obtained connected to this matrix elements we will give a proof for the analogue of Plancherer formula. We will prove that the representation U_a is irreducible.

The representation has the following form

$$(U_{a^{-1}} f)(z) := e^{i\psi} \frac{1 - |b|^2}{(1 - \bar{b}z)^2} f\left(e^{i\psi} \frac{z - b}{1 - \bar{b}z} \right) \quad (a = (b, e^{i\psi}) \in \mathbb{B}) \tag{3.3}$$

and is unitary regarding to the scalar product

$$\langle f, g\rangle := \int_{\mathbb{D}} f(z)\overline{g(z)}\, d\sigma(z). \tag{3.4}$$

The voice transform induced by the representation U_a is defined by

$$(V_\rho f)(a^{-1}) := \langle f, U_{a^{-1}}\rho\rangle \quad (a = (b, e^{i\psi}) = (re^{i\varphi}, e^{i\psi}) \in \mathbb{B}). \tag{3.5}$$

The matrix elements of the representation in the base $h_n(z) := z^n$ ($n \in \mathbb{N}$, $z \in \mathbb{D}$) are of the form

$$\begin{aligned} v_{mn}(a^{-1}) := \langle h_m, U_{a^{-1}} h_n\rangle = e^{-i(n+1)\psi}(1-|b|^2) \\ \times \int_{\mathbb{D}} \overline{\left[\frac{(z-b)^n}{(1-\bar{b}z)^{n+2}}\right]} z^m\, d\sigma(z). \end{aligned} \tag{3.6}$$

Theorem 4. *The matrix elements of the representation* (3.3) *have the following form*

$$v_{mn}(a^{-1}) = (1-r^2)e^{-i(n+1)\psi}e^{-i(n-m)\varphi}\alpha_{mn}(r) \quad (m, n \in \mathbb{N}), \tag{3.7}$$

where $\alpha_{mn}(r)$ can be expressed with the Jacobi polynomials in the following way

$$\alpha_{mn}(r) := \frac{r^{n-m}}{(m+1)(n+1)!}\left[(1-u)^n u^{m+1}\right]^{(n+1)}_{u=r^2} \tag{3.8}$$

and $\alpha_{mn}(r)$ satisfy the following relations

$$\alpha_{mn}(r) = (-1)^{m+n}\alpha_{nm}(r), \tag{3.9}$$

$$\int_0^1 r|\alpha_{mn}(r)|^2\, dr = \frac{1}{2(n+1)(m+1)} \quad (m, n \in \mathbb{N}). \tag{3.10}$$

It is known that the matrix elements of the representations satisfy the following so called addition formula:

$$v_{mn}(a_1 \circ a_2) = \sum_k v_{mk}(a_1)v_{kn}(a_2).$$

From this relation we obtain the following addition formulae:

$$\begin{aligned} (1-r^2)e^{i(n+1)\psi}e^{i(n-m)\varphi}\alpha_{mn}(r) \\ = \sum_k (1-r_1^2)e^{i(k+1)\psi_1}e^{i(k-m)\varphi_1}\alpha_{mk}(r_1)(1-r_2^2)e^{i(n+1)\psi_2}e^{i(n-k)\varphi_2}\alpha_{kn}(r_2) \end{aligned} \tag{3.11}$$

where $a_j := (r_j e^{i\varphi_j}, e^{i\psi_j})$, $j \in \{1, 2\}$ and $a := (re^{i\varphi}, e^{i\psi}) = a_1 \circ a_2$.

Theorem 5. *The representation U_a is irreducible on the space $H^2(\mathbb{D})$.*

Theorem 6. *The voice transform induced by U_a satisfies*

$$[V_{\rho_1} f, V_{\rho_2} g] = 4\pi\langle\rho_1, \rho_2\rangle \ \langle f, g\rangle \quad (f, g, \rho_1, \rho_2 \in H^2(\mathbb{D})) \tag{3.12}$$

where

$$[F, G] := \int_{\mathbb{B}} F(a)\overline{G(a)}\, dm(a),$$

m is the Haar measure of the group $\mathbb{B}$ and every element $\rho \in H^2(\mathbb{D})$ is admissible.

Remark 1. The space $\mathbb{B} = \mathbb{D}\times\mathbb{T}$ is $G/K\times K$, where $G = SU(1,1)$. Therefore, the representation constructed in this paper is related to one of $SU(1,1)$. Actually, it corresponds to one of the holomorphic discrete series of $SU(1,1)$. Hence, by noting another discrete series and using the weighted Bergman space, we can construct another voice transform similarly.

Remark 2. Since G/K is KA as a space, according to a Cartan decomposition of G, as related topic is treated by T. Kawazoe in [8].

3.3. Construction of orthogonal rational wavelets. Let consider the shift operator

$$(S\varphi)(z) = z\varphi(z), \quad (\varphi \in H^2(\mathbb{D})). \tag{3.13}$$

Denote by

$$\varphi_{a,m}(z) := \sqrt{(m+1)}(U_{a^{-1}} S^m \varphi)(z)$$

$$a \in \mathbb{B},\ m \in \mathbb{N},\ \varphi \in H^2(\mathbb{D}). \tag{3.14}$$

If we consider as *mother wavelet* $\varphi = 1 \in H^2(\mathbb{D})$ then the corresponding rational wavelets are :

$$\varphi_{a,m}(z) = \sqrt{(m+1)}\epsilon\frac{1-|b|^2}{(1-\bar{b}z)^2}\left(\frac{\epsilon(z-b)}{1-\bar{b}z}\right)^m. \tag{3.15}$$

They form an orthonormal system in $H^2(\mathbb{D})$, for all a $\in \mathbb{B}$. Indeed taking into account the unitarity of the representation U_a

$$\begin{aligned}\langle\varphi_{a,m}(z), \varphi_{a,n}(z)\rangle &= \langle\sqrt{(m+1)}U_{a^{-1}}z^m, \sqrt{(n+1)}U_{a^{-1}}z^n\rangle \\ &= \sqrt{(m+1)}\sqrt{(n+1)}\langle z^m, z^n\rangle = \delta_{mn}, \quad \forall a \in \mathbb{B}, \quad n, m \in \mathbb{N}.\end{aligned} \tag{3.16}$$

We observe that if $a = e = (0,1) \in \mathbb{B}$ then

$$\varphi_{e,m}(z) = \sqrt{(m+1)}z^m =: \varphi_m(z) \quad e = (0,1) \in \mathbb{B}.$$

Theorem 7. *For all* $z \in \mathbb{D}$ *and* $a \in \mathbb{B}$ *the Bergman projection operator* $P : L^2(\mathbb{D}) \to H^2(\mathbb{D})$ *can be written in the following way*

$$Pf(z) = \sum_{m=0}^{\infty} V_{\varphi_m} f(a^{-1}) U_{a^{-1}} \varphi_m(z) \quad (a \in \mathbb{B}). \tag{3.17}$$

Consequence 1. *Every* f *from* $H^2(\mathbb{D})$ *can be represented as*

$$f(z) = \sum_{m=0}^{\infty} V_{\varphi_m} f(a^{-1}) U_{a^{-1}} \varphi_m(z) \quad (a \in \mathbb{B}).$$

Consequence 2. *For every* $a \in \mathbb{B}$ *the collection*

$$\varphi_{a,m} := \sqrt{(m+1)} U_{a^{-1}} \varphi_m \quad (m \in \mathbb{N}) \tag{3.18}$$

is an orthonormal basis in $H^2(\mathbb{D})$.

From Theorem 6 it follows that the coefficients $V_{\varphi_m} f(a^{-1})$ satisfy the following relations:

$$\int_{\mathbb{B}} V_{\varphi_m} f(a^{-1}) \overline{V_{\varphi_n} g(a^{-1})} dm(a) = 4\pi \langle \varphi_m, \varphi_n \rangle \langle f, g \rangle = 4\pi \delta_{mn} \langle f, g \rangle \quad (f, g \in H^2(\mathbb{D})).$$

If $e = (0,1) \in \mathbb{B}$ is the unit element of the group $\mathbb{B}$, and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$, then

$$V_{\varphi_m} f(e) = \frac{a_m}{\sqrt{(m+1)}}.$$

We observe that the matrix elements of the representation can be written as

$$v_{mn}(a^{-1}) = \frac{1}{\sqrt{(n+1)(m+1)}} V_{\varphi_n} \varphi_m(a^{-1})$$

and due to (3.12) we obtain that they form a double indexed orthogonal system in $L^2(\mathbb{B})$, namely:

$$\int_{\mathbb{B}} v_{mk}(a) \overline{v_{n\ell}(a)} dm(a) = \frac{4\pi}{(n+1)(k+1)} \delta_{mn} \delta_{k\ell}.$$

3.4. The voice transform of the kernel function. Let

$$K_z(\xi) := K(\xi, z) = \frac{1}{(1-\xi\overline{z})^2}, \quad a = (b, e^{i\psi}) \in \mathbb{B}.$$

Then

$$(U_{a^{-1}}K_z)(\xi) = e^{i\psi}\frac{1-|b|^2}{(1-\overline{b}\xi)^2}\frac{1}{\left(1-e^{i\psi}\frac{\xi-b}{1-\overline{b}\xi}\overline{z}\right)^2}.$$

Using the fact that the kernel function belongs to the Bergman space we can apply the reproducing formulae (2.11) when we compute the voice transform of the kernel function, and we will obtain the following:

$$\begin{aligned}(V_{K_z}K_z)(a^{-1}) &= \langle K_z, (U_{a^{-1}}K_z)\rangle = \frac{1}{\pi}\int_{\mathbb{D}} K_z(\xi)\overline{(U_{a^{-1}}K_z)(\xi)}d\xi_1 d\xi_2 \\ &= \overline{\frac{1}{\pi}\int_{\mathbb{D}}(U_{a^{-1}}K_z)(\xi)\frac{1}{(1-z\overline{\xi})^2}d\xi_1 d\xi_2} \\ &= \overline{(U_{a^{-1}}K_z)(z)} = e^{-i\psi}\frac{1-|b|^2}{(1-b\overline{z})^2}\frac{1}{\left(1-e^{-i\psi}\frac{\overline{z}-\overline{b}}{1-b\overline{z}}z\right)^2}.\end{aligned}$$

From this it follows that

$$(V_{K_z}K_z)(a) = e^{i\psi}\frac{1-|b|^2}{(1+e^{i\psi}b\overline{z})^2}\frac{1}{\left(1-e^{i\psi}\frac{\overline{z}+\overline{b}e^{-i\psi}}{1+be^{i\psi}\overline{z}}z\right)^2} = (U_aK_z)(z).$$

Theorem 8. *For every $\rho \in H^\infty(\mathbb{D})$*

$$\int_{\mathbb{B}} |V_\rho\rho(a)|^2 dm(a) < \infty,$$

which means that every element ρ from $H^\infty(\mathbb{D})$ is admissible.

4. Proofs

Proof of Theorem 3. The set of functions $(F_a, a \in \mathbb{B})$ defined by (3.1) satisfies the following relation

$$(F_{a_1} \circ B_{a_2}) \cdot F_{a_2} = F_{a_1 \circ a_2} \quad (a_1, a_2 \in \mathbb{B}). \tag{4.1}$$

To prove this we will use the identity

$$|1+b_1\overline{b}_2\overline{\epsilon}_2|\sqrt{1-|b|^2} = \sqrt{|1+b_1\overline{b}_2\overline{\epsilon}_2|^2 - |b_1\overline{\epsilon}_2+b_2|^2} = \sqrt{(1-|b_1|^2)(1-|b_2|^2)}.$$

Using this we obtain that

$$\begin{aligned} F_{a_1}(B_{a_2}(z)) \cdot F_{a_2}(z) &= \sqrt{\epsilon_1\epsilon_2(1-|b_1|^2)(1-|b_2|^2)}\frac{1}{1-\overline{b}_1\epsilon_2\frac{z-b_2}{1-\overline{b}_2 z}}\frac{1}{1-\overline{b}_2 z} \\ &= \frac{\sqrt{\epsilon_1\epsilon_2(1-|b|^2)}|1+b_1\overline{b}_2\overline{\epsilon}_2|}{1+\overline{b}_1 b_2\epsilon_2}\frac{1}{1-\overline{b}z} \\ &= \sqrt{\epsilon(1-|b|^2)}\frac{1}{1-\overline{b}z} = F_{a_1\circ a_2}(z), \end{aligned}$$

where $a = a_1 \circ a_2 = (b, \epsilon)$, is given by (2.3) and $z \in \mathbb{D}$. From this it follows that

$$\begin{aligned} U_{a_1}(U_{a_2}f) &= U_{a_1}([F_{a_2^{-1}}]^2 \cdot f \circ B_{a_2}^{-1}) = [F_{a_1^{-1}}]^2 \cdot [F_{a_2^{-1}} \circ B_{a_1^{-1}}]^2 \cdot f \circ B_{a_2}^{-1} \circ B_{a_1}^{-1} \\ &= [F_{(a_1\circ a_2)^{-1}}]^2 \cdot (f \circ B_{a_1\circ a_2}^{-1}) = U_{a_1\circ a_2} f. \end{aligned}$$

In what follows we will show that the restriction of linear application U_a on the Hilbert space $H^2(\mathbb{D})$ is unitary with respect to the inner product defined by (2.8) which implies that: if $f \in H^2(\mathbb{D})$, then $U_a f \in H^2(\mathbb{D})$. To prove this we will use the following result: if $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic bijection, then the determinant of the corresponding Jacobi matrix is equal to $|\varphi'(z)|^2$. Making the change of variable $w = \varphi(z)$ we get the following integral transformation formulae

$$\int_{\mathbb{D}} F(w)\, d\sigma(w) = \int_{\mathbb{D}} F(\varphi(z))|\varphi'(z)|^2\, d\sigma(z).$$

In the special case $\varphi = B_a$ we get

$$B_a'(z) = \epsilon\frac{1-|b|^2}{(1-\overline{b}z)^2} = [F_a(z)]^2 \quad (a = (b, \epsilon) \in \mathbb{B}).$$

We want to show that U_a is a unitary representation, namely

$$\langle U_a f, U_a g\rangle = \langle f, g\rangle \quad (f, g \in H^2(\mathbb{D})).$$

Indeed, making the change of variable $w = B_a(z)$ in the integral on the left hand side we obtain that

$$\begin{aligned} \langle U_a f, U_a g\rangle &= \int_{\mathbb{D}} |F_{a^{-1}}(w)|^4 f(B_a^{-1}(w))\overline{g(B_a^{-1}(w))}\, d\sigma(w) \\ &= \int_{\mathbb{D}} |F_a(z)|^4 |F_{a^{-1}}(B_a(z))|^4 f(z)\overline{g(z)}\, d\sigma(z). \end{aligned}$$

Using (4.1) we obtain that

$$\langle U_a f, U_a g\rangle = \langle f, g\rangle.$$

In a similar way we obtain that

$$\int_{\mathbb{D}} f(w)\frac{d\sigma(w)}{(1-|w|^2)^2} = \int_{\mathbb{D}} f(B_a(z))\frac{d\sigma(z)}{(1-|z|^2)^2} \quad (a \in \mathbb{B}),$$

which means that the integral is invariant under the transformation B_a. Indeed, if we make the change of variables $w = B_a(z)$ in the integral on the left hand side we obtain that

$$\int_{\mathbb{D}} f(w)\frac{d\sigma(w)}{(1-|w|^2)^2} = \int_{\mathbb{D}} f(B_a(z))\frac{|F_a(z)|^4}{(1-|B_a(z)|^2)^2}\, d\sigma(z) = \int_{\mathbb{D}} f(B_a(z))\frac{d\sigma(z)}{(1-|z|^2)^2}.$$

Proof of Theorem 4. Let denote by $z = \rho e^{it}$, $b = re^{i\varphi}$, $(t, \varphi \in \mathbb{I})$. To compute the integral which appear in (3.6) first we apply the polar transformation and then we make the following change of variable, we replace t by $t+\varphi$ and we obtain that

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{D}} \overline{\Big(\frac{(z-b)^n}{(1-\bar{b}z)^{n+2}}\Big)} z^m \, dxdy &= \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi} \frac{(\rho e^{-it} - re^{-i\varphi})^n}{(1-r\rho e^{-i(t-\varphi)})^{n+2}}\rho^{m+1}e^{imt}\, d\rho dt \\
&= e^{-in\varphi}\frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi} \frac{(\rho e^{-i(t-\varphi)} - r)^n}{(1-r\rho e^{-i(t-\varphi)})^{n+2}}\rho^{m+1}e^{imt}\, d\rho dt \\
&= e^{-i(n-m)\varphi}\frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi} \frac{(\rho e^{-it} - r)^n}{(1-\rho re^{-it})^{n+2}}\rho^{m+1}e^{imt}\, d\rho dt \\
&= e^{-i(n-m)\varphi}\frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi} \frac{(\rho e^{it} - r)^n}{(1-\rho re^{it})^{n+2}}\rho^{m+1}e^{-imt}\, d\rho\, dt.
\end{aligned}$$

Let denote by

$$\alpha_{mn}(r) := \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi} \frac{(\rho e^{it} - r)^n}{(1-r\rho e^{it})^{n+2}}\rho^{m+1}e^{-imt}\, d\rho dt$$

then due to (3.6) $v_{mn}(a^{-1})$ can be written as

$$v_{mn}(a^{-1}) = (1-r^2)e^{-i(n+1)\psi}e^{-i(n-m)\varphi}\alpha_{mn}(r) \quad (m, n \in \mathbb{N}).$$

We will show that $\alpha_{mn}(r)$ can be expressed by the Jacobi polynomials in the following way:

$$\alpha_{mn}(r) := \frac{r^{n-m}}{(m+1)(n+1)!}\left[(1-u)^n u^{m+1}\right]^{(n+1)}_{u=r^2}.$$

Indeed, if we substitute e^{-it} by ξ, on the base of Cauchy integral formulae we obtain that

$$\begin{aligned}\alpha_{mn}(r) &:= \int_0^1 \rho^{m+1}\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{(\rho - re^{-it})^n}{(e^{-it}-r\rho)^n}\frac{e^{-2it}}{(e^{-it}-r\rho)^2}e^{-imt}\,d\rho dt \\ &= 2\int_0^1 \rho^{m+1}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(\rho - r\zeta)^n}{(\zeta - r\rho)^{n+2}}\zeta^{m+1}\,d\zeta d\rho \\ &= 2\int_0^1 \frac{\rho^{m+1}}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}}\left[(\rho - rz)^n z^{m+1}\right]_{z=r\rho}\,d\rho \\ &= 2\int_0^1 \frac{\rho^{m+1+n}\left(\frac{r}{\rho}\right)^{-m-1}}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}}\left[\left(1-\frac{r}{\rho}z\right)^n\left(\frac{r}{\rho}z\right)^{m+1}\right]_{z=r\rho}\,d\rho.\end{aligned}$$

From this, using the notation

$$\frac{d^{n+1}}{du^{n+1}}\left[(1-u)^n u^{m+1}\right] = \left[(1-u)^n u^{m+1}\right]^{(n+1)}$$

it follows that

$$\begin{aligned}\alpha_{mn}(r) &= 2\int_0^1 \rho^{2m+1}r^{n-m}\frac{1}{(n+1)!}\left[(1-u)^n u^{m+1}\right]^{(n+1)}_{u=r^2}\,d\rho \\ &= \frac{r^{n-m}}{(m+1)(n+1)!}\left[(1-u)^n u^{m+1}\right]^{(n+1)}_{u=r^2}.\end{aligned}$$

To show i) let use the following relation

$$v_{mn}(a^{-1}) = \langle h_m, U_{a^{-1}}h_n\rangle = \overline{\langle U_{a^{-1}}h_n, h_m\rangle} = \overline{\langle h_n, U_a h_m\rangle} = \overline{v_{nm}(a)} \quad (m,n\in\mathbb{N}).$$

Takeing into account that $a^{-1} = (be^{i(\psi+\pi)}, e^{-i\psi})$, then for $a = (r,1)$, namely when $\varphi = \psi = 0$ we obtain that

$$(1-r^2)\alpha_{mn}(r) = v_{mn}(a) = \overline{v_{nm}(a^{-1})} = (-1)^{m+n}(1-r^2)\alpha_{nm}(r)$$

which implies that i) is true.

To prove ii) in the integral on the left side let make the change of variable $x = r^2$ and let use the notation $m = n+\ell$ and we obtain that

$$I = \int_0^1 r|\alpha_{mn}(r)|^2 dr = \frac{1}{2((m+1)(n+1)!)^2}\int_0^1 x^{-\ell}\left(\frac{d^{n+1}}{dx^{n+1}}[(1-x)^n x^{n+\ell+1}]\right)^2 dx.$$

We observe that

$$x^{-\ell}[(1-x)^n x^{n+\ell+1}]^{(n+1)} = (-1)^n(n+1)(n+\ell+1)n!x^n + (1-x)Q_n(x) =: P_n(x)$$

where and P_n is polynomials of degree n and Q_n is a polynomial of degree $n-1$. Using the partial integration we obtain that

$$\int_0^1 x^{-\ell}\big([(1-x)^n x^{n+\ell+1}]^{(n+1)}\big)^2\,dx = \int_0^1 [(1-x)^n x^{n+\ell+1}]^{(n+1)} P_n(x)\,dx$$
$$= [(1-x)^n x^{n+\ell+1}]^{(n)} P_n(x)|_{x=0}^1 - \int_0^1 [(1-x)^n x^{n+\ell+1}]^{(n)} P_n'(x)\,dx$$
$$= (n+1)!\ n!\ (n+\ell+1) - \int_0^1 [(1-x)^n x^{n+\ell+1}]^{(n)} P_n'(x)\,dx.$$

Taking into account that 0 and 1 are roots of the polynomial $u(x) := [(1-x)^n x^{n+\ell+1}]^{(n-1)}$ on the base of partial integration we obtain that this last integral is equal by 0. Indeed,

$$\int_0^1 [(1-x)^n x^{n+\ell+1}]^{(n)} P_n'(x)\,dx = u(x)P_n'(x)|_{x=0}^1 - \int_0^1 u(x)P_n''(x)\,dx$$
$$= -\int_0^1 u(x)P_n''(x)\,dx.$$

Repeating this procedure $n-1$ times we obtain that the integral is 0. Consequently

$$I = \frac{1}{2(n+1)(m+1)}.$$

Proof of Theorem 5. Let consider the power functions $h_n(z) := z^n$ $(z \in \mathbb{C},\ n \in \mathbb{N})$. First we take the images of these functions under the representation

$$b \to (U_a h_n)(z) := \epsilon^{n+1}(1-|b|^2)\frac{(z-b)^n}{(1-\bar{b}z)^{n+2}} \quad (b = b_1 + ib_2 \in \mathbb{D}),$$

and we compute the partial derivatives regarding to the variables b_1 and b_2, and we take the values of this partial derivatives in $e = (0,1) \in \mathbb{B}$. An easy computation gives that

$$\frac{\partial}{\partial b_1} U_e h_n = -nh_{n-1} + (n+2)h_{n+1}$$
$$\frac{\partial}{\partial b_2} U_e h_n = -inh_{n-1} - i(n+2)h_{n+1}.$$

From this we obtain that

$$\left(\frac{\partial}{\partial b_1} + i\frac{\partial}{\partial b_2}\right) U_e h_n = 2(n+2)h_{n+1}, \quad \left(\frac{\partial}{\partial b_1} - i\frac{\partial}{\partial b_2}\right) U_e h_n = 2nh_{n-1}.$$

From the definition of U_a follows that for the one parameter subgroup $\alpha(t) = (0, e^{it})$ $(t \in \mathbb{R})$ of $\mathbb{B}$

$$U_{\alpha(t)} p_n = e^{i(n+1)t} h_n \quad (n \in \mathbb{N},\ t \in \mathbb{R}),$$

which means that the subspace wrapped by the function h_n is an invariant one dimensional subspace of the restricted representation on the commutative subgroup mentioned bellow. It is known that any invariant subspace of the representation can be written as the direct sum of this kind of subspaces. From these it follows that the representation U_a $(a \in \mathbb{B})$ is irreducible. Indeed let be H at least one dimensional closed invariant subspace of the representation. This is also invariant subspace of the restricted representation U_a to the one parameter subgroup, consequently it contains one of the power functions h_n . On the base of the definition of invariant subspaces it is evident that

$$\frac{1}{b_1}(U_{(b_1,1)} h_n - U_{(0,1)} h_n) \in H, \quad \frac{1}{b_2}(U_{(ib_2,1)} p_n - U_{(0,1)} h_n) \in H \quad (b_1, b_2 \in \mathbb{R}).$$

From the closeness of the subspace it follows that the limit of this expressions when $b_1 \to 0, b_2 \to 0$ is also in H, namely

$$\left(\frac{\partial}{\partial b_1} + i\frac{\partial}{\partial b_2}\right) U_e h_n = 2(n+2) h_{n+1} \in H, \quad \left(\frac{\partial}{\partial b_1} - i\frac{\partial}{\partial b_2}\right) U_e h_n = 2n h_{n-1} \in H.$$

From this evidently follows that $h_k \in H$, if $k \geq n$, and $h_k \in H$, if $k < n$ and $k \geq 0$. This implies that $H = H^2(\mathbb{D})$, and the irreducibility of the representation U_a $(a \in \mathbb{B})$ is proved.

Proof of Theorem 6. To prove Theorem 6 let consider the function $f,\ \rho \in H^2(\mathbb{D})$, then they can be represented in the base $h_n(z) = z^n$ $(z \in \mathbb{D},\ n \in \mathbb{N})$ in the following way

$$f = \sum_{n=0}^{\infty} a_n h_n, \quad \rho = \sum_{n=0}^{\infty} b_n h_n,$$

where the series are convergent in norm therefore are uniformly convergent on every compact subset of $\mathbb{D}$. Consequently

$$\langle f, f\rangle = \frac{1}{2}\sum_{n=0} \frac{|a_n|^2}{n+1} < \infty, \quad \langle \rho, \rho\rangle = \frac{1}{2}\sum_{n=0} \frac{|b_n|^2}{n+1} < \infty.$$

From the definition of the voice transform and from the continuity of the scalar product it follows that

$$(V_\rho f)(a) := \langle f, U_a \rho\rangle = \sum_{m=0}^{\infty} \overline{b_m} \sum_{n=0}^{\infty} a_n \langle h_n, U_a h_m\rangle = \sum_{m=0}^{\infty} \overline{b_m} \sum_{n=0}^{\infty} a_n v_{nm}(a).$$

From Theorem 4 it follows that

$$v_{nm}(a) = (1 - r^2)\alpha_{nm}(r)u_{nm}(\psi, \varphi),$$

where

$$u_{nm}(\psi, \varphi) := e^{i(n+1)\psi} e^{i(n-m)\varphi} \quad (m, n \in \mathbb{N})$$

is an orthonormal system in the square $\mathbb{I} \times \mathbb{I}$. Namely

$$\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} u_{mn}(\psi, \varphi) \overline{u_{k\ell}(\psi, \varphi)}\, d\psi d\varphi = \delta_{mk}\delta_{n\ell} \quad (k, \ell, m, n \in \mathbb{N}).$$

Using the definition of the Haar measure of the Blaschke group we obtain that

$$\begin{aligned}
\frac{1}{2\pi}[(V_\rho f), (V_\rho f)] &= \frac{1}{2\pi} \int_{\mathbb{B}} |(V_\rho f)(a)|^2 \, dm(a) \\
&= \frac{1}{4\pi^2} \int_0^1 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{r|V_\rho(re^{i\varphi}, e^{i\psi})|^2}{(1 - r^2)^2} \, d\psi d\varphi dr \\
&= \sum_{m,n=0}^{\infty} |a_n|^2 |b_m|^2 \int_0^1 r|\alpha_{mn}(r)|^2 \, dr \\
&= \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \right) \left(\sum_{m=0}^{\infty} \frac{|b_m|^2}{m+1} \right) = 2\langle f, f\rangle \langle \rho, \rho\rangle.
\end{aligned}$$

Thus the following analogue of the Plancherel formulae hold:

$$[V_\rho f, V_\rho f] = 4\pi \langle f, f\rangle \langle \rho, \rho\rangle \quad (f,\, \rho \in H^2(\mathbb{D})).$$

From this it follows that for every $\rho \in H^2(\mathbb{D})$

$$[V_\rho \rho, V_\rho \rho] = 4\pi \|\rho\|_{H^2}^4 < \infty,$$

which means that every element $\rho \in H^2(D)$ is admissible. Let suppose that $\rho \in H^2(\mathbb{D}) \setminus \{0\}$, then $V_\rho f = 0$ implies that $\langle f, f\rangle = 0$ consequently $f = 0$. This means that the voice transform is injective. The relation (3.12) implies that for the positive bilinear map from Theorem 2 $B(\rho, \rho) = 4\pi\langle \rho, \rho\rangle$ is satisfied. This relation and the unimodularity of the Blaschke group implies that $B(\rho_1, \rho_2) = 4\pi\langle \rho_1, \rho_2\rangle$. Due to Theorem 2 we obtain that:

$$[V_{\rho_1} f, V_{\rho_2} g] = 4\pi\langle f, g\rangle \langle \rho_1, \rho_2\rangle, \quad (f, g, \rho_1, \rho_2 \in H^2(\mathbb{D})).$$

Proof of Theorem 7. Let consider the following infinite series

$$\sum_{m=0}^{\infty}(m+1)\overline{(U_{a^{-1}}\varphi_m)(y)}(U_{a^{-1}}\varphi_m)(z).$$

Since

$$|\overline{(U_{a^{-1}}\varphi_m)(y)}(U_{a^{-1}}\varphi_m)(z)| \leqq \frac{(1-r^2)^2}{(1-r_1r)^2(1-r_2r)^2}\left(\frac{r+r_1}{1+r_1r}\right)^m$$

$$(z=r_1e^{it},\ y=r_2e^{it}\in\mathbb{D},\ a=(re^{i\varphi},e^{i\psi})\in\mathbb{B}),$$

and $\frac{r+r_1}{1+r_1r}<1$, we obtain that for a fix $z\in\mathbb{D}$ and for a fix $a=(re^{i\varphi},e^{i\psi})\in\mathbb{B}$ the series converges absolutely and uniformly in $y\in\mathbb{D}$. This permits the interchange of summation and integration in the following expression

$$\begin{aligned}\sum_{m=0}^{\infty} V_{\rho_m}f(a^{-1})(U_{a^{-1}}\varphi_m)(z) &= \sum_{m=0}^{\infty}\langle f, U_{a^{-1}}\varphi_m\rangle(U_{a^{-1}}\varphi_m)(z)\\ &= \frac{1}{\pi}\int_{\mathbb{D}} f(y)\left(\sum_{m=0}^{\infty}(m+1)\overline{(U_{a^{-1}}\varphi_m)(y)}(U_{a^{-1}}\varphi_m)(z)\right)dy_1dy_2.\end{aligned}$$

We observe that

$$\begin{aligned}&\sum_{m=0}^{\infty}(m+1)\overline{(U_{a^{-1}}\varphi_m)(y)}(U_{a^{-1}}\varphi_m)(z)\\ &\quad=\frac{(1-|b|^2)^2}{(1-b\overline{y})^2(1-\overline{b}z)^2}\sum_{m=0}^{\infty}(m+1)\overline{\left(\frac{\epsilon(y-b)}{1-\overline{b}y}\right)^m}\left(\frac{\epsilon(z-b)}{1-\overline{b}z}\right)^m\\ &\quad=\frac{(1-|b|^2)^2}{(1-b\overline{y})^2(1-\overline{b}z)^2}\frac{1}{\left(1-\overline{\left(\frac{y-b}{1-\overline{b}y}\right)}\frac{z-b}{1-\overline{b}z}\right)^2}=\frac{1}{(1-\overline{y}z)^2}=K(y,z).\end{aligned}$$

Consequently

$$\sum_{m=0}^{\infty}V_{\varphi_m}f(a^{-1})(U_{a^{-1}}\varphi_m)(z)=\frac{1}{\pi}\int_{\mathbb{D}}f(y)\frac{1}{(1-\overline{y}z)^2}dy_1dy_2=(Pf)(z).$$

Proof of Theorem 8. Using the definition of the voice transform and the unitarity of the representation U we obtain that

$$|(V_\rho\rho)(a)|=|\langle U_{a^{-1}}\rho,\rho\rangle|\le\frac{1}{\pi}(1-|b|^2)\|\rho\|_{H^\infty}^2\int_{\mathbb{D}}\frac{dxdy}{|1-\overline{b}z|^2}.$$

Using the Parseval-formulae from

$$\frac{1}{1-\bar{b}re^{i\varphi}} = \sum_{n=0}^{\infty}(\bar{b}r)^n e^{in\varphi} \quad \bigl(b\in\mathbb{D},\ z=re^{i\varphi},\ \varphi\in[-\pi,\pi],\ 0\le r\le 1\bigr),$$

we obtain that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\varphi}{|1-\bar{b}re^{i\varphi}|^2} = \sum_{n=0}^{\infty}|br|^{2n},$$

$$\frac{1}{2\pi}\int_{\mathbb{D}}\frac{dxdy}{|1-\bar{b}z|^2} = \sum_{n=0}^{\infty}|b|^{2n}\int_0^1 r^{2n+1}\,dr = \frac{1}{2}\sum_{n=0}^{\infty}\frac{|b|^{2n}}{n+1} =: G(|b|).$$

Consequently

$$\int_{\mathbb{B}}|V_\rho\rho(a)|^2 dm(a) \le \|\rho\|_{H^\infty}^4\frac{4}{2\pi}\int_{\mathbb{I}}\int_{\mathbb{D}}(1-|b|^2)^2G^2(|b|)\frac{1}{(1-|b|^2)^2}\,db_1db_2dt$$
$$= 8\pi\|\rho\|_{H^\infty}^4\int_0^1 G^2(r)r\,dr.$$

We will show that

$$c^2 := \int_0^1 G^2(r)r\,dr < \infty.$$

Indeed, using

$$G^2(r) = \frac{1}{4}\sum_{n=0}^{\infty}r^{2n}\sum_{k+\ell=n}\frac{1}{(k+1)(\ell+1)},$$

we obtain that

$$\int_0^1 G^2(r)rdr = \frac{1}{8}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k+\ell=n}\frac{1}{(k+1)(\ell+1)} \le \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{1}{n/2+1}\sum_{k=0}^{n/2}\frac{1}{k+1}$$
$$\le \frac{1}{2}\sum_{n=0}^{\infty}\frac{1+\log n}{(n+1)^2} < \infty,$$

References

[1] C. K. Chui, An Introduction to Wavelets publ Academic Press, New York, London, Toronto, Sydney, San Francisco, 1992.

[2] N. F. Dudley Ward and J. R. Partington, Robust identification in the disc algebra using rational wavelets and orthonormal basis functions, *INT. J. Control* **64**, no. 3 (1996), 409–423.

[3] P. Duren and A. Schuster, Bergman Spaces, Mathematical Surveys and Monographs,Vol. 100, *American Mathematical Society, Providence, RI*, 2004.

[4] G. Evangelista and S. Cavaliere, Discrete Frequency Warped Wavelets: Theory and Applications, IEEE Transactions on Signal Processing, vol. 46, 4. April. 1998.

[5] H. Hedenmalm, B. Korenblum and K. Yhu, Theory of Bergman Spaces, Graduate Texts in Mathematics 199, *Springer Verlag, New York*, 2000.

[6] C. E. Heil and D. F. Walnut, Continuous and discrete wavelet transforms, *SIAM Review* **31**, no. 4 (1989), 628–666.

[7] T. J. de Hoog, Rational Orthonormal Basis and Related Transforms in Linear System Modeling, *Netherlands by Ponsen and Looijn b.v., Netherlandes*, 2001.

[8] T. Kawazoe, KA wavelets on semisimple Lie groups and quasi-orthogonality of matrix coefficients, Proceedings of Infinite Dimensional Harmonic Analysis, (H. Heyer, T. Hirai and N. Obata, eds.), *D.+M. Gröbner, Tübingen*, 2000, 238–252.

[9] V. V. Kissil, Wavelets in Applied and Pure Mathematics, `http://maths.leeds.ac.uk/ kisilv/courses/wavelets.htlm`.

[10] M. Pap and F. Schipp, The voice transform on the Blaschke group I., *Pure Math. Appl.* **17**, no. 3–4 (2006), 387–395.

[11] M. Pap and F. Schipp, The voice transform on the Blaschke group II., *Annales Univ. Sci. Budapest., Sect. Comp.* **29** (2008), 157–173.

[12] F. Schipp and W. R. Wade, Transforms on Normed Fields, Leaflets in Mathematics, *Janus Pannonius University Pécs*, 1995.

[13] A. Wawrzyńczyk, Group Representation and Special Functions, D. Reidel Publshing Company, *PWN-Polish Scientific Publishers, Warszawa*, 1984.

MARGIT PAP
UNIVERSITY OF PÉCS,
IFJÚSÁG ÚTJA 6,
7634 PÉCS
HUNGARY

E-mail: papm@ttk.pte.hu

FERENC SCHIPP
ÖTVÖS L. UNIVERSITY, MTA SZTAKI, BUDAPEST
PÁZMÁNY SÉTÁNY I/D
H-1117 BUDAPEST
HUNGARY

E-mail: schipp@ludens.elte.hu

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