On Daróczy's problem for additive functions

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Dedicated to Professor Zoltán Daróczy on the occasion of his 70th birthday

Abstract. In this paper we investigate the functional equation

$$\sum_{i=1}^{n} \alpha_i A(\beta_i x) = 0$$

which holds for all $x \in \mathbb{R}$ with an unknown additive function $A : \mathbb{R} \to \mathbb{R}$ and fixed real parameters α_i, β_i, where $i = 1, \ldots, n$. The case $n = 2$ is discussed by Z. DARÓCZY [1]. Here we formulate sufficient conditions for the existence of nontrivial solutions in terms of the parameters $\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}$ and $\frac{\alpha_1}{\alpha_n}, \ldots, \frac{\alpha_{n-1}}{\alpha_n}$.

1. Introduction and preliminaries

Let α_i, β_i be fixed real parameters, $i = 1, \ldots, n$. Consider the functional equation

$$\sum_{i=1}^{n} \alpha_i A(\beta_i x) = 0 \quad (x \in \mathbb{R}) \tag{1.1}$$

with an unknown additive function $A : \mathbb{R} \to \mathbb{R}$, i.e.

$$A(x + y) = A(x) + A(y)$$

is satisfied for all x and $y \in \mathbb{R}$. Since any additive function vanishes at $x = 0$, without loss of generality we can suppose that none of the parameters equals zero.

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The case $n = 2$ has been investigated in DARÓCZY [1]. His fundamental result states that the functional equation $\alpha_1 A(\beta_1 x) + \alpha_2 A(\beta_2 x) = 0$ has non-trivial solutions if and only if both the parameters

$$\lambda := \frac{\beta_1}{\beta_2} \quad \text{and} \quad \mu := -\frac{\alpha_2}{\alpha_1}$$

are transcendental or they are algebraic with the same defining polynomial. It is important to see that this condition is equivalent to the existence of a field isomorphism $\delta : \mathbb{Q}(\lambda) \to \mathbb{Q}(\mu)$ such that $\delta(\lambda) = \mu$. Consider now $\mathbb{R}$ as the vector space over $\mathbb{Q}(\lambda)$ and $\mathbb{Q}(\mu)$, respectively. We can use the procedure of semilinear extension to construct an additive function $A : \mathbb{R} \to \mathbb{R}$ such that $A(\lambda x) = \mu A(x)$ for all $x \in \mathbb{R}$. This is obviously equivalent to equation (1.1) in case of $n = 2$. In this paper we investigate the case $n \geq 3$ which is a natural extension of the original problem. The theory of functional equations

$$\sum_{i=0}^{n} a_i f(\alpha_i x + (1 - \alpha_i)y) = 0 \quad (x \in I)$$

containing weighted arithmetic means also gives important motivations. Here $I \subset \mathbb{R}$ is a non-void open interval, $f : I \to \mathbb{R}$ is an unknown function, the parameters $\alpha_i \in [0, 1]$ are arbitrarily fixed and $i = 0, 1, \ldots, n$. The particular case $n = 3$, $a_0 = a_1 = 1$, $a_2 = a_3 = -1$ and $\alpha_2 = 1$, $\alpha_3 = 0$ has been investigated in DARÓCZY–MAKSA–PÁLES [3], DARÓCZY–LAJKÓ–LOVAS–MAKSA–PÁLES [8], and also in MAKSA [9] in connection with the equivalence of certain functional equations involving means. The result have been extended for the case of arbitrary $\alpha_2, \alpha_3 \in (0, 1)$ in the paper [10]. The investigation of the general case can be found in [11] with proving that f is a solution if and only if it has the form

$$f(x) = A_0 + A_1(x) + \cdots + A_{n-1}(x, \ldots, x),$$

where A_0 is a constant and, for any $k = 1, \ldots, n-1$, $A_k : \mathbb{R}^k \to \mathbb{R}$ is a symmetric k-additive function such that the first order condition

$$a_1 A_1(\beta_1 x) + \cdots + a_{n-1} A_1(\beta_{n-1} x) + a_n A_1(\beta_n x) = 0,$$

the second order conditions

$$a_1 A_2(x, \beta_1 y) + \cdots + a_{n-1} A_2(x, \beta_{n-1} y) + a_n A_2(x, \beta_n y) = 0 \quad \text{and}$$

$$a_1 A_2(\beta_1 x, \beta_1 x) + \cdots + a_{n-1} A_2(\beta_{n-1} x, \beta_{n-1} x) + a_n A_2(\beta_n x, \beta_n x) = 0$$

and so on are also satisfied with the parameters

$$\beta_i := \frac{\alpha_i - \alpha_0}{\alpha_n - \alpha_0}$$

for all real numbers x and y. It can be easily seen that the condition for the additive function A_1 is the same as equation (1.1). First of all we introduce some basic notions we need in the following.

Definition 1.1. Let m be a positive integer and consider the elements $\vec{\lambda} := (\lambda_1, \ldots, \lambda_m)$ and $\vec{\mu} := (\mu_1, \ldots, \mu_m)$ of the coordinate space $\mathbb{R}^m$.

(i) The ideal $\mathcal{I}(\vec{\lambda}) := \{p \in \mathbb{Q}[x_1, \ldots, x_m] \mid p(\lambda_1, \ldots, \lambda_m) = 0\}$ of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$ is called the *defining ideal* of $\vec{\lambda} := (\lambda_1, \ldots, \lambda_m)$.

(ii) If the defining ideals of $\vec{\lambda}$ and $\vec{\mu}$ are the same then we say that they are *algebraic conjugate* of each other.

Remark 1.2. An important special case when the defining ideal of

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_m)$$

contains only the zero polynomial, i.e. the coordinates are algebraically independent. Otherwise they are algebraically dependent. In the particular case $m = 1$ the ideal $\mathcal{I}(\lambda)$ can be generated by the minimal polynomial and we have that λ and μ are algebraic conjugate if both of them are transcendent or they are algebraic and their defining polynomials are the same.

Lemma 1.3. *Suppose that $2 \leq n \in \mathbb{N}$ and let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{n-1})$ and $\vec{\mu} := (\mu_1, \ldots, \mu_{n-1})$ be arbitrarily fixed. There exists a field isomorphism $\delta : \mathbb{Q}(\mu_1, \ldots, \mu_{n-1}) \to \mathbb{Q}(\lambda_1, \ldots, \lambda_{n-1})$ such that $\delta(\mu_i) = \lambda_i$ for all $i = 1, \ldots, n-1$ if and only if $\vec{\lambda}$ and $\vec{\mu}$ are algebraic conjugate.*

For the proof see A. VARGA and CS. VINCZE [11].

Definition 1.4. A translate of a k - dimensional linear subspace of $\mathbb{R}^m$ is called a k-*flat*. If $k = 1$ then we speak about a *line*. In case of $k = m - 1$ we have a *hyperplane*. A k-flat F_k is called *algebraic* if there exists a not identically zero element P of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$ such that P vanishes at all points of the k-flat, i.e.

$$P \in \cap_{\vec{\lambda} \in F_k} \mathcal{I}(\vec{\lambda}).$$

Remark 1.5. In case of $k = 0$, i.e. if the flat reduces to a point we can refer to Definition 1.1.

2. Some observations on algebraic flats

Lemma 2.1. *A k-flat F_k in $\mathbb{R}^m$ is the union of zero's of not identically zero elements of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$ if and only if it is algebraic.*

PROOF. If F_k is algebraic then the statement is trivial. The proof of the converse is by the induction on k as follows. Since a 0-flat F_0 is a singleton, the statement is obvious for $k = 0$. Now let us assume that $0 \leq k \leq m - 1$ and the statement is valid for all k-flats in $\mathbb{R}^m$. We consider a representation of an arbitrary $(k+1)$-flat F_{k+1} in the form

$$F_{k+1} = \{v_0 + tv + t_1 v_1 + \cdots + t_k v_k \mid (t, t_1, \ldots, t_k) \in \mathbb{R}^{k+1}\},$$

where $v \in \mathbb{R}^m$ and $v_j \in \mathbb{R}^m$ $(j = 0, 1, \ldots, k)$ are fixed. Let

$$F_k(t) = \{v_0 + tv + t_1 v_1 + \cdots + t_k v_k \mid (t_1, \ldots, t_k) \in \mathbb{R}^k\}.$$

Then $F_k(t) = F_k(0) + tv$ is a k-flat such that $F_k(t) \subset F_{k+1}$. Geometrically we interpret F_{k+1} as the union of parallel k-flats $F_k(t)$ by running the parameter t along the set of the real numbers. Now suppose that F_{k+1} is the union of zero's of not identically zero elements of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$. Then, for every real number t, so is $F_k(t)$, and due to the inductive hypothesis there exists a not identically zero polynomial $P_t \in \mathbb{Q}[x_1, \ldots, x_m]$ such that $P_t(u + tv) = 0$ for every $u \in F_k(0)$. Since $\mathbb{Q}[x_1, \ldots, x_m]$ is countable, there exists an uncountable (and thus infinite) set $I \in \mathbb{R}$ and a not identically zero polynomial $P \in \mathbb{Q}[x_1, \ldots, x_m]$ such that $P_t = P$ for every $t \in I$. Hence we have

$$P(u + tv) = 0$$

for every $u \in F_k(0)$ and $t \in I$. Consequently, for each $u \in F_k(0)$, the polynomial $Q_u(t) = P(u + tv)$ has infinitely many zeros, and thus Q_u is identically zero. This yields the equation $P(u + tv) = 0$ for every $u \in F_k(0)$ and $t \in \mathbb{R}$, therefore P vanishes at all points of F_{k+1}. $\square$

Remark 2.2. The result says that a k-flat F_k is algebraic if and only if the coordinates of any point of F_k are algebraic dependent.

Lemma 2.3. *The hyperplane in $\mathbb{R}^m$ defined by the equation*

$$\lambda_1 x_1 + \ldots \lambda_{m-1} x_{m-1} + \lambda_m = x_m$$

is algebraic if and only if all of the coefficients $\lambda_1, \ldots, \lambda_m$ is algebraic.

PROOF. Suppose that the hyperplane is algebraic. Then we have a not identically zero element P of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$ which vanishes at the points of the hyperplane. Therefore

$$P(x_1, x_2, \ldots, x_{m-1}, \lambda_1 x_1 + \cdots + \lambda_{m-1} x_{m-1} + \lambda_m) = 0. \qquad (2.1)$$

Substituting $x_1 = \cdots = x_{m-1} = 0$ we have that λ_m is the root of the polynomial $p(t) := P(0, \ldots, 0, t)$. Suppose, in contrary, that λ_m is not algebraic. Then $P(0, \ldots, 0, t) = 0 \Rightarrow D_m P(0, \ldots, 0, t) = 0$, where $D_m P$ means the usual partial derivative of the polynomial P with respect to x_m. Differentiating (2.1.) by x_i, we have that for any indices $i = 1, \ldots, m-1$

$$D_i P(0, \ldots, 0, \lambda_m) + \lambda_i D_m P(0, \ldots, 0, \lambda_m) = 0$$

and, consequently, $D_i P(0, \ldots, 0, \lambda_m) = 0$. Because λ_m is not algebraic

$$D_i P(0, \ldots, 0, t) = 0 \Rightarrow D_m D_i P(0, \ldots, 0, t) = 0$$

for all indices $i = 1, \ldots, m-1$ and $t \in \mathbb{R}$. Of course, the same is true if $i = m$. Differentiating again (2.1) by x_i and x_j, we have that

$$D_j D_i P(0, \ldots, 0, \lambda_m) + \lambda_j D_m D_i P(0, \ldots, 0, \lambda_m)$$
$$+ \lambda_i D_j D_m P(0, \ldots, 0, \lambda_m) + \lambda_i \lambda_j D_m D_m P(0, \ldots, 0, \lambda_m) = 0$$

and, consequently, $D_j D_i P(0, \ldots, 0, \lambda_m) = 0$. Since λ_m is not algebraic, it follows that $D_i D_j P(0, \ldots, 0, t) = 0$ where i, j are arbitrary indices including the cases $i = m$ or $j = m$ too. According to the Taylor formula at $(0, 0, \ldots, 0)$, the process shows that P must be identically zero, which is a contradiction.

Let now, for example, $x_1 := 1$ and $x_2 = \cdots = x_m = 0$. Then we have $P(1, 0, \ldots, 0, \lambda_1 + \lambda_m) = 0$, i.e. $\lambda_1 + \lambda_m$ is the root of the polynomial $p(t) := P(1, 0, \ldots, 0, t)$. A similar reasoning as above gives that $\lambda_1 + \lambda_m$ is algebraic. Because the algebraic numbers form a field it follows that $\lambda_1 = \lambda_1 + \lambda_m - \lambda_m$ is algebraic. The proof is similar for any further coefficients.

Conversely suppose that the coefficients are algebraic numbers and consider their defining polynomials in the form

$$\Omega_1(t) := (t - \lambda_{1,1})(t - \lambda_{1,2}) \cdots (t - \lambda_{1,k_1}),$$

$$\Omega_2(t) := (t - \lambda_{2,1})(t - \lambda_{2,2}) \cdots (t - \lambda_{2,k_2}), \ldots,$$

$$\Omega_{m-1}(t) := (t - \lambda_{m-1,1})(t - \lambda_{m-1,2}) \cdots (t - \lambda_{m-1,k_{m-1}}),$$

$$\Omega_m(t) := (t - \lambda_{m,1})(t - \lambda_{m,2}) \cdots (t - \lambda_{m,k_m}),$$

where $\lambda_{i,1} := \lambda_i$ and for any further index j, $\lambda_{i,j}$ are the algebraic conjugates of λ_i. Let

$$P(x_1, \ldots, x_m)$$
$$:= \prod_{i_1=1}^{k_1} \prod_{i_2=1}^{k_2} \cdots \prod_{i_m=1}^{k_m} (x_m - \lambda_{1,i_1}x_1 - \cdots - \lambda_{m-1,i_{m-1}}x_{m-1} - \lambda_{m,i_m}) \quad (2.2)$$

which obviously vanishes at the points of the hyperplane. On the other hand, for any fixed $x_1, \ldots, x_m$, it can be considered as a symmetric polynomial of the variables $\lambda_{1,1}, \ldots, \lambda_{1,k_1}$. Using the fundamental theorem of symmetric polynomials it has a unique representation as the polynomial of the elementary symmetric polynomials

$$E_0(\lambda_{1,1}, \ldots, \lambda_{1,k_1}) = 1,$$
$$E_1(\lambda_{1,1}, \ldots, \lambda_{1,k_1}) = \lambda_{1,1} + \cdots + \lambda_{1,k_1},$$
$$E_2(\lambda_{1,1}, \ldots, \lambda_{1,k_1}) = \lambda_{1,1}\lambda_{1,2} + \cdots + \lambda_{1,1}\lambda_{1,k_1} + \lambda_{1,2}\lambda_{1,3} + \ldots$$
$$+ \lambda_{1,2}\lambda_{1,k_1} + \cdots + \lambda_{1,k_1-1}\lambda_{1,k_1}, \ldots$$
$$E_{k_1}(\lambda_{1,1}, \ldots, \lambda_{1,k_1}) = \lambda_{1,1}\lambda_{1,2} \ldots \lambda_{1,k_1}.$$

According to the relations between the coefficients of the polynomials and its roots we have that P is a polynomial of the variables $x_1, x_2, \ldots, x_m, \lambda_{2,1}, \ldots, \lambda_{2,k_2}, \ldots,$ λ_{m,k_m} with rational coefficients. Repeating the procedure as above we have that P is an element of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_m]$.   □

Example 2.4. Consider the line $x_2 = \sqrt{3}x_1 + \sqrt{2}$ in $\mathbb{R}^2$. An easy calculation shows that $P(x_1, x_2) = x_2^4 - 6x_1^2x_2^2 - 4x_2^2 + 9x_1^4 - 12x_1^2 + 4$.

3. Sufficient conditions for the existence of non-trivial solutions

Before we can state our main theorems in their final forms we need the following result like Theorem 3.2 in [11].

Lemma 3.1. *Let $3 \leq n \in \mathbb{N}$ be arbitrarily fixed and $\beta_i, \delta_i, \beta_n \in \mathbb{R}$ be nonzero real numbers, $i = 1, \ldots, n-1$. If there exists a field isomorphism*

$$\delta : \mathbb{Q}\left(\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}\right) \to \mathbb{Q}\left(\delta_1, \ldots, \delta_{n-1}\right)$$

such that

$$\delta\left(\frac{\beta_i}{\beta_n}\right) = \delta_i \quad and \quad \frac{\alpha_1}{\alpha_n}\delta_1 + \cdots + \frac{\alpha_{n-1}}{\alpha_n}\delta_{n-1} = -1, \tag{3.1}$$

where $i = 1, \ldots, n-1$, then there exists a not identically zero additive function $A : \mathbb{R} \to \mathbb{R}$ such that $\sum_{i=1}^{n} \alpha_i\, A(\beta_i x) = 0$ $(x \in \mathbb{R})$.

PROOF. Consider $\mathbb{R}$ as the vector space over $\mathbb{Q}\left(\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}\right)$ with the basis $\mathcal{H}$. Define $A : \mathbb{R} \to \mathbb{R}$ as follows: on the elements of $\mathcal{H}$ we define it arbitrarily and for $x = \sum_j c_j h_j$, where $c_j \in \mathbb{Q}\left(\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}\right)$ and $h_j \in \mathcal{H}$, let $A(x) := \sum_j \delta(c_j) A(h_j)$; it is easy to see that A is additive and for any $i = 1, \ldots, n-1$

$$A\left(\frac{\beta_i}{\beta_n}x\right) = \delta_i A(x) \quad (x \in \mathbb{R}).$$

Indeed,

$$A\left(\frac{\beta_i}{\beta_n}x\right) = A\left(\sum_j \frac{\beta_i}{\beta_n}c_j h_j\right) = \sum_j \delta\left(\frac{\beta_i}{\beta_n}c_j\right) A(h_j)$$

$$= \sum_j \delta_i \delta(c_j) A(h_j) = \delta_i \sum_j \delta(c_j) A(h_j) = \delta_i A(x)$$

holds for all $x \in \mathbb{R}$, where $i = 1, \ldots, n-1$. Therefore

$$\frac{\alpha_1}{\alpha_n}A\left(\frac{\beta_1}{\beta_n}x\right) + \cdots + \frac{\alpha_{n-1}}{\alpha_n}A\left(\frac{\beta_{n-1}}{\beta_n}x\right) + A(x)$$

$$= \left(\frac{\alpha_1}{\alpha_n}\delta_1, \ldots, + \frac{\alpha_{n-1}}{\alpha_n}\delta_{n-1} + 1\right)A(x) = 0$$

for all $x \in \mathbb{R}$, i.e. $\sum_{i=1}^{n} \frac{\alpha_i}{\alpha_n}A\left(\frac{\beta_i}{\beta_n}x\right) = 0$. Multiplying by α_n and substituting x by $\beta_n x$ this equation is equivalent to $\sum_{i=1}^{n} \alpha_i\, A(\beta_i x) = 0$ $(x \in \mathbb{R})$ which was to be stated. $\square$

Theorem 3.2. *Suppose that $n \geq 3$. If the parameters $\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}$ are algebraically independent and at least one of the parameters $\frac{\alpha_1}{\alpha_n}, \ldots, \frac{\alpha_{n-1}}{\alpha_n}$ is transcendental then equation (1.1) always has a nontrivial additive solution which is semi-homogeneous in the sense that $A(\frac{\beta_i}{\beta_n}x) = \delta_i A(x)$ for some δ_i's, where $x \in \mathbb{R}$ and $i = 1, \ldots n-1$.*

PROOF. Let $\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}$ be fixed such that they are algebraic independent. According to Lemma 1.3 and Lemma 3.1 equation (1.1) has nontrivial additive solutions if there exist $\delta_1, \ldots, \delta_{n-1} \in \mathbb{R}^{n-1}$ such that they satisfy the equation

$$\frac{\alpha_1}{\alpha_n}x_1 + \cdots + \frac{\alpha_{n-1}}{\alpha_n}x_{n-1} = -1 \tag{3.2}$$

and the defining ideal of $\vec{\delta} := (\delta_1, \ldots, \delta_{n-1})$ contains only the zero polynomial. Because of Lemma 2.3 the hyperplane defined by (3.2) is not algebraic which means, by Lemma 2.1 that it is not the union of zero's of nontrivial elements of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_{n-1}]$. This implies the existence of δ_i's as was to be stated. $\qquad\square$

Theorem 3.3. *Suppose that $n \geq 3$. If the parameters $\frac{\alpha_1}{\alpha_n}, \ldots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraically independent and at least one of the parameters $\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}$ is transcendental then equation (1.1) always has nontrivial additive solution which is semi-homogeneous in the sense that $A(\delta_i x) = \frac{\alpha_i}{\alpha_n} A(x)$ for some δ_i's, where $x \in \mathbb{R}$ and $i = 1, \ldots n - 1$.*

PROOF. The proof is similar to that of Theorem 3.2 using the field isomorphism $\delta : \mathbb{Q}(\delta_1, \ldots, \delta_{n-1}) \to \mathbb{Q}(\frac{\alpha_1}{\alpha_n}, \ldots, \frac{\alpha_{n-1}}{\alpha_n})$ such that

$$\delta(\delta_i) = \frac{\alpha_i}{\alpha_n} \quad \text{and} \quad \delta_1 \frac{\beta_1}{\beta_n} + \cdots + \delta_{n-1} \frac{\beta_{n-1}}{\beta_n} = -1, \tag{3.3}$$

where $i = 1, \ldots, n - 1$. $\qquad\square$

Remark 3.4. First of all note that β_n and α_n can be substituted with any other coefficients β_i and α_i, respectively, where $i = 1, \ldots, n - 1$. On the other hand the reasoning in the proofs shows that points with algebraic independent coordinates can be find almost everywhere on the hyperplanes defined by (3.2).

Remark 3.5. If a not identically zero additive function A is semi-homogeneous in the sense that

$$A\left(\frac{\beta_i}{\beta_n} x\right) = \delta_i A(x) \quad (x \in \mathbb{R})$$

with some δ_i's, $i = 1, \ldots, n-1$ then for any $\beta \in \mathbb{Q}(\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n})$, $A(\beta x) = \delta A(x)$ with some $\delta \in \mathbb{Q}(\delta_1, \ldots, \delta_{n-1})$. Explicitly, if

$$\beta = \frac{w\left(\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}\right)}{k\left(\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}\right)}, \quad \text{then} \quad \delta = \frac{w(\delta_1, \ldots, \delta_{n-1})}{k(\delta_1, \ldots, \delta_{n-1})},$$

where w, k are the elements of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_{n-1}]$. According to Lemma 3.1 in [11] the fields

$$\mathbb{Q}\left(\frac{\beta_1}{\beta_n}, \ldots, \frac{\beta_{n-1}}{\beta_n}\right) \quad \text{and} \quad \mathbb{Q}(\delta_1, \ldots, \delta_{n-1})$$

are isomorphic to each other. Like the *homogeneity field* [5] we can define the *inner semi-homogeneity field*

$$IS(A) := \{\beta \in \mathbb{R} \mid A(\beta x) = \delta A(x) \text{ for all real number } x\}$$

and the *outer semi-homogeneity field*

$$OS(A) := \{\delta \in \mathbb{R} \mid A(\beta x) = \delta A(x) \text{for all real number } x\}.$$

The semi-homogeneity field is unique in the sense that $IS(A)$ and $OS(A)$ is isomorphic. According to the rational homogeneity of the additive functions $\mathbb{Q} \subset IS(A) \cap OS(A)$.

In terms of the semi-homogeneity field the following converse of Theorem 3.2 is natural.

Theorem 3.6. *Suppose that $n \geq 3$. If the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ are algebraically independent and equation (1.1) has a nontrivial semi-homogeneous additive solution with $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ in its inner semi-homogeneity field, then at least one of the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ is transcendental.*

In a similar way we have the following result.

Theorem 3.7. *Suppose that $n \geq 3$. If the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraically independent and equation (1.1) has a nontrivial semi-homogeneous additive solution with $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ in its outer semi-homogeneity field then at least one of the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ is transcendental.*

4. Examples for the remaining cases

In the previous section we formulated existence theorems for the nontrivial solution of equation (1.1) under certain conditions. According to Theorems 3.2 and 3.3 we have the following remaining cases:

(i) $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ and $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebrically dependent,

(ii) $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ are algebraic independent and all of the parameters $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraic, or

(iii) $\frac{\alpha_1}{\alpha_n}, \dots, \frac{\alpha_{n-1}}{\alpha_n}$ are algebraic independent and all of the parameters $\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}$ are algebraic.

Here we investigate only an example for the case (ii). Consider the functional equation

$$\sqrt{d_1}A\left(\frac{\beta_1}{\beta_3}x\right) + \sqrt{d_2}A\left(\frac{\beta_2}{\beta_3}x\right) + A(x) = 0,$$

where d_1 and d_2 are positive rational and the parameters $\frac{\beta_1}{\beta_3}$, $\frac{\beta_2}{\beta_3}$ are algebraic independent. It is equivalent to equation (1.1) under $n = 3$, $\frac{\alpha_1}{\alpha_3} = \sqrt{d_1}$, $\frac{\alpha_2}{\alpha_3} = \sqrt{d_2}$. To solve the functional equation consider the following procedure. Multiplying by $\sqrt{d_1}$

$$A\left(d_1\frac{\beta_1}{\beta_3}x\right) + \sqrt{d_1 d_2}A\left(\frac{\beta_2}{\beta_3}x\right) + \sqrt{d_1}A(x) = 0.$$

Substituting $\frac{\beta_1}{\beta_3}x$ we have

$$A\left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 x\right) + \sqrt{d_1 d_2}A\left(\frac{\beta_1\beta_2}{\beta_3^2}x\right) + \sqrt{d_1}A\left(\frac{\beta_1}{\beta_3}x\right) = 0. \qquad (4.1)$$

In a similar way

$$\sqrt{d_1 d_2}A\left(\frac{\beta_1}{\beta_3}x\right) + A\left(d_2\frac{\beta_2}{\beta_3}x\right) + \sqrt{d_2}A(x) = 0$$

and

$$\sqrt{d_1 d_2}A\left(\frac{\beta_1\beta_2}{\beta_3^2}x\right) + A\left(d_2\left(\frac{\beta_2}{\beta_3}\right)^2 x\right) + \sqrt{d_2}A\left(\frac{\beta_2}{\beta_3}x\right) = 0 \qquad (4.2)$$

follows immediately. Taking the difference of (4.2) and (4.1)

$$\sqrt{d_2}A\left(\frac{\beta_2}{\beta_3}x\right) - \sqrt{d_1}A\left(\frac{\beta_1}{\beta_3}x\right) = A\left(\left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2\right)x\right).$$

On the other hand

$$\sqrt{d_1}A\left(\frac{\beta_1}{\beta_3}x\right) + \sqrt{d_2}A\left(\frac{\beta_2}{\beta_3}x\right) = -A(x)$$

as the original functional equation shows. Taking the sum

$$2\sqrt{d_2}A\left(\frac{\beta_2}{\beta_3}x\right) = A\left(\left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 - 1\right)x\right).$$

Substituting $\frac{\beta_3}{2\beta_2}x$ we have that

$$\sqrt{d_2}A(x) = A\left(\left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 - 1\right)\frac{\beta_3}{2\beta_2}x\right)$$

which means, by Daróczy theorem, that

$$\sqrt{d_2} \ \text{ and } \ \left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 - 1\right)\frac{\beta_3}{2\beta_2},$$

are algebraic conjugate to each other, i.e.

$$\pm\sqrt{d_2} = \left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 - 1\right)\frac{\beta_3}{2\beta_2}$$

provided that A is not identically zero. In a similar way (taking the difference instead of the sum) it follows that

$$-2\sqrt{d_1}A\left(\frac{\beta_1}{\beta_3}x\right) = A\left(\left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 + 1\right)x\right)$$

and, consequently, $\pm\sqrt{d_1} = \left(d_1\left(\frac{\beta_1}{\beta_3}\right)^2 - d_2\left(\frac{\beta_2}{\beta_3}\right)^2 + 1\right)\frac{\beta_3}{2\beta_1}$ provided that A is not identically zero. It can be easily seen that both of the final results contradicts to the condition of algebrically independence of the parameters $\frac{\beta_1}{\beta_3}$ and $\frac{\beta_2}{\beta_3}$. Therefore the only solution is the identically zero function. The method can be easily generalized for solving the functional equation

$$\frac{\alpha_1}{\alpha_3}A\left(\frac{\beta_1}{\beta_3}x\right) + \frac{\alpha_2}{\alpha_3}A\left(\frac{\beta_2}{\beta_3}x\right) + A(x) = 0$$

if the parameters $\frac{\alpha_1}{\alpha_3}$ and $\frac{\alpha_2}{\alpha_3}$ are algebraic of degree at most 2. Then all of them have the form $\frac{\alpha_i}{\alpha_3} = r_i + s_i\sqrt{d_i}$, where r_i, s_i and d_i are rational and $i = 1, 2$.

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