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On the stability of the translation equation

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Dedicated to Professor Zoltán Daróczy on his seventieth birthday

Abstract. In this paper the stability of the translation equation, $F(t, F(s,x)) = F(s+t,x)$, where $F : (0,\infty)\times \to I$, and I is a real interval, is investigated.

1. Introduction

The translation equation, i.e. functional equation of the form

$$F(t, F(s,x)) = F(s+t,x), \tag{1.1}$$

can be considered in a very general setting, $t,s \in G$, where G is a grupoid, and $x \in X$, where X is an arbitrary space. This equation is of great importance both in the theory of functional equations and iteration theory (see [4], [5] and references there). If X is a metric space, with metric ρ, there arises a problem of the stability of the functional equation (1.1), that is a question whether for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every function $H : G \times X \to X$ satisfying "approximate translation equation", up to δ, i.e. the inequality

$$\rho(H(t, H(s,x)), H(s+t,x)) < \delta, \tag{1.2}$$

we can find a solution F of (1.1), which is "close" to H, more precisely, such that

$$\rho(F(t,x), H(t,x)) < \varepsilon, \tag{1.3}$$

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for every $x \in X$ and $t \in G$. Actually, this is only one of the possible approaches to the problem of stability of the functional equation, (see [6]), but in this paper we restrict our attention to that definition of stability. Up to now, the problem of stability of the translation equation was considered in few papers.

In [2] W. Jabłoński and L. Reich obtained the stability of (1.1) in rings of formal power series.

A. Mach and Z. Moszner in [3] investigated the stability of the translation equation in two classes, **CB** and **CI**. More precisely, with G a monoid with unit element 0 and X an arbitrary space, let

$$\mathbf{CB} = \{H : G \times X \to X;\ H(\cdot, \alpha) \text{ is a bijection for a certain } \alpha \in X\},$$

$$\mathbf{CI} = \{H : G \times X \to X;\ H(\cdot, \alpha) \text{ is an injection for a certain } \alpha \in X, \text{ and } H(G, \alpha) = H(0, X)\}.$$

Theorem 1.1 ([3])**.** *Let $\rho : X \times X \to \mathbb{R}$ be an arbitrary function.*

1. *For every function $H \in$ **CB** there exist a function $F \in$ **CB** satisfying* (1.1) *such that for every $\varepsilon > 0$ if* (1.2) *with $\delta = \varepsilon$ holds, then* (1.3) *also is true.*

2. *If the function ρ satisfies the triangle inequality in X then for every function $H \in$ **CI** there exists a solution $F \in$ **CI** of* (1.1) *such that for every $\varepsilon > 0$ if* (1.2) *is fulfilled with $\delta = \frac{\varepsilon}{2}$, we have* (1.3)*.*

J. Chudziak considered iteration groups on a real interval and obtained the following result.

Theorem 1.2 ([1])**.** *Let I be a real interval and $\varepsilon > 0$. Assume that $H : \mathbb{R} \times I \to I$ satisfies the inequality*

$$|H(s, H(t, x)) - H(t + s, x)| \le \varepsilon, \quad x \in I,\ s, t \in \mathbb{R},$$

and for some $x_0 \in I$ the function $H(\cdot, x_0)$ is a continuous surjection of $\mathbb{R}$ onto I. Then there exists a homeomorphism $f : \mathbb{R} \to I$ (and continuous iteration group $F(t, x) = f(t + f^{-1}(x))$) such that

$$|H(t, x) - f(t + f^{-1}(x))| \le 9\varepsilon, \quad x \in I,\ t \in \mathbb{R}.$$

In this paper we restrict our attention to continuous iteration semigroups, that is continuous solutions $F : (0, \infty) \times I \to I$ of (1.1), where I, as it was in paper [1], is a real interval. Here without any assumption about surjectivity or injectivity, we get the approximation of H, by the exact solution F of (1.1), but not on the whole interval I (see Theorem 3.2). However, if H satisfies some additional conditions, which are, after all, satisfied by any $F \in \mathcal{F}$ (see Remarks 2.1, 2.2, 2.3), we can find a continuous iteration semigroup F such that $|F(t,x) - H(t,x)| < \varepsilon$, for every $x \in I$ and $t \in (0,\infty)$ (see Theorem 3.1).

In the next section we fix notation and prove some lemmas, the main theorems will be formulated in the last section of this paper.

2. Preparatory work

Let I be a real interval. Denote by $\mathcal{F}$ the family of all continuous iteration semigroups $F : (0,\infty) \times I \to I$ and by $\mathcal{F}_\delta$ the family of all continuous approximate solutions of (1.1), that is

$$\mathcal{F} = \{F : (0,\infty) \times I \to I;\ F \text{ satisfies (1.1) and } F \text{ is continuous}\},$$

$$\mathcal{F}_\delta = \{H : (0,\infty) \times I \to I;\ |H(t, H(s,x)) - H(s+t,x)| < \delta \text{ and } H \text{ is continuous}\}.$$

Let me remind that continuity of $F \in \mathcal{F}$ is equivalent to continuity with respect to each variable. By $V = V_H$ we mean the set of values of function H.

Lemma 2.1. *Let $H \in \mathcal{F}_\delta$. The following assertions hold true:*

(i) *if $x \in \operatorname{cl} V$ then $|H(t,x) - x| < 2\delta$, $t < T$, for some $T > 0$;*

(ii) *if $H(a,x) = H(b,x)$, for some $0 < a < b < \infty$ and $x \in I$, then $|H(t+T,x) - H(t,x)| < 2\delta$ for $t \geq b$ and $0 \leq T \leq b-a$;*

(iii) *for every $t_1 < t_2 < t_3$ and for every $x \in I$, $H(t_1,x) = H(t_3,x)$ implies $|H(t_2,x) - H(t_1,x)| < 4\delta$*

Proof. The first assertion was proved in [8, Lemma 2.2] whereas the second in [7, Lemma 2.1]. Last assertion follows from inequalities:

$$\begin{aligned}|H(t_2,x) - H(t_2 - t_1 + t_3, x)| &\leq |H(t_2,x) - H(t_2 - t_1, H(t_1,x))| \\ &+ |H(t_2 - t_1, H(t_3,x)) - H(t_2 - t_1 + t_3, x)| < 2\delta\end{aligned}$$

and

$$|H(t_2 - t_1 + t_3, x) - H(t_3,x)| < 2\delta,$$

which results from (ii) with $a = t_1$, $b = t_3$, $T = t_2 - t_1$ and $t = t_3$. □

Now we are going to differentiate points of I according to "monotonicity" of their trajectories, i.e. functions $H(\cdot, x)$. Namely, for the given function $H \in \mathcal{F}_\delta$ and positive l, put

$$\mathcal{I}_l = \{x \in I;\ \text{there exist } t_1 < t_2 \text{ such that } H(t_2,x) - H(t_1,x) > l\},$$

$$\mathcal{D}_l = \{x \in I;\ \text{there exist } t_1 < t_2 \text{ such that } H(t_1,x) - H(t_2,x) > l\},$$

$$\mathcal{C}_l = \{x \in I;\ \text{for every } t_1,t_2 \in (0,\infty) \text{ we have } |H(t_2,x) - H(t_1,x)| \le l\}.$$

It is quite obvious that $I = \mathcal{I}_l \cup \mathcal{D}_l \cup \mathcal{C}_l$, $\mathcal{I}_l \cap \mathcal{C}_l = \emptyset$, $\mathcal{D}_l \cap \mathcal{C}_l = \emptyset$ and the sets $\mathcal{I}_l$, $\mathcal{D}_l$ are open (in I). Moreover, as follows from Lemma 2.1 (iii),

$$\mathcal{I}_{4\delta} \cap \mathcal{D}_{4\delta} = \emptyset. \tag{2.1}$$

Lemma 2.2. *Let $H \in \mathcal{F}_\delta$ and $x_0 \in V \cap A$, where A is a component of $\mathcal{I}_l \cup \mathcal{D}_l$. For every $t \in (0,\infty)$ we have* $\operatorname{dist}(H(t,x_0),A) < 4\delta + l$.

Proof. We have $x_0 = H(t_0,y) \in A$ for some $t_0 \in (0,\infty)$ and $y \in I$. Either $H(t+t_0,y) \in A$ for every $t \in (0,\infty)$ and then $\operatorname{dist}(H(t,x_0),A) < \delta$, or there is $\bar{t} \in (0,\infty)$ such that $H(t+t_0,y) \in A$ for every $t \in (0,\bar{t})$ and $H(\bar{t}+t_0,y) =: z$ is the endpoint of A (the case in which the endpoint of A belongs to A, and is, in fact, the endpoint of interval I, can be easily considered separately). Therefore $z \in V \cap \mathcal{C}_l$, which due to definition of $\mathcal{C}_l$ and part (i) of Lemma 2.1 results in $|H(t,z) - z| < l + 2\delta$, $t \in (0,\infty)$. This together with $|H(t,z) - H(t+\bar{t}+t_0,y)| < \delta$, $t \in (0,\infty)$, gives $|z - H(t+\bar{t}+t_0,y)| < l + 3\delta$, $t \in (0,\infty)$. Taking into account $|H(t,x_0) - H(t+t_0,y)| < \delta$ we get the assertion. □

Now we can notice, as a corollary, that trajectories of points $x \in V$, which are either in $\mathcal{C}_l$ or in components of $\mathcal{I}_l \cup \mathcal{D}_l$ of "small" length, are "close" to x.

Corollary 2.1. *Let $H \in \mathcal{F}_\delta$ and $x_0 \in V$.*

(i) *if $x_0 \in \mathcal{C}_l$ then $|H(t,x_0) - x_0| < l + 2\delta$;*

(ii) *if $x_0 \in A$, where A is a component of $\mathcal{I}_l \cup \mathcal{D}_l$ then $|H(t,x_0) - x_0| < l + |A| + 4\delta$.*

Lemma 2.3. *Let $H \in \mathcal{F}_\delta$ and $x_0 \in A$, where A is a component of $\mathcal{I}_{7\delta}$ [$\mathcal{D}_{7\delta}$] of length $|A| > 11\delta$. Assume that $\inf A \in \mathcal{C}_{7\delta}$ and $(\inf A, x_0) \subset V$ [$(x_0, \sup A) \subset V$, resp.]. Then there exists $y \in A$, $y < x_0$, such that $[x_0, \sup A) \subset \{H(t,y);\ t \in (0,\infty)\}$ [$(\inf A, x_0]) \subset \{H(t,y);\ t \in (0,\infty)\}$, resp.].*

Proof. We assume that $A \subset \mathcal{I}_{7\delta}$. Notice that for every $x \in A \cap \operatorname{cl} V$ we have

$$\sup\{H(t,x);\ t \in (0,\infty)\} \ge \sup A. \tag{2.2}$$

Indeed, fix an $x \in A \cap \operatorname{cl} V$ and suppose, on the contrary that

$$\sup\{H(t,x);\ t \in (0,\infty)\} < \sup A. \tag{2.3}$$

From Lemma 2.1 (i), (2.1) and $x \in \mathcal{I}_{7\delta}$ we infer that $\sup\{H(t,x);\ t \in (0,\infty)\} > x$ which together with (2.3) gives $\sup\{H(t,x);\ t \in (0,\infty)\} \in A$. Choose $H(\bar{t},x) =: \bar{x}$ such that $\sup\{H(t,x);\ t \in (0,\infty)\} - \frac{\delta}{2} < \bar{x}$. We have $|H(t,\bar{x}) - H(t+\bar{t},x)| < \delta$, for $t \in (0,\infty)$ and $\bar{x} - 4\delta < H(t+\bar{t},x)$, for $t \in (0,\infty)$, which follows again from (2.1). These inequalities contradicts $\bar{x} \in \mathcal{I}_{7\delta}$.

Now denote $\hat{x} := \inf A$ and we will prove that

$$H(t,\hat{x}) = \hat{x}, \quad t \in (0,\infty). \tag{2.4}$$

Since $\hat{x} \in \mathcal{C}_{7\delta} \cap \operatorname{cl} V$, we infer that

$$H(t,\hat{x}) < \hat{x} + 9\delta, \qquad t \in (0,\infty). \tag{2.5}$$

If $H(t_0,\hat{x}) \in A$ for a $t_0 \in (0,\infty)$, than, taking into account (2.2), we would have a contradiction with (2.5). On the other hand, $H(t_0,\hat{x}) < \hat{x}$, for a $t_0 \in (0,\infty)$, continuity of H, together with (2.2), implies that $H(t,x) = \hat{x}$ for an $x \in A$ and a $t \in (0,\infty)$, which, due to (2.5), contradicts (2.2).

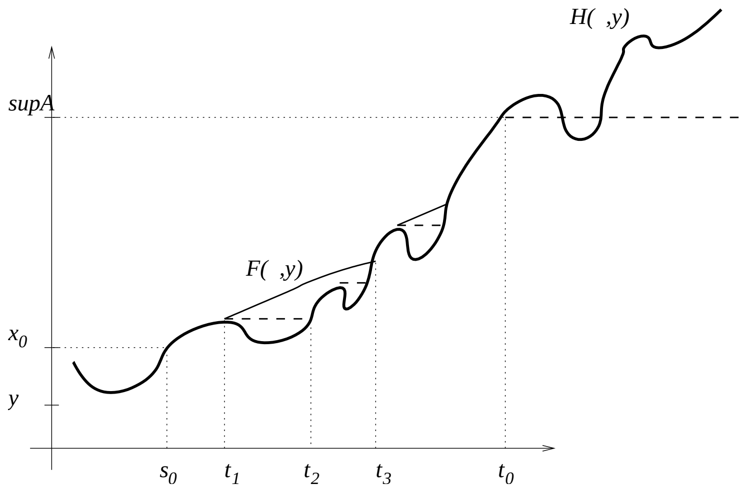
Finally, the assertion of this Lemma follows from (2.4), continuity of H and (2.2). □

Now we pass to construction of a continuous iteration semigroup F which is close to given $H \in \mathcal{F}_\delta$ on the "almost whole" interval A, which is a component of $\mathcal{I}_{7\delta} \cup \mathcal{D}_{7\delta}$.

Construction 2.1. Let $H \in \mathcal{F}_\delta$*,* A *be a component of* $\mathcal{I}_{7\delta}$ *[*$\mathcal{D}_{7\delta}$ *can be considered analogously],* $y \in A$*,* $y < H(s_0,y) =: x_0 \in A$ *(if* $\lim_{t\to 0} H(t,y) = y$ *it is possible to consider* $x_0 = y$ *and put* $s_0 = 0$ *in such case) and* $\sup\{H(t,y); t \in (0,\infty)\} \geq \sup A$*.*

Let $t_0 > s_0$ be the smallest such that $H(t_0,y) = \sup A$, if such a point does not exist, then we take $t_0 = \infty$. Define $\tilde{F}(t,y) := \max\{H(l,y);\ s_0 \leq l \leq t\}$ for $t \in [s_0,t_0]$ and $\tilde{F}(t,y) := \sup A$ for $t \geq t_0$. Of course, $\tilde{F}$ is nondecreasing with respect to first variable. Let $f : [s_0,\infty) \to \operatorname{cl} A$ be a continuous function with $f \geq \tilde{F}(\cdot,y)$, strictly increasing on interval $[s_0,t_0]$, such that $f(t) = \tilde{F}(t,y) = \sup A$, for $t \geq t_0$, and for every maximal interval $J = (t_1,t_3)$ on which $f \neq \tilde{F}(\cdot,y)$ there exists $t_2 \in (t_1,t_3)$, such that $\tilde{F}(t_1,y) = \tilde{F}(t_2,y)$ and $t_3 - t_2 \leq t_2 - t_1$, (see Figure 2.1).

Notice that $f(t_1) = \tilde{F}(t_1,y)$ and $f(t_3) = \tilde{F}(t_3,y)$. Put $F(t,y) = f(t)$ for $t \in [s_0,t_0]$ and $F(t,y) = \sup A$ for $t \geq t_0$. For every $x \in A$, $x \geq x_0$, there exists only one $t_x \geq s_0$ such that $F(t_x,y) = x$. For such x and $t \in (0,\infty)$ we define $F(t,x) := F(t_x + t, y)$.



We have $|F(t,y) - H(t,y)| < 10\delta$ for $t \geq s_0$. Indeed, as a consequence of Lemma 2.1 (ii), we get $H(t,y) - H(t_2,y) < 2\delta$, $t \in [t_2,t_3]$, moreover (2.1) implies $H(t,y) > H(t_1,y) - 4\delta$ for $t \in (t_1,t_3)$, which yields $|F(t,y) - H(t,y)| < 6\delta$ for $t \in [s_0,t_0]$. For $t > t_0$ we have estimations $H(t,y) > \sup A - 4\delta$ (by (2.1)), and $H(t,y) < \sup A + \delta + 2\delta + 7\delta$ (similar reasoning as in Lemma 2.2), which ends the proof of the desired inequality.

Now we will show that

$$|F(t,x) - H(t,x)| < 19\delta, \quad x \in A,\ x \geq x_0,\ t \in (0,\infty). \tag{2.6}$$

Fix such x and t and consider two possibilities. Either $x = F(t_x,y) = H(t_x,y)$ and then

$$\begin{aligned} |F(t,x) - H(t,x)| &\leq |F(t+t_x,y) - H(t+t_x,y)| \\ &+ |H(t+t_x,y) - H(t,x)| < 10\delta + \delta. \end{aligned}$$

Or $x = F(t_x,y) \neq H(t_x,y)$ and then there exist t_1, t_2, t_3, $t_1 < t_2 < t_3$, such that $H(t_1,y) = H(t_2,y)$, $t_3 - t_2 \leq t_2 - t_1$, $t_x \in (t_1,t_3)$, and $\hat{t} \in (t_2,t_3)$ such that $x = H(\hat{t},y)$. Of course $|H(t,x) - H(t+\hat{t},y)| < \delta$, $F(t,x) = F(t+t_x,y)$ and, what was already shown, $|F(t+t_x,y) - H(t+t_x,y)| < 10\delta$. So it is enough to prove that $|H(t+\hat{t},y) - H(t+t_x,y)| < 8\delta$. If $t + t_x > t_2$, using twice Lemma 2.1 (ii) we estimate

$$\begin{aligned} |H(t+\hat{t},y) - H(t+t_x,y)| &\leq \left| H(t+\hat{t},y) - H\left(t + \hat{t} - \frac{\hat{t}-t_x}{2}, y\right)\right| \\ &+ \left| H\left(t + t_x + \frac{\hat{t}-t_x}{2}, y\right) - H(t+t_x,y)\right| < 2\delta + 2\delta. \end{aligned}$$

However, if $t + t_x \le t_2$, using again twice Lemma 2.1 (ii) and (2.1), we conclude that

$$|H(t+\hat{t},y) - H(t+t_x,y)| \le |H(t+\hat{t},y) - H(\hat{t},y)| + |H(\hat{t},y) - H(t_2,y)| \\ +|H(t_2,y) - H(t+t_x,y)| < 2\delta + 2\delta + 4\delta,$$

which ends the proof of (2.6).

Now we are going to describe two sets of conditions, which, if satisfied by trajectories of points of A, guarantee the possibility of extending defined earlier continuous iteration semigroup F on the whole interval A, such that the difference $|F(t,x) - H(t,x)|$ is "small" for every $x \in A$ and $t \in (0,\infty)$.

1. Let $H \in \mathcal{F}_\delta$, A be a component of $\mathcal{I}_{7\delta}$ (we consider similarly the case when A is a component of $\mathcal{D}_{7\delta}$) and $x_0 \in \operatorname{int}(A \cap V)$. We say that condition (E_1^l) is satisfied for A if $|B| < l$, where $B := V \cap A \cap (-\infty, x_0)$, and there exists a continuous strictly decreasing function $T : B \to (0,\infty)$ such that $H(T(x),x) = x_0$, $\lim_{x\to x_0^-} T(x) = 0$, and, if $\inf B \notin B$, then $\lim_{x\to \inf B^+} T(x) = \infty$.

Remark 2.1. If $F \in \mathcal{F}$ than for every $x_0 \in V_F$ there exists (unique) such a function T.

Proposition 2.1. *Let $H \in \mathcal{F}_\delta$, A be a component of $\mathcal{I}_{7\delta}$ $[\,\mathcal{D}_{7\delta}]$ and $x_0 \in \operatorname{int}(A \cap V)$. If A satisfies condition (E_1^l) then there exists a continuous iteration semigroup $F : (0,\infty) \times \operatorname{cl}(V \cap A) \to \operatorname{cl}(V \cap A)$ such that $|F(t,x) - H(t,x)| \le \max\{20\delta, 6\delta + l\}$, $t \in (0,\infty)$, $x \in \operatorname{cl}(V \cap A)$.*

Proof. We start with defining $F : (0,\infty) \times [x_0, \sup A] \to \operatorname{cl} A$ according to Construction 2.1 with $y = x_0$ (and $s_0 = 0$). We extend F putting for $x \in B$

$$F(t,x) = \begin{cases} T^{-1}(T(x) - t), & \text{for } t < T(x); \\ x_0, & \text{for } t = T(x); \\ F(t - T(x), x_0), & \text{for } t > T(x). \end{cases}$$

Moreover, if $\inf B \notin B$ we put $F(t, \inf B) = \inf B$, for every $t \in (0,\infty)$. It is easy to verify that F is continuous.

The translation equation is satisfied also for $x \in B$. Indeed, let us consider the following cases:

- $t + s < T(x)$
  $F(t, F(s,x)) = F(t, T^{-1}(T(x) - s)) = T^{-1}(T(x) - s - t) = F(s+t, x)$;

- $t+s=T(x)$
  $F(t,F(s,x)) = F(t,T^{-1}(T(x)-s)) = F(t,T^{-1}(t)) = x_0 = F(T(x),x) = F(s+t,x)$;
- $t+s>T(x)$, $s<T(x)$
  $F(t,F(s,x)) = F(t,T^{-1}(T(x)-s)) = F(t-T(T^{-1}(T(x)-s)),x_0) = F(t-T(x)+s,x_0) = F(s+t,x)$;
- $s=T(x)$
  $F(t,F(s,x)) = F(t,x_0) = F(t+T(x)-T(x),x_0) = F(t+T(x),x) = F(t+s,x)$;
- $s>T(x)$
  $F(t,F(s,x)) = F(t,F(s-T(x),x_0)) = F(t+s-T(x),x_0) = F(t+s,x)$.

Finally we check the distance between F and H. In view of (2.6), it is enough to consider it for $x<x_0$.

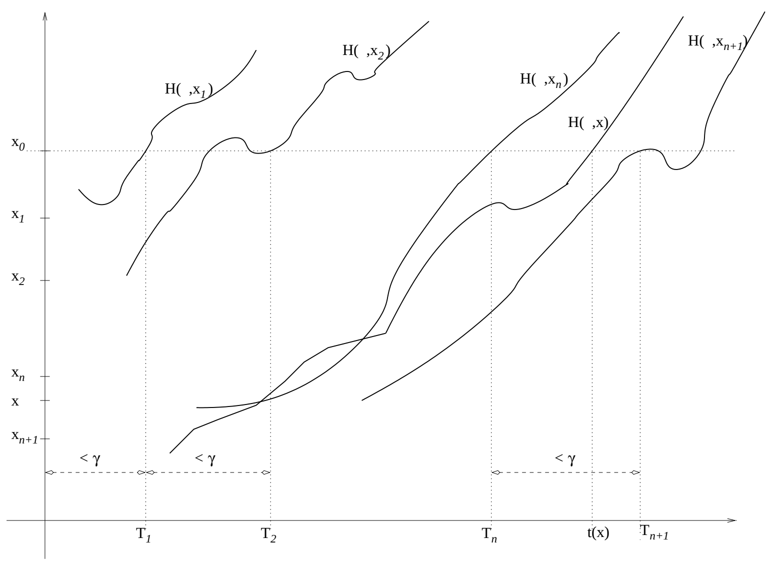
- $t>T(x)$
  $|F(t,x)-H(t,x)| \leq |F(t-T(x),x_0)-H(t-T(x),x_0)| + |H(t-T(x),x_0)-H(t,x)| < 19\delta+\delta$
- $t=T(x)$
  $F(t,x)=x_0=H(t,x)$
- $t<T(x)$
  $F(t,x)=T^{-1}(T(x)-t)\in(x,x_0)$; $H(t,x)<x_0+4\delta$, due to (2.1); $H(t,x)>x-2\delta-4\delta$, by Lemma 2.1 (i) and (2.1). Taking above into account and $x_0-x<l$, we infer that $|F(t,x)-H(t,x)|<l+6\delta$.

The proof is completed. □

2. Let $H\in\mathcal{F}_\delta$, A be a component of $\mathcal{I}_{7\delta}$ (we consider similarly the case when A is a component of $\mathcal{D}_{7\delta}$) and $x_0\in\operatorname{int}(A\cap V)$. We say that condition $(E_2^{l,\eta})$ is satisfied for A if $|B|<l$, where $B:=V\cap A\cap(-\infty,x_0)$, there exist a decreasing sequence (x_n) of points of $[\inf B,x_0)$ and an increasing sequence (T_n) of positive numbers, both finite or infinite, depending whether $\inf B\in B$ or not, respectively, which satisfy the following conditions.

- $\lim_{n\to\infty}x_n = \inf B$ if the sequence is infinite, otherwise the last element $=\inf B$;
- there exists a positive γ such that $|t-s|<\gamma$ implies $|H(t,x_n)-H(s,x_n)|<\eta$, for every n;
- $T_1<\gamma$, $T_{n+1}-T_n<\gamma$ and T_n tends to infinity, if infinite;
- $H(T_n,x_n)=x_0$;

- for every $x \in (x_{n+1}, x_n)$ there is a $t(x) \in [T_n, T_{n+1}]$ with $H(t(x), x) = x_0$. (see Figure 2.2)



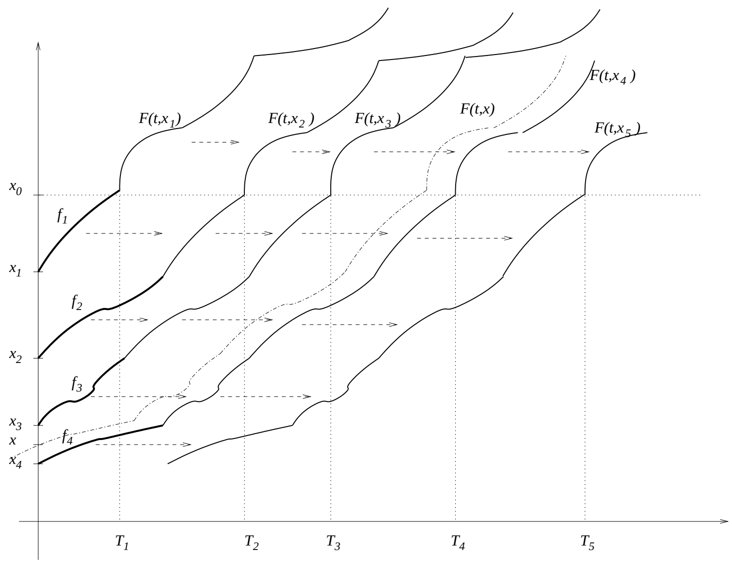
Remark 2.2. Let $F : (0, \infty) \times X \to X$ be a continuous iteration semigroup and X a compact metric space then the function $(0, \infty) \ni t \mapsto F(t, \cdot) \in \mathcal{C}(X, X)$ is continuous. So, if $H \in \mathcal{F}$ then condition $(E_2^{l,\eta})$ is satisfied with every x_0, suitable for l, and η.

Proposition 2.2. *Let $H \in \mathcal{F}_\delta$, A be a component of $\mathcal{I}_{7\delta}$ $[\mathcal{D}_{7\delta}]$ and $x_0 \in \operatorname{int}(A \cap V)$. If A satisfies condition $(E_2^{l,\eta})$ then there exists a continuous iteration semigroup $F : (0, \infty) \times \operatorname{cl}(V \cap A) \to \operatorname{cl}(V \cap A)$ such that $|F(t, x) - H(t, x)| \leq 2\eta + 26\delta + l$, $t \in (0, \infty)$, $x \in \operatorname{cl}(V \cap A)$.*

Proof. As previously, we define $F(t, x_1)$, for $t \geq T_1$, and $F(t, x)$ for $t \in (0, \infty)$ and $x \geq x_0$ according to Construction 2.1 with $y = x_1$, $s_0 = T_1$. Next we choose any strictly increasing continuous functions $f_n : (0, T_n - T_{n-1}] \to (x_n, x_{n-1}]$ (we put $T_0 = 0$) and define $F(t, x_n) := f_n(t)$ for $t \in (0, T_n - T_{n-1}]$ and $F(t, x)$ on the rest of domain of the desired continuous iteration semigroup in an unique way determined by the translation equation (see Figure 2.3). Moreover, if $\inf B \notin B$ we put $F(t, \inf B) = \inf B$.

We pass to verifying the estimation of the difference between F and H. First for $x = x_n$ and $t \leq T_n$. We have $F(t, x_n) \in (x_n, x_0]$ and

$$\inf B - 2\delta - 4\delta < H(t, x_n) < x_0 + 4\delta, \tag{2.7}$$



which follows from Lemma 2.1 (i) and (2.1). These gives

$$|F(t, x_n) - H(t, x_n)| < 6\delta + l.$$

For $t > T_n$ we get

$$|F(t, x_n) - H(t, x_n)| \leq |F(t - T_n, x_0) - H(t - T_n, x_0)| \\ +|H(t - T_n, x_0) - H(t, x_n)| < 19\delta + \delta.$$

Fix $x \in B$, $x_n < x < x_{n-1}$ and put $\tilde{t} = f_n^{-1}(x)$. Consider the case $t > t(x)$. Then $x = F(\tilde{t}, x_n)$ and $F(t, x) = F(t + \tilde{t}, x_n)$. We have the following inequalities.

$$|F(t + \tilde{t}, x_n) - H(t + \tilde{t}, x_n)| < \max\{20\delta, 6\delta + l\},$$

by what was already shown;

$$|H(t + \tilde{t}, x_n) - H(t, x_n)| < \eta$$

and

$$|H(t, x_n) - H(t + (T_n - t(x)), x_n)| < \eta,$$

since $(E_2^{l,\eta})$ is satisfied;

$$|H(t - t(x) + T_n, x_n) - H(t - t(x), x_0)| < \delta$$

and

$$|H(t - t(x), x_0) - H(t, x)| < \delta.$$

They result in

$$|F(t, x) - H(t, x)| < 2\eta + \max\{22\delta, 8\delta + l\}.$$

If $t < t(x)$ then, the same reasoning as in (2.7), gives

$$\inf B - 6\delta < H(t, x) < x_0 + 4\delta.$$

Notice that $F(T_n - \tilde{t}, x) = x_0$. If $t \le T_n - \tilde{t}$ then $F(t, x) \in (x, x_0]$, which implies

$$|F(t, x) - H(t, x)| < 6\delta + l.$$

Otherwise $T_n - \tilde{t} < t < t(x)$ and then

$$\begin{aligned} 0 < F(t, x) - x_0 \le &|F(t - (T_n - \tilde{t}), x_0) - H(t - (T_n - \tilde{t}), x_0)| \\ &+ |H(t - (T_n - \tilde{t}), x_0) - H(T_1 + t - (T_n - \tilde{t}), x_1)| \\ &+ |H(T_1 + (t - (T_n - \tilde{t})), x_1) - H(T_1, x_1)| < 19\delta + \delta + \eta. \end{aligned}$$

That is why

$$|F(t, x) - H(t, x)| < 26\delta + l + \eta.$$

This ends the proof. □

After that we pass to formulation of a condition under which we can extend continuous iteration semigroup F, defined on $\operatorname{cl} V$, approximating H, to the whole interval I, such that the extension is also close to H.

3. Let $H \in \mathcal{F}_\delta$. We say that H satisfies condition (E_3) if

- $\liminf_{t\to 0} H(t, \inf V) = \inf V$,
- $\limsup_{t\to 0} H(t, \sup V) = \sup V$

and there exists a continuous function $e : I \setminus \operatorname{cl} V \to \operatorname{cl} V$ with

- $e(x) \in [\liminf_{t\to 0} H(t, x), \limsup_{t\to 0} H(t, x)]$,
- $\lim_{x\to \inf V^-} e(x) = \inf V$,
- $\lim_{x\to \sup V^+} e(x) = \sup V$.

Remark 2.3. Notice that every $F \in \mathcal{F}$ satisfies condition (E_3) (see [9]).

Lemma 2.4. *Let $H \in \mathcal{F}_\delta$ satisfies condition (E_3). Let $\tilde{F} : (0, \infty) \times \operatorname{cl} V \to \operatorname{cl} V$ be a continuous iteration semigroup which satisfies $|H(t, x) - \tilde{F}(t, x)| \le \tilde{\varepsilon}$, $t \in (0, \infty)$, $x \in \operatorname{cl} V$, for some $\tilde{\varepsilon} > 0$. Then there is an extension $F : (0, \infty) \times I \to I$ of $\tilde{F}$ such that $|H(t, x) - F(t, x)| \le \tilde{\varepsilon} + \delta$, $t \in (0, \infty)$, $x \in I$.*

Proof. Fix H, $\tilde{F}$ and e, as in assumptions. Put $F(t,x) = \tilde{F}(t, e(x))$, for $x \in I \setminus \mathrm{cl}\, V$, $t \in (0,\infty)$, and, obviously, $F(t,x) = \tilde{F}(t,x)$, for $x \in \mathrm{cl}\, V$, $t \in (0,\infty)$. It is easy to check that $F \in \mathcal{F}$. To complete the proof, fix $x \in I \setminus \mathrm{cl}\, V$ and choose a decreasing to 0 sequence (t_n) such that $\lim_{n\to\infty} H(t_n, x) = e(x)$. We have $|H(t,x) - H(t - t_n, H(t_n, x))| < \delta$, which yields $|H(t,x) - H(t, e(x))| \le \delta$, and furthermore, $|F(t,x) - H(t,x)| \le \delta + \tilde{\varepsilon}$. □

3. The main theorems

Now we are in position to formulate and prove the main results of this paper.

Let $H \in \mathcal{F}_\delta$ and (A_n) be the sequence of all components of $\mathcal{I}_{7\delta} \cup \mathcal{D}_{7\delta}$. We say that H satisfies condition $(E^{l,\eta})$ if condition (E_3) holds as well as, for every n, A_n satisfies either (E_1^l) or $(E_2^{l,\eta})$, provided $\mathrm{int}(A_n \cap V) \neq \emptyset$ and $|A_n| > 11\delta$.

Theorem 3.1. *Let $\varepsilon, \delta, l, \eta > 0$ be such that $27\delta + l + 2\eta < \varepsilon$. Then for every $H \in \mathcal{F}_\delta$ satisfying $(E^{l,\eta})$ there exists a continuous iteration semigroup $F : (0,\infty) \times I \to I$ such that $|F(t,x) - H(t,x)| < \varepsilon$ for $x \in I$ and $t \in (0,\infty)$.*

Proof. Let ε, δ, l, η and H be such as in assumptions. Let (A_n) be a sequence of all components of $\mathcal{I}_{7\delta} \cup \mathcal{D}_{7\delta}$ of length greater than 11δ which have nonempty intersection with V. For every n we construct $F : (0,\infty) \times \mathrm{cl}(A_n \cap V) \to \mathrm{cl}(A_n \cap V)$ according to Proposition 2.1 or 2.2, depending on which of the conditions (E_1^l) or $(E_2^{l,\eta})$ is satisfied for A_n. For the rest of $\mathrm{cl}\, V$ we put $F(t,x) = x$, and we extend F on the whole interval I using Lemma 2.4. Defined in that way, $F \in \mathcal{F}$. Lemma 2.4 together with Propositions 2.1 and 2.2, as well as Corollary 2.1, give the assertion. □

Next theorem is about approximation of H by F, without assuming the condition $(E^{l,\eta})$, but only for $x \in V \setminus L$, where L is of positive length, however as small as we wish.

Theorem 3.2. *For every $\varepsilon, \zeta > 0$ there exists $\delta > 0$ such that for every $H \in \mathcal{F}_\delta$ there exist $L \subset I$, $|L| < \zeta$, and $F \in \mathcal{F}$ such that $|F(t,x) - H(t,x)| \le \varepsilon$ for $t \in (0,\infty)$ and $x \in \mathrm{cl}\, V \setminus L$.*

Proof. Fix $\varepsilon, \zeta > 0$ and choose $\delta > 0$, in order to $22\delta < \varepsilon$ and $6\delta < \zeta$. Let $H \in \mathcal{F}_\delta$, and (A_n) be a sequence of all components of $\mathcal{I}_{7\delta} \cup \mathcal{D}_{7\delta}$ of length greater than 11δ which have nonempty intersection with V. Choose a sequence (δ_n) of positive numbers, with $\sum_n \delta_n < \zeta - 6\delta$. If the assumptions of Lemma 2.3 are satisfied for A_n then let $x_0^n \in A_n$ be such that $x_0^n - \inf(A_n \cap V) < \delta_n$ or

$\sup(A_n \cap V) - x_0^n < \delta_n$, and find $y_n \in A_n$ and $s_0^n \in (0, \infty)$ such that $y_n < H(s_0^n, y_n) = x_0^n$ or $y_n > H(s_0^n, y_n) = x_0^n$, depending whether $A_n \in \mathcal{I}_{7\delta}$ or $A_n \in \mathcal{D}_{7\delta}$, respectively. Otherwise, choose an arbitrary $A_n \cap V \ni y_n < \inf V + \delta$ and $x_0^n = H(s_0^n, y_n) < y_n + 2\delta$, or $A_n \cap V \ni y_n > \sup V - \delta$ and $x_0^n = H(s_0^n, y_n) > y_n - 2\delta$, if $A_n \in \mathcal{I}_{7\delta}$ or $A_n \in \mathcal{D}_{7\delta}$, respectively. We define F according to Construction 2.1 with $y = y_n$, $x_0 = x_0^n$ and $s_0 = s_0^n$, and extend it on the whole interval $A_n \cap V$ for instance, as in Figure 2.3, with an arbitrary sequences (x_n), (T_n) and (f_n). We have the estimation (2.6) with $x_0 = x_0^n$. For the rest of V we put $F(t, x) = x$, which gives the estimation $|F(t, x) - H(t, x)| < 22\delta$, according to Corollary 2.1. Finally we extend F on the whole interval I in order to $F \in \mathcal{F}$. Put $L_n = [\inf(A_n \cap V), x_0^n)$ or $(x_0^n, \sup(A_n \cap V)]$, if $A_n \in \mathcal{I}_{7\delta}$ or $A_n \in \mathcal{D}_{7\delta}$, respectively, and notice that $|L_n| < \delta_n$ or $|L_n| < 3\delta$, but the second possibility holds only for at most two indexes n. With $L = \bigcup_n L_n$ we get the assertion. □

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