

## Geometry of space-time and generalized Lagrange spaces

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*To the memory of Professor András Rapcsák*

**Abstract.** It has recently been shown that starting from the Finslerian geometrical framework and demanding Riemannian conformal and projective structures together with the constancy of norm of vectors under parallel displacement do not reduce the geometry of the space-time to be Riemannian, but there results rather a Berwald space subject to a constraint. Here we study what happens if the starting geometrical framework is further generalized to be that of a generalized Lagrange space  $GL^n$  (introduced by the first author, [4]). In particular, we employ the  $GL^n$  space endowed with fundamental tensor  $g_{ij}(x, y) = e^{2\sigma(x, y)}\gamma_{ij}(x)$  and study the consequences of imposing the above conditions. We determine the canonical metrical  $d$ -connection for this space and study some of its properties with the help of concrete examples. This further generalization can be of potential importance, especially in view of the recent discovery of the relevance of such generalized spaces in the study of  $W$ -gravity, [8].

### 1. Introduction

The point of view that the underlying geometry of space-time is Riemannian, is usually thought to be further strengthened by the constructive-axiomatic formulation of general relativity given by EHLERS, PIRANI and SCHILD, [1]. Within this scheme the geometry of space-time is viewed in terms of the main substructures: conformal and projective structures, which are in turn thought to be fixed by light propagation and freely falling non-rotating neutral test particles respectively. The main result of EPS (EPS means “Ehlers, Pirani and Schild”) is that these substructures together with the constancy of the norm of vectors under parallel displacement are sufficient to ensure the Riemannian nature of space-time.

Recently, the second author together with VAN DEN BERGH, [2,3] has shown: A necessary and sufficient condition for a Finsler space endowed with the fundamental function  $F(x, y) = e^{2\sigma(x, y)}\gamma_{ij}(x)y^iy^j$  to satisfy the EPS conditions is for the function  $\sigma(x, y)$  to satisfy the equation

$$(1.1) \quad \frac{\partial \sigma}{\partial x^i} - \frac{\partial \sigma}{\partial y^k} \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} y^j = 0,$$

where  $\gamma_{ij}(x)$  is a Riemannian metric and  $\{\overset{i}{j}k\}$  are its Christoffel symbols.

The question then arises as to what happens if the starting geometrical framework is allowed to be even more general than the Finslerian one, namely a generalized Lagrange space.

Before turning to this question we give a brief account of these spaces, sufficient for our purposes here.

A generalized Lagrange space  $GL^n$  is a pair  $(M, g_{ij}(x, y))$  formed by the  $C^\infty$ -real  $n$ -dimensional differentiable manifold  $M$  and a  $d$ -tensor field (where “ $d$ ” denotes “distinguished”)  $g_{ij}(x, y)$  differentiable on  $TM = TM \setminus \{0\}$ , covariant of order 2, symmetric with the rank  $\|g_{ij}(x, y)\| = n$  and of constant signature, called the fundamental tensor field of  $GL^n$ . Clearly a Finsler space  $F^n$  or a Lagrange space  $L^n$  is a  $GL^n$  space. A  $GL^n$  is said to be reducible to a Lagrange space if there exists a regular Lagrangian

$L : TM \rightarrow R$  such that  $g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}$ . It is important to note that, as opposed to the Finslerian case, the fundamental tensor field  $g_{ij}$  is not homogeneous in  $y^i$ . The geometry of  $GL^n$  can be developed by the same methods as those employed in Lagrange spaces, [7].

Let  $N$  be a nonlinear connection on  $TM$  with the local coefficients  $N^i_j(x, y)$ . Then the vector fields  $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ , ( $i = 1, \dots, n$ ), with

$$(1.2) \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$$

are a local basis of the module of vector fields on  $\widetilde{TM}$ . The dual basis is given by  $(dx^i, \delta y^i)$ , where

$$(1.2)' \quad \delta y^i = dy^i + N^i_j dx^j.$$

Then, the autoparallel curves of the nonlinear connection  $N$  are given by  $\frac{\delta y^i}{dt} = 0$ ,  $\frac{dx^i}{dt} = y^i$ . The absolute energy of  $GL^n$  is

$$(1.3) \quad \mathcal{E}(x, y) = g_{ij}(x, y) y^i y^j.$$

In a generalized Lagrange space  $GL^n$  endowed with a nonlinear connection  $N$  we can deduce a so called  $d$ -connection, [7]  $CT(N) = (L^i_{jk}, C^i_{jk})$ . This allows the  $h$ - and  $v$ -covariant derivatives, denoted by “ $|$ ” and “ $|$ ” respectively. For example the  $h$ - and  $v$ -covariant derivatives of the fundamental tensor field  $g_{ij}$  are given by:

$$(1.4) \quad g_{ij|k} = \frac{\delta g_{ij}}{\delta x^k} - L^s_{ik} g_{sj} - L^s_{jk} g_{is}; \quad g^{ij}|_k = \frac{\partial g^{ij}}{\partial y^k} - C^s_{ik} g^{sj} - C^s_{jk} g^{is}.$$

Denoting by  $g^{ij}(x, y)$  the reciprocal of  $g_{ij}(x, y)$  we have:

In the generalized Lagrange space  $GL^n = (M, g_{ij}(x, y))$  there exists a unique  $d$ -connection  $CT(N)$  with the properties: (a)  $g_{ij|k} = 0$ ,  $g_{ij}|_k = 0$ ; (b)  $T^i_{jk} = S^i_{jk} = 0$ . The coefficients of the  $d$ -connection  $CT(N)$  are given by the generalized Christoffel symbols

$$(1.5) \quad \begin{aligned} L^i_{jk} &= \frac{1}{2} g^{is} \left( \frac{\delta g_{js}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s} \right), \\ C^i_{jk} &= \frac{1}{2} g^{is} \left( \frac{\partial g_{js}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right). \end{aligned}$$

Therefore, once  $N$  is given,  $CT(N)$  is determined solely by  $g_{ij}(x, y)$ . It is called the *canonical metrical  $d$ -connection* of the space  $GL^n$ . In particular, if  $GL^n$  is a Finsler space  $F^n$  and  $N$  is the nonlinear Cartan connection, then  $CT(N)$  is the metrical Cartan connection.

The connection  $CT(N)$  has three torsion tensor fields and three curvature tensor fields.

Now some new remarks and properties. A  $d$ -tensor  $t$  is called  $h$ -covariant constant ( $h$ -c.c.) if  $t|_k = 0$ . If  $t$  is a scalar field on  $\widetilde{TM}$  and has the property  $t|_k = 0$ , it is said to be  $h$ -constant ( $h$ -c.).

Now a  $h$ -c function  $\sigma(x, y)$  is a solution of the equation

$$(1.6) \quad \frac{\partial \sigma}{\partial x^i} - N^j_i(x, y) \frac{\partial \sigma}{\partial y^j} = 0,$$

which in the case  $N^i_j = \{^i_{jk}\} y^k$  is given by (1.1). This makes sense geometrically as  $\sigma|_k$  is a covector field and equation (1.6) is given by  $\sigma|_k = 0$ . We have the following results concerning  $h$ -c functions:

**Proposition 1.1.** *If the Liouville vector field  $c = y^i \frac{\partial}{\partial y^i}$  is  $h$ -c.c. then the absolute energy (1.3) of the space  $GL^n$  is an  $h$ -c function.*

It is more important that we have.

**Theorem 1.1.** *Every  $h$ -c function  $\sigma(x, y)$  is a constant on the autoparallel curves of the nonlinear connection  $N$ .*

PROOF. Recall that the differential  $d\sigma$  of the function  $\sigma(x, y)$  can be written in the form  $d\sigma = \sigma|_k dx^k + \sigma|_k \delta y^k$  and that on an autoparallel curve of  $N$  we have  $\frac{\delta y^k}{dt} = 0$ ,  $y^k = \frac{dx^k}{dt}$ . The condition  $\sigma|_k = 0$  then implies  $d\sigma = 0$ .

Now considering the following 2-form on  $\widetilde{TM}$ :  $\theta = g_{ij}(x, y) \delta y^i \wedge dx^j$  we have: The pair  $(\widetilde{TM}, \theta)$  is an almost symplectic space.

Generally  $\theta$  is not integrable. If, however,  $GL^n$  is a Finsler or a Lagrange space and  $N$  is a canonical nonlinear connection, then  $\theta$  is a closed 2-form. The above almost symplectic structure  $\theta$  imposes an almost Hermitian space  $H^{2n} = (\widetilde{TM}, G, \mathbb{F})$  which is *the almost Hermitian model* of the generalized Lagrange space  $GL^n$ .

## 2. Geometry of space-time and generalized Lagrange spaces

In this section we show that the EPS conditions can be satisfied by a special generalized Lagrange space  $GL^n$  endowed with a nonlinear connection uniquely determined by its fundamental tensor  $g_{ij}$ .

Let us start with a  $GL^n = (M, g_{ij}(x, y))$  and assume that the following axioms hold:

(a.1) The fundamental tensor field  $g_{ij}(x, y)$  is of the form

$$(2.1) \quad g_{ij}(x, y) = e^{2\sigma(x, y)} \gamma_{ij}(x)$$

where  $\sigma : TM \rightarrow R$  is a  $C^\infty$  function on  $\widetilde{TM}$ , at least continuous on the null section 0, and  $\gamma_{ij}(x)$  is the metric tensor of a given Riemannian space  $\mathcal{R}^n$ .

It should be noted here that these assumptions also eliminate the usual complications related to the causal structure of the space  $\mathcal{R}^n$  and the related problem of nullity of vectors. Furthermore,  $\sigma(x, y)$  does not satisfy any homogeneity conditions with respect to  $y^i$ .

(a.2) The space  $GL^n$  is endowed with the nonlinear connection  $N$  with the coefficients

$$(2.2) \quad N^i_j(x, y) = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} (x) y^k.$$

where  $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$  are the Christoffel symbols of the Riemannian metric  $\gamma_{ij}(x)$ .

(a.3) The space  $GL^n$  is endowed with the canonical metrical  $d$ -connection  $CT(N)$ .

We then have

**Theorem 2.1.** *Axioms (a.1), (a.2) and (a.3) are equivalent to EPS conditions.*

PROOF. Axiom (a.1) affirms that the space  $GL^n$  with the metric given in (2.1) has the same conformal structure as the Riemannian space specified by  $\gamma_{ij}(x)$ . Axiom (a.2) shows (cf. section 1) that the autoparallel curves of the nonlinear connection  $N$  with the coefficients (2.2) are coincident with the autoparallel curves of the Riemannian space  $\mathcal{R}^n$ . And finally, the constancy of the norm of vectors under parallel transport is ensured by the metricity conditions satisfied by the canonical metrical  $d$ -connection  $CT(N)$  mentioned in axiom (a.3).

It is worthwhile to give a geometrical interpretation of the axiom (a.1). Let  $(\Gamma, \gamma)$  be a smooth parametrized curve  $\gamma : t \in I \rightarrow x(t) \in M$  and  $\text{Im}\gamma = \Gamma$ . Using the tangent mapping of  $\gamma$ , we consider the curve  $(\tilde{\Gamma}, \tilde{\gamma})$ , on  $\widetilde{TM}$ , given by  $\tilde{\gamma} : t \in I \rightarrow (x(t), \dot{x}(t)) \in \widetilde{TM}$  and  $\tilde{\Gamma} = \text{Im}\tilde{\gamma}$ . Therefore the element of the arc of the parametrized curve  $(\Gamma, \gamma)$  in  $\mathcal{R}^n$  is  $d_r s^2 = \gamma_{ij}(x) \dot{x}^i \dot{x}^j dt^2$  and the element of the arc of the parametrized curve  $(\tilde{\Gamma}, \tilde{\gamma})$  in  $GL^n$  is  $ds^2 = g_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j dt^2$ . Axiom (a.1) which implies  $ds^2 = e^{2\sigma(x, y)} d_r s^2$  can then be interpreted geometrically as representing a homothety (a gauge transformation, dilatation or contraction). The homothety is fixed in terms of  $\sigma(x, y)$  which in turn is to be specified by the physical nature of the space. The parameter  $t$  can also be given physically, for example it may be the proper time.

We close this section with a number of *examples* for the functional form of  $\sigma$ .

(I)  $\sigma(x, y)$  may be taken to be any solution of the equation (1.1), which is in turn strongly related to the axiom (a.2).

(II) Let  $A_i(x)$  be an electromagnetic covector field on  $M$ . Then we may choose

$$(2.3) \quad \sigma(x, y) = A_i(x) y^i.$$

(III) Recalling the Liouville vector field, we can choose

$$(2.4) \quad \sigma(x, y) = \gamma_{ij}(x) y^i y^j.$$

(IV) If  $(M, V^i(x), n(x, V(x)))$  is a dispersive medium, [6], with  $V^i(x)$  the velocity of the particle and  $n(x, V(x))$  the refractive index, we may consider

$$(2.5) \quad \sigma(x, y) = \alpha \left( 1 - \frac{1}{n^2(x, y)} \right), \quad \alpha \in R^*_{+}.$$

The metric tensor  $g_{ij}(x, y) = e^{2\alpha \left( 1 - \frac{1}{n^2(x, y)} \right)} \gamma_{ij}(x)$  can serve as a new theory of relativistic optics, which reduces to the classical theory obtained when  $\frac{\partial \sigma}{\partial y^i} = 0$ , (i.e. the medium is non-dispersive). When  $\frac{\partial \gamma_{ij}}{\partial x^k} = 0$ , and  $\sigma(x, y)$  is a solution of equation (1.1), then the medium is strongly dispersive, (assuming  $\frac{\partial \sigma}{\partial y^i} \neq 0$ ). Indeed, if  $\mathcal{R}^n$  is a Minkowski metric, then  $\sigma$ , with  $\frac{\partial \sigma}{\partial y^i} \neq 0$ , satisfies (1.1) if and only if  $\frac{\partial \sigma}{\partial x^i} = 0$ . But in this case  $\sigma$  is purely  $y^i$  (or velocity) dependent.

It is worthwhile to note that the axioms (a.1)–(a.3) imply

**Proposition 2.1.** *The nonlinear connection  $N$  with the coefficients (2.2) is uniquely determined by the fundamental tensor  $g_{ij}(x, y)$  of the space  $GL^n$ .*

We can prove, without difficulties

**Theorem 2.2.** *The generalized Lagrange space  $GL^n$  with the fundamental tensor (2.1) and  $\frac{\partial \sigma}{\partial y^i} \neq 0$  is not reducible to a Lagrange or a Finsler space.*

### 3. Canonocal metrical $D$ -connection of the space $GL^n$

In this section we briefly study the canonical metrical  $d$ -connection  $CT(N)$  postulated in (a.3). We have

**Theorem 3.1.** *The canonical metrical  $d$ -connection  $CT(N)$  of the space  $GL^n$  has the coefficients*

$$(3.1) \quad \begin{aligned} L_{jk}^i &= \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \delta_j^i \sigma_k + \delta_k^i \sigma_j - g_{jk} \sigma^i; \\ C_{jk}^i &= \delta_j^i \dot{\sigma}_k + \sigma_k^i \dot{\sigma}_j - g_{jk} \dot{\sigma}^i, \end{aligned}$$

where  $\sigma_k = \sigma|_k$ ,  $\sigma^i = g^{ij} \sigma_j$ ,  $\dot{\sigma}_k = \frac{\partial \sigma}{\partial y^k}$ ,  $\dot{\sigma}^i = g^{ij} \dot{\sigma}_j$ .

Now clearly  $CT(N)$  has the properties  $g_{ij|k} = 0$ ,  $g_{ij}|_k = 0$ .

The torsion of  $CT(N)$  is given by  $C_{jk}^i$  in (3.1) and by

$$(3.2) \quad R_{hk}^i = y^j \mathcal{R}_j^i{}_{hk}, \quad P_{jk}^i = -(\delta_j^i \sigma_k + \delta_k^i \sigma_j - g_{jk} \sigma^i),$$

where  $\mathcal{R}_j^i{}_{hk}$  is the curvature tensor of the Riemannian metric  $\gamma_{ij}(x)$ . A further interesting property of  $CT(N)$ , which shows the importance of the result established by R. Tavakol and Van den Bergh, [2,3], is given by

**Theorem 3.2.** *The canonical metrical  $d$ -connection  $CT(N)$  has the property  $L_{jk}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ , if and only if the function  $\sigma(x, y)$  is a solution of the equation (1.1).*

PROOF. Using (3.1), the property  $L_{jk}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$  holds if and only if  $\delta_j^i \sigma_k + \delta_k^i \sigma_j - g_{jk} \sigma^i = 0$ . Contracting the indices  $i$  and  $j$  gives  $\sigma_k = 0$ , which, by using  $\sigma_k = \frac{\delta \sigma}{\delta x^k}$ , is equivalent to (1.1).

Theorem 1.1 has as consequence the

**Theorem 3.3.** *If the function  $\sigma(x, y)$  is a solution of the equation (1.1), then the absolute energy  $\mathcal{E}(x, y)$  of the space  $GL^n$  is also a solution of this equation. Furthermore, the absolute energy is constant on the autoparallel curves of the nonlinear connection  $N$ .*

**Proposition 3.1.** *If  $\frac{\partial \sigma}{\partial y^k} \neq 0$ , then the coefficients  $C_{jk}^i$  of  $CT(N)$  cannot vanish.*

Also, regarding the  $d$ -tensor  $R_{jk}^i$  in (3.2) we obtain

**Theorem 3.4.** *The nonlinear connection  $N$  of the space  $GL^n$  is integrable if and only if the Riemannian space  $\mathcal{R}^n$  is locally Euclidean.*

We end this section with two *examples*:

(I) If  $\sigma(x, y) = A_i(x)y^i$  and  $A_{i;j} = 0$ , where “;” denotes covariant differentiation with respect to the Christoffel symbols  $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ , then we obtain:

- (a)  $\sigma(x, y)$  satisfies the equation (1.1).
- (b) The coefficients of the canonical metrical connection  $CT(N)$  are  $L_{jk}^i = \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ ,  $C_{jk}^i = \delta_j^i A_k + \delta_k^i A_j - g_{jk} A^i$ , ( $A^i = g^{ij} A_j$ ).
- (c) The absolute energy  $\mathcal{E}(x, y) = e^{2A_k(x)y^k} \gamma_{ij} y^i y^j$  satisfies (1.1).
- (d)  $\mathcal{E}(x, y)$  is a constant on the autoparallels of  $N$ .

(II) If  $\sigma(x, y)$  is given by (2.4), then:

- (a)  $\sigma(x, y)$  satisfies the equation (1.1).
- (b) The coefficients of  $CT(N)$  are given by

$$L_{jk}^i = \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}, \quad C_{jk}^i = \delta_j^i y_k + \delta_k^i y_j - g_{jk} g^{is} y_s, \quad (y_i = \gamma_{ij} y^j).$$

- (c)  $\mathcal{E}(x, y) = e^{\|y\|^2} \|y\|^2$ , ( $\|y\|^2 = \gamma_{ij} y^i y^j$ ) satisfies (1.1).
- (d)  $\mathcal{E}(x, y)$  is a constant on the autoparallels of  $N$ .

#### 4. Conclusion

The results of the previous sections show that the usual EPS criteria are not sufficient to fix the geometry of the space-time to be Riemannian. Furthermore, the comparison between the results here and those in [2,3] show that the more general the starting geometrical framework, the more general will be the non-Riemannian geometrical framework that can be made compatible with EPS conditions. This could have potentially important consequences in generalizations of general relativity, especially in the light of the recent discovery of the relevance of such a generalized geometrical framework in the study of  $W$ -gravity [8].

It is clearly of interest to develop further the ideas presented in this paper by determination of the curvature tensor fields, Einstein equations

within  $GL^n$  with the canonical metrical  $d$ -connection (3.1), and the corresponding electromagnetic tensor fields and the Maxwell equations, as well as post-Newtonian approximation. We shall return to these questions in future publications, in collaboration with Prof. I. ROXBURGH and Prof. V. BALAN.

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