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Einstein Landsberg metrics

By NASRIN SADEGHZADEH (Qom), ASADOLLAH RAZAVI (Tehran)
and BAHMAN REZAEI (Urmia)

Abstract. The paper focuses on the study of Ricci curvature of Einstein Finsler metrics in order to prove the Schur type lemma for Einstein spaces. In this paper, Einstein Finsler metrics of Berwald and SCR type are mainly considered, we will prove that the Ricci scalar of any Einstein Landsberg metric of SCR type is necessarily constant, we will also show that every Einstein Finsler metric of SCR type in dimension 3 is of sectional (flag) curvature. Moreover we will prove that every Einstein Berwald space of non-zero Ricci scalar is Riemannian.

1. Introduction

The Einstein metrics comprise a major focus in differential geometry, these metrics are more general than those with constant curvature. The well-known Ricci tensor was introduced in 1904 by G. Ricci, which was used to formulate the Einstein's theory of gravitation in 1913. The so-called Einstein manifolds whose Ricci tensors are proportional to the metric have been studied extensively, specially in general relativity.

Define the Ricci scalar by

$$R = \frac{\mathrm{Ric}(x,y)}{F^2} := \frac{R^i_i}{F^2},$$

where R^i_j are the coefficients of the spray curvature on M. In fact we can think of Ric as $(n-1)$ times the average curvature at x in the direction y.

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In Finsler geometry an Einstein metric is a metric F with the property

$$\mathrm{Ric} = (n-1)K(x)F^2,$$

where K is a function on M.

We used AKBAR-ZADEH's version of Ricci tensor which is defined by [1]

$$\mathrm{Ric}_{ij} = \frac{1}{2}(\mathrm{Ric}(x,y))_{y^i y^j}.$$

The reciprocal relationship between the Ricci scalar and the Ricci tensor tells us that the Ricci tensor of an Einstein metric is

$$\mathrm{Ric}_{ij} = (n-1)K(x)g_{ij}.$$

F is said to be Ricci flat if the average $\left(\frac{\mathrm{Ric}}{F^2}\right)$ does not depend on the location x, in this case, the function K is constant.

For Riemannian case, this average does not depend on the location x as we have

Theorem ([9]). *If g is an Einstein Riemannian metric on a connected manifold of dimension $n \geq 3$, its scalar curvature is constant.*

The above theorem is known as Einstein Schur theorem. The following question can be raised in Finsler geometry when F is an Einstein metric:
Does the Einstein Schur theorem hold for an arbitrary Finsler metric?
In Finsler geometry this theorem is called Schur lemma, we have chosen this terminology throughout the paper. ROBLES succeeded to prove this lemma for Randers metrics [8]. In this paper we are going to generalize this lemma for more Finsler metrics. A straight answer to this question is to find Einstein Finsler metrics, which are reduced to the Riemannian cases or Finsler metrics of constant flag curvature. In this paper we consider Finsler metrics with $R_{j\ mk}^{m} y^j = R_{k\ mj}^{m} y^j$ (so-called of SCR type), and our main results are as follows

Theorem 1.1. *Every Einstein Landsberg metric of SCR type on a compact manifold M is Ricci constant.*

Theorem 1.2. *Every Einstein Finsler metric of SCR type is of sectional (flag) curvature ($n = 3$).*

Some Einstein (α, β) metrics and projectively related Einstein metric have been considered in [14] and [19].

2. Preliminaries

Let M be a connected n-dimensional C^∞ manifold, denote the tangent space at $x \in M$ by T_xM, and the tangent bundle of M by $TM = \cup_{x\in M} T_xM$. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$, the natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$. The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber at $v \in TM_0$ is just T_xM, where $\pi(v) = x$, then

$$\pi^*TM = \{(x, y, v) \mid y \in T_xM_0,\ v \in T_xM\}$$

A Finsler metric on a manifold M is a function $F : TM \to [0, \infty)$ with the following properties:

(i) F is C^∞ on TM_0,

(ii) $F(x, \lambda y) = \lambda F(x, y)$ $x \in M$, $y \in T_xM$ and $\lambda > 0$,

(iii) For any tangent vector $y \in T_xM$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$g_{ij}(x, y) = \left[\frac{1}{2}F^2\right]_{y^iy^j},$$

is positive definite.

We obtain a symmetric tensor C, Cartan tensor, on π^*TM defined by

$$C(u, v, w) = C_{ijk}(y)u^iv^jw^k,$$

where $u = u^i\partial_i$, $v = v^i\partial_i$, $w = w^i\partial_i$ and $C_{ijk} = \frac{1}{4}[F^2]_{y^iy^jy^k}(y)$, here we have put $\partial_i := \frac{\partial}{\partial x^i}$. It is well known that $C = 0$ if and only if F is Riemannian [18]. Define

$$L_{ijk} = C_{ijk|m}y^m,$$

$$L_y(u, v, w) = L_{ijk}(y)u^iv^jw^k.$$

Definition 2.1. A Finsler metric F is called isotropic Landsberg if

$$L + c(x)FC = 0,$$

where $c(x)$ is a scalar function on M. The Finsler metric F is called Landsberg if $c(x) = 0$ i.e. $L = 0$.

Define

$$I_y(u) := \sum g^{ij}(y) C_y(b_i, b_j, u) = \sum g^{ij}(y) C_{ijk}(y) u^k,$$

where $\{b_i\}$ is an arbitrary basis for T_xM and $(g^{ij}(y)) := (g_{ij}(y))^{-1}$. The family $I = \{I_y\}_{y \in T_xM \setminus \{0\}}$ is called the mean Cartan torsion.

Definition 2.2. A Finsler metric F is called isotropic weakly Landsberg if

$$J + c(x) F I = 0,$$

where $J_i = I_{i|m} y^m$, $J_y(u) = \sum J_k(y) u^k$ and $c(x)$ is a scalar function on M. The Finsler metric F is called weakly Landsberg if $c(x) = 0$ i.e. $J = 0$.

Theorem 2.1 ([18]). *A Minkowski norm on a vector space V is Euclidean if and only if $I_y = 0$ for any $y \in V \setminus \{0\}$.*

Every Finsler metric F induces a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i}$ by

$$G^i(x,y) := \frac{1}{4} g^{il}(x,y) \left\{ 2 \frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y) \right\} y^j y^k.$$

Definition 2.3. A Finsler metric F on a manifold M is called Berwald metric if in a local coordinate system (x^i, y^i) on TM, the spray coefficients G^i are quadratic in $y \in T_xM$ for all $x \in M$.

The Riemann curvature $R_y = R^i_k dx^k \otimes \frac{\partial}{\partial x^i}|_p : T_pM \to T_pM$ is defined by

$$R^i_k(y) := 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \tag{1}$$

It has the following properties:
For any non-zero vectors $y, u, v \in T_pM$,

(a) $R_y(y) = 0$, $g_y(R_y(u), v) = g_y(u, R_y(v))$,

(b) $R^i_{kl} = \frac{1}{3} \{ \frac{\partial R^i_k}{\partial y_l} - \frac{\partial R^i_l}{\partial y_k} \}$,

(c) $R_j{}^i{}_{kl} = R^i_{kl.j} + L^i_{kj|l} - L^i_{lj|k} + L^i_{lm} L^m_{kj} - L^i_{km} L^m_{lj}$. (2)

Put $\mathrm{Ric}_{kl} = g^{jr} R_k{}^i{}_{jl} g_{ir}$ (cf. the Ricci tensor in Riemannian geometry) and also

$$\mathrm{Ric}_{0k} = \mathrm{Ric}_{lk}\, y^l, \quad \mathrm{Ric}_{k0} = \mathrm{Ric}_{kl}\, y^l.$$

Definition 2.4. A Finsler metric is called of Symmetric Corresponding Ricci tensor or Finsler metric of SCR type if $\mathrm{Ric}_{0k} = \mathrm{Ric}_{k0}$.

Example 1. Let $F = F(x, y, u, v)$ be a Finsler metric on an open subset $U \subset R^2$ in the form

$$F = e^{2\rho(x,y)} \exp\left[2Q \arctan\left(\frac{u}{v}\right)\right](u^2 + v^2),$$

where ρ is a function and $Q > 0$ is a constant. Then the mean Cartan torsion of F satisfies

$$I^2 = \frac{4Q^2}{1 + Q^2} < 4.$$

The spray coefficients $G^1 = G$ and $G^2 = H$ of F are given by

$$G = -\frac{1}{2(1 + Q^2)}\{(\rho_x - Q\rho_y)u^2 + 2(\rho_y + Q\rho_x)uv - (\rho_x - Q\rho_y)v^2\}, \tag{3}$$

$$H = -\frac{1}{2(1 + Q^2)}\{-(\rho_y + Q\rho_x)u^2 + 2(\rho_x - 2Q\rho_y)uv + (\rho_y + Q\rho_x)v^2\}. \tag{4}$$

Its Ricci scalar is

$$R = -\frac{\rho_{.x.x} + \rho_{.y.y}}{1 + Q^2}(u^2 + v^2).$$

hence F is a Berwald metric. Thus its geodesic spray coefficients are

$$G^i = \Gamma^i_{jk}(x)y^j y^k.$$

and we can show that [16]

$$R_j{}^m{}_{ml} y^j = \left\{2\frac{\partial \Gamma^m_{jk}}{\partial x^m} - 2\frac{\partial \Gamma^m_{mk}}{\partial x^j} + 4\Gamma^m_{lm}\Gamma^l_{jk} - 4\Gamma^m_{lk}\Gamma^l_{mj}\right\} y^j = \eta_{jk}(x)y^j.$$

Now, this metric is of SCR type if and only if $\eta_{jk}(x)y^j = \eta_{kj}(x)y^j$, which is equivalent to

$$\frac{\partial \Gamma^m_{mk}}{\partial x^j} = \frac{\partial \Gamma^m_{mj}}{\partial x^k},$$

and reduces to

$$\frac{\partial \Gamma^m_{m1}}{\partial x^2} = \frac{\partial \Gamma^m_{m2}}{\partial x^1}.$$

We can calculate the coefficients of connection by (3) and (4), then we get

$$\Gamma^m_{m1} = \Gamma^1_{11} + \Gamma^2_{21} = \frac{1}{4}\left[\frac{\partial^2 G}{\partial u \partial u} + \frac{\partial^2 H}{\partial v \partial u}\right].$$

Therefore the above metric is of SCR type if and only if

$$[G_{.u} + H_{.v}]_{.u.y} = [G_{.u} + H_{.v}]_{.v.x}.$$

Moreover calculations show that ([16], [17])

$$2\rho_{.x.y} - 3Q\rho_{.y.y} = 2\rho_{.y.x} + 2Q\rho_{.x.x}.$$

Now if $Q > 0$ (i.e. F is not Riemannian) then $\rho_{.y.y} = -\frac{2}{3}\rho_{.x.x}$.
As every two dimensional Finsler metric is of scalar curvature, the Ricci curvature R is related to flag curvature $K(y)$ by $K(y) = \frac{R}{L}$ (which is defined below). Assuming $Q = 1$ and $\rho_{.x.x} = -\frac{3}{2}\rho_{.y.y} \neq 0$ we have

$$K(y) = \frac{\rho_{.y.y}}{2e^{2\rho}e^{2arc\tan\frac{u}{v}}}.$$

Evidently, its flag curvature is not constant, which means that the class of SCR type Finsler spaces is not contained in the class of Finsler spaces of constant flag curvature.

Definition 2.5. A Finsler metric is called R-quadratic if R_y is quadratic in y, namely, in local coordinates, $R^i_k(y)$ are quadratic in $y \in T_xM$.

For a two-dimensional plane $P \subset T_pM$ and a non-zero vector $y \in T_pM$, the flag curvature $K(P, y)$ is defined by

$$K(P, y) := \frac{g_y(u, R_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2},$$

where $P = \mathrm{span}\{y, u\}$. F is said to be of scalar curvature $K = \lambda(x, y)$ if for any $y \in T_pM$, the flag curvature $K(P, y) = \lambda(x, y)$ is independent of P containing $y \in T_pM$, which in a local coordinate system (x^i, y^i) on TM, is equivalent to the following:

$$R^i_k = \lambda(x, y)\ F^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}.$$

Moreover if λ is a constant, then F is said to be of constant flag curvature.
A Finsler metric is of sectional (flag) curvature if its flag curvature depends only on the plane section.

A Finsler manifold is said to be negatively curved if for all non-zero vectors $y, v \in T_xM$ with $g_y(u, y) = 0$,

$$g_y(u, R_y(u)) < 0.$$

The Ricci scalar function of F is given by

$$\rho := \frac{1}{F^2}R^i_i.$$

Therefore, the Ricci scalar function is positive homogeneous of degree 0 in y. This means that $\rho(x,y)$ depends on the direction of the flag pole y but not its length.

The Ricci tensor of a Finsler metric F is defined by

$$\mathrm{Ric}_{ij} := \left\{\frac{1}{2}R^k_k\right\}_{y^i y^j}.$$

Ricci-flat manifolds are those with vanishing Ricci tensor. In physics, they are important because they represent vacuum solutions to Einstein's equation.

Definition 2.6. A Finsler metric F is said to be an Einstein metric if the Ricci scalar is a function of x alone, equivalently

$$\mathrm{Ric} = \rho(x)F^2,$$

In fact, Ricci-flat manifolds are special cases of Einstein manifolds.

3. Proof of theorems

The properties of Einstein Riemannian metrics are most known, for example the Einstein Schur lemma holds for them. In Finsler geometry this is an open problem in general, Robles proved this problem for Randers metrics [8]. For a different class namely SCR type we are going to prove Einstein Schur lemma. Now, we are going to give the proofs of our main theorems. First we quote some lemmas.

Lemma 3.0 ([3]). *Let (M,g) be a Landsberg manifold of dimension n. Then we have*

$$\|I_{j|i}\|^2 = F^2 I^i{}_{|i} I^j{}_{|j} - (n-2) I_m R^m_k I^k F^{-2} + \mathrm{Div} \ \text{ on } S(M). \tag{5}$$

Proof. See [3] for details. □

Lemma 3.1. *Let (M,F) be a Landsberg manifold and $T = T^{pq} b_p \otimes b_q$ a section of $\pi^*T^*M \otimes \pi^*T^*M$, then*

$$T^{pq}{}_{.j|i} - T^{pq}{}_{|i.j} = T^{sq} P^p_{s\,ij} + T^{ps} P s^q{}_{ij}.$$

Proof. As $T^{pq}{}_{|k}$ and $T^{pq}{}_{.l}$ are defined by

$$dT^{pq} + T^{sq}\omega^p_s + T^{ps}\omega^q_s = T^{pq}{}_{|k}\omega^k + T^{pq}{}_{.l}\omega^{n+l}, \tag{6}$$

differentiating and using the structure equations one deduces

$$\begin{aligned}(dT^{tq} + T^{sq}\omega_s^t) \wedge \omega_t^p + (dT^{pt} + T^{ps}\omega_s^t) \wedge \omega_t^q + T^{sq}\Omega_s^p + T^{ps}\Omega_s^q \\ = (dT^{pq}{}_{|k} - T^{pq}{}_{|r}\omega_k^r) \wedge \omega^k + (dT^{pq}{}_{.l} - T^{pq}{}_{.r}\omega_l^r) \wedge \omega^{n+l}.\end{aligned} \tag{7}$$

Recall that Ω_j^i has the following form

$$\Omega_j^i = \frac{1}{2}R_{j\,kl}^{i}\omega^k \wedge \omega^l + P_{j\,kl}^{i}\omega^k \wedge \omega^{n+l}.$$

Hence as F is Landsberg by taking the components of $\omega^i \wedge \omega^{n+j}$ and $\omega^i \wedge \omega^j$ we get

$$T^{pq}{}_{.j|i} - T^{pq}{}_{|i.j} = T^{sq}P_{s\,ij}^{p} + T^{ps}P_{s\,ij}^{q}. \tag{8}$$

□

Lemma 3.2. *Let (M, F) be a Berwald space. If F is Einstein of non-zero Ricci scalar then it is Riemannian.*

PROOF. Let (M, F) be an Einstein Berwald metric of the non-zero Ricci scalar $\lambda(x)$, then for its geodesic spray we have

$$G^i = \Gamma_{jk}^i(x)y^jy^k.$$

Applying (1) we obtain

$$\mathrm{Ric} = R_m^m = \left\{2\frac{\partial\Gamma_{jk}^m}{\partial x^m} - 2\frac{\partial\Gamma_{mk}^m}{\partial x^j} + 4\Gamma_{lm}^m\Gamma_{jk}^l - 4\Gamma_{lk}^m\Gamma_{mj}^l\right\}y^jy^k = \eta_{jk}(x)y^jy^k.$$

As F has a non-zero Ricci scalar $\lambda(x)$ for any direction $y \in T_xM$, $\frac{\eta_{jk}(x)}{\lambda(x)}y^jy^k$ is always positive regardless of the sign of $\lambda(x)$. Thus

$$F = \sqrt{\frac{\eta_{jk}(x)}{\lambda(x)}y^jy^k},$$

which means (M, F) is Riemannian. □

PROOF OF THEOREM 1.1. First we prove

(a) $I_{|j}^i = 0$,

(b) $C_{ijk|l} = 0$.

Let compute $I_m R^m_k I^k$. For a positively homogeneous function $f\colon TM\backslash\{0\}\to\mathbb{R}$, one has

$$df = f_{|i}\omega^i + f_{.i}\omega^{n+i}.$$

By the equation (13–3) in [18] we have

$$0 = d^2 f = \frac{1}{2}\left(f_{|l|k} - f_{|k|l} + \frac{1}{2} f_{.i} R^i_{kl}\right)\omega^k\wedge\omega^l + (f_{|i.j} - f_{.j|i} + f_{.k} L^k_{ij})\omega^{n+j}\wedge\omega^i. \quad (9)$$

This yields the following Ricci identities

$$f_{|k|l} - f_{|l|k} = f_{.i} R^i_{kl}. \quad (10)$$

$$f_{|i.j} - f_{.j|i} = -f_{.k} L^k_{ij}. \quad (11)$$

Now we show that $I_m R^m_k = 0$, for Finsler metrics of SCR type. According to [18] we have

$$R_{jikl} + R_{ijkl} + 2C_{ijm} R^m_{kl} = 0, \quad (12)$$

contracting with g^{ij} and y^l yields

$$2I_m R^m_k = -2R^m_{mk0}. \quad (13)$$

By contracted Bianchi identity and being of SCR type we have

$$0 = R_j{}^i{}_{i0} - R_0{}^i{}_{ij} = R_i{}^i{}_{j0}.$$

This gives

$$I_m R^m_k = 0. \quad (14)$$

Now we use Lemma 3.1 by taking $T^{pq} = g^{pq}$ we deduce that $g^{pq}{}_{.j|i} = P^{qp}{}_{ij} + P^{qp}{}_{ij}$, contracting by y^i after some calculation ([7]) we get

$$g^{pq}{}_{.j|0} = 0. \quad (15)$$

The task is now to show $I^i_{|m.i} y^m = 0$ and then $I^i_{|i} = 0$. For a compact manifold M we have

$$\langle J, J\rangle = -\frac{1}{3}\langle b, R\rangle, \quad (16)$$

where $\langle\,,\rangle$ denotes the global products over $S(M)$, this product is introduced in [1], $R = R_y$ as before and

$$F^{-2} b^k_i = 2[I^k{}_{.i} + F^{-2}(y^k I_i + y_i I^k)] + I_i I^k + g^{rs}(\delta_i{}^k - F^{-2} y^k y_i) I_{s.r}. \quad (17)$$

The best reference here is [3].

Now we show that

$$Kg^{rs}I_{s.r} = 0, \tag{18}$$

where $K = \frac{1}{n-1}R^m{}_m$, using (16) and F being Landsberg metric. Taking $f = \tau$ as distortion function defined in [18] and using (11) we get

$$\tau_{|k|l}y^l = \tau_{|l|k}y^l.$$

As F is Landsberg noting the equation (13)–(20) in [18],we obtain $\tau_{|k} = S_{.k}$. Moreover we have

$$S_{.k|l}y^l - S_{|k} = 0.$$

As $S_{.k|l}y^l - S_{|k} = -\frac{1}{3}(2R^m_{k.m} + R^m_{m.k})$ (by [18]) we have

$$2R^m_{k.m} = -R^m_{m.k}. \tag{19}$$

Since F is Einstein this means that $I^kR^i{}_{k.i} = 0$.
Using (14) we have

$$I^k_{.i}R^i{}_k = (I^kR^i{}_k)_{.i} - I^kR^i{}_{k.i} = 0.$$

Combining this equation with (16) and (17) yields (18).
If $K \neq 0$ by (15) we have

$$0 = (g^{rs}I_{s.r})_{|m}y^m = (I^r{}_{.r} - I_sg^{rs}{}_{.r})_{|0} = I^r{}_{.r|0} - I_s(g^{rs}{}_{.r|0}) = I^r{}_{.r|0}.$$

Now in the same way as in Lemma 3.1, we can prove that $I^p{}_{.j|i}y^j = I^p{}_{|i.j}y^j$ when (M, F) is Landsberg. Hence

$$0 = I^r{}_{.r|m}y^m = g^{rs}I_{r.s|m}y^m = g^{rs}I_{r|m.s}y^m = I^r{}_{|m.r}y^m.$$

By (5) we get

$$\langle I_{i|j}, I_{i|j}\rangle = \|I_{i|j}\|^2 = \mathrm{Div} \ \text{ on } S(M).$$

By integrating we have $\int_{S(M)} \|I_{i|j}\|^2 dV = 0$ and hence $I_{i|j} = 0$. This completes the proof of (a). In order to prove (b), first we consider the equation (1–4) of Chapter 5 in [1], i.e.

$$\nabla_0\nabla_0 C^i_{km} + C^r_{km}R_0{}^i{}_{r0} + \nabla^{\cdot}{}_m R_j{}^i{}_{kl}y^jy^l = 0.$$

As F is Landsberg by equation (3.5.4) in [6] we have $\nabla^{\cdot}_m R_j{}^i{}_{kl}y^j{=}R^i_{kl\,.y^m.y^j}y^j{=}0$, hence

$$C^r_{km}R^i{}_r = 0. \tag{20}$$

By (12) we have

$$R_{ijk0} + R_{jik0} = 0. \tag{21}$$

The following equation can be found in Chapter 5 of [1]

$$\|FC_{ijk|l}\|^2 = \|I_{k|l}\|^2 - (n-2)C^l{}_{is}C^{is}_r R^r{}_l - 2F^2 C^l{}_{is}C^{jrs}R_r{}^i{}_{jl} + \text{Div on } S(M). \tag{22}$$

Using $I_{k|l} = 0$ we need to show that second and third sentences in the right side of (22) are equal to zero. As before $C_{sil}R^l_r = 0$ then $C^l_{is}R_{rl} = C_{lis}R^l_r = 0$.

Now we prove that $C^l{}_{is}C^{jrs}R_r{}^i{}_{jl} = 0$.

Applying (12) and changing the indices k and l yield

$$0 = C^{li}{}_s C^{jks}(R_{ijkl} + R_{jikl}) = C^{jks}C^{li}{}_s C_{ijm}R^m{}_{kl}. \tag{23}$$

Now we show that $C^{jk}{}_s C^{ls}{}_i R^i{}_{kl} = 0$. By (20) and (23) we have

$$\begin{aligned} C^{jks}C^l{}_{is}R^i{}_{l.k} &= C^{jks}(\underbrace{C^l{}_{is}R^i{}_l}_{0})_{.k} - C^{jks}C^l{}_{is.k}R^i{}_l \\ &= -C^{jks}R^i{}_l\{g^{lp}C_{psi.k} - C^{lp}{}_k C^l{}_{si}\} = -C^{jks}C_{lis.k}R^{il}, \end{aligned}$$

and

$$C^{jis}C^l{}_{si}R^k{}_{l.k} = C^{jis}(\underbrace{C^l{}_{si}R^k{}_l}_{0})_{.k} - C^{jis}C^l{}_{sik}R^k{}_l$$

$$= -C^{jis}R^k{}_l\{g^{lp}C_{psi.k} - C^l p_k C_{psi}\} = -C^{jis}C_{lsi.k}R^{kl} = -C^{jks}C_{lsi.k}R^{il}.$$

Comparing the two above equations and by (19) one sees immediately that

$$C^{jks}C^l{}_{si}R^i{}_{l.k} = \lambda(x)C^{jis}C^l{}_{si}y_l = 0. \tag{24}$$

We prove that $C^{jks}C^l{}_{is}R^i{}_{k.l} = 0$ and then by (24) we show that $C_{jks}C^l{}_{si}R^i{}_{kl} = 0$.

$$C^{jks}C^l{}_{si}R^i{}_{k.l} = C^{jks}(\underbrace{C^l{}_{si}R^i{}_k}_{0})_{.l} - C^{jks}C^l{}_{si.l}R^i{}_k = -C^l{}_{si.k}\underbrace{C^{jks}R^i{}_k}_{0} = 0.$$

Now we show that

$$C^{jks}C^l{}_{si}R^i_{j\,kl} = 0.$$

As F is Landsberg we have

$$\begin{aligned} C^{jks}C^l{}_{si}R^i_{j\,kl} &= C^{jks}C^l si R^i{}_{kl.j} = (\underbrace{C^{jks}C^l{}_{is}R^i{}_{kl}}_{0})_{.j} - (C^{jks}C^l si)_{.j}R^i{}_{kl} \\ &- (C^{jks}C^l{}_{si})_{.j}R^i{}_{kl} = (C^{jls}C^k{}_{si})_{.j}R^i{}_{kl} = -2(C^{jl}{}_s g^{ks}{}_{.i})_{.j}R^i{}_{kl} \\ &- 2\underbrace{(C^{jl}{}_s g^{ks})_{.i.j}R^i{}_{kl}}_{A} + 2\underbrace{(C^{jl}{}_{s.i}g^{ks})_{.j}R^i{}_{kl}}_{B}. \end{aligned}$$

In the following we show that $A = B = 0$.

$$\begin{aligned}
A &= (C^{jl}{}_{s}g^{ks})_{.i.j}R^{i}{}_{kl} = (C^{jlk})_{.i.j}R^{i}{}_{kl} = (C^{jlk}R^{i}{}_{kl})_{.i.j} - C^{jlk}{}_{.i}R^{i}_{j\,kl} \\
&= (\underbrace{C^{jlk}R^{i}{}_{kl}}_{0})_{.i.j} - (C^{jlk}\underbrace{R^{i}_{i\,kl}}_{0})_{.j} - C^{jlk}{}_{.i}R^{i}_{j\,kl} \\
&= \underbrace{g^{lp}g^{kq}C^{j}{}_{pq.i}R^{i}_{j\,kl}}_{0} - 2C^{lp}{}_{i}C^{jk}{}_{p}R^{i}_{j\,kl} - 2C^{kq}{}_{i}C^{lj}{}_{q}R^{i}_{j\,kl} \\
&= -2C^{lp}{}_{i}C^{jk}{}_{p}R^{i}_{j\,kl} + 2C^{lp}{}_{i}C^{kj}{}_{p}R^{i}_{j\,kl} = 0.
\end{aligned}$$

And

$$\begin{aligned}
B &= (C^{jl}{}_{s.i}g^{ks})_{.j}R^{i}{}_{kl} = (g^{lp}g^{jq}g^{ks}C_{pqs.i})_{.j}R^{i}{}_{kl} = \underbrace{-2C_{pqs.i}R^{i}{}_{kl}C^{lpq}g^{ks}}_{D} \\
&\underbrace{-2C_{pqs.i}R^{i}{}_{kl}I^{q}g^{lp}g^{ks}}_{E}\underbrace{-2C_{pqs.i}R^{i}{}_{kl}C^{ksq}g^{lq}}_{F}\underbrace{-g^{lp}g^{jq}g^{ks}C_{pqs.i.j}R^{i}{}_{kl}}_{H},
\end{aligned}$$

and we show that $D + F = E = H = 0$.

$$E = -2C_{pqs.i}R^{i}{}_{kl}g^{lp}g^{ks}I^{q} = -2C_{pqs.i}R^{i}{}_{kl}g^{ls}g^{kp}I^{q} = 2C_{pqs.i}R^{i}{}_{kl}g^{lp}g^{ks}I^{q} = -E,$$

it means that $E = 0$.

$$\begin{aligned}
D + F = -2C_{pqs.i}R^{i}{}_{kl}C^{lpq}g^{ks} - 2C_{pqs.i}R^{i}{}_{kl}C^{ksq}g^{lp} = 2C_{pqs.i}R^{i}{}_{kl}C^{kpq}g^{ls} \\
+2C_{pqs.i}R^{i}{}_{kl}C^{lsq}g^{kp} = 2C_{pqs.i}R^{i}{}_{kl}C^{ksq}g^{lp} + 2C_{pqs.i}R^{i}{}_{kl}C^{lpq}g^{ks} = -(D + F),
\end{aligned}$$

then $D + F = 0$. Now we show that $H = 0$.

$$\begin{aligned}
H &= -2g^{lp}g^{jq}g^{ks}C_{pqs.i.j}R^{i}{}_{kl} = 2g^{kp}g^{jq}g^{ls}C_{pqs.i.j}R^{i}{}_{kl} \\
&= 2g^{ks}g^{jq}g^{lp}C_{pqs.i.j}R^{i}{}_{kl} = -H,
\end{aligned}$$

thus $H = 0$.

Now we can rewrite (22) as $\|FC_{ijk|l}\|^2 = \mathrm{Div}$ on $S(M)$. By integration we get $C_{ijk|l} = 0$. F is Landsberg metric, then $0 = FC^{i}_{j\,k|l} = P^{i}_{j\,kl} = -F\frac{\partial \Gamma^{i}{}_{jk}}{\partial y^{l}}$, this means that F is Berwald. Now Lemma 3.2. completes the proof of theorem. □

Corollary 3.1 (Extension of Numata Theorem)**.** *Let (M, F) be Landsberg metric of SCR type on a compact manifold M of dimension ≥ 3. Suppose that F is Einstein metric with non-zero Ricci scalar. Then F is Riemannian.*

Corollary 3.2. *Every negatively curved Finsler space of SCR type is Riemannian.*

Proof. Assume that F is negatively curved at a point $x \in M$. Since the vector I_y is orthogonal to y with respect to g_y, it follows from $R_y(I_y) = 0$ that $g_y(R_y(I), I) = 0$ but F is negatively curved at the point x therefore $I_y = 0$ for any $y \in T_xM \backslash \{0\}$. By Deicke's theorem, F is Riemannian.
In [14], it has been proved that some Einstein (α, β)-metrics such as Matsumoto and Kropina metrics with $s_i = 0$ and $r_{ij} = 0$ must be Riemannian or Ricci flat. This results are exactly the same as Theorem 1.1 when F being the above (α, β)-metrics.
Now we give an example of Einstein Landsberg metric. □

Example 2 ([17]). Let $F = F(x, y, u, v)$ be a Finsler metric on an open subset $U \subset R^2$ in the form

$$F^2 = e^{2\rho(x,y)} v^2 \exp\left(2a\frac{u}{v}\right),$$

where a is a constant. The mean Cartan torsion of F satisfies

$$I^2 = 4.$$

The spray coefficients $G^1 = G$ and $G^2 = H$ by F are given by

$$G = -a\rho_x uv + \frac{1}{2}(\rho_x - a\rho_y)v^2,$$
$$H = -\frac{1}{2}a\rho_y v^2.$$

Thus F is a Berwald metric. Its Ricci scalar is

$$R = \rho_{xx} v^2.$$

As F is Einstein if and only if $\left(\frac{R}{F^2}\right)_{.k} = 0$, this yields $\rho_{.x.x} = 0$ or $a = 0$. These are equivalent to F being Riemannian or Ricci flat.

Proof of Theorem 1.2. Let (M, F) be a 3-dimensional Einstein Finsler space of SCR type with Ricci scalar $\lambda(x)$. The Riemann curvature in a direction $y \in T_xM$ is a linear transformation $R_y : T_xM \to T_xM$ and the Ricci curvature is defined as the trace of the Riemann curvature. In the other hand, for an arbitrary basis $\{b_i\}_{i=1}^3$ for T_xM it can be expressed as

$$\mathrm{Ric}(y) = \sum_{i=1}^{3} R_i^i(y).$$

Assume that $\{b_i\}_{i=1}^3$ is an orthonormal basis with respect to g_y such that

$$b_2 = \frac{I}{F(I)}, \quad b_3 = \frac{y}{F(y)}.$$

Let

$$P_i = \mathrm{span}\{b_i, y\}, \quad i = 1, 2, 3$$

$$K(P_i, y) = \frac{1}{F^2(y)} g_y(R_y(b_i), b_i).$$

Using $R_y(y) = 0$ and (14) we have

$$\mathrm{Ric}(y) = F^2(y) K(P_1, y).$$

As F is Einstein then

$$\lambda(x) F^2(y) = F^2(y) K(P_1, y) = g_y(R_y(b_1), b_1).$$

Thus the flag curvature of the flag P_1 is a scalar function of x alone. We will show that the flag curvature of every arbitrary flag P at x containing y depends only on the section. Let $P = \mathrm{span}\{V, y\}$, where V is an arbitrary vector in $T_x M$ such that $g_y(V, V) = 1$ and $g_y(V, y) = 0$, put $v^1 = g_y(V, b_1)$, $v^2 = g_y(V, I)$ and $v^3 = g_y(V, y) = 0$. As R_y is a linear function

$$R_y(V) = R_y(v^1 b_1 + v^2 b_2 + v^3 b_3) = v^1 R_y(b_1) + v^2 R_y(I) + v^3 R_y(y) = v^1 R_y(b_1).$$

Hence

$$\begin{aligned} g_y(R_y(V), V) &= g_y(v^1 R_y(b_1), V) = (v^1)^2 g_y(R_y(b_1), b_1) + v^1 v^2 g_y(R_y(b_1), I) \\ &\quad + v^1 v^3 g_y(R_y(b_1), y) = (v^1)^2 g_y(R_y(b_1), b_1) = (v^1)^2 \lambda(x) F^2. \end{aligned}$$

Thus

$$K(P, y) = \lambda(x)(v^1)^2. \qquad \square$$

Theorem 3.1 (Ricci rigidity)**.** *Let F be an Einstein Landsberg metric of SCR type in dimension 3 on a compact manifold M. Then F is locally Minkowski if it is Ricci flat.*

PROOF. As it is proved in the proof of Theorem 1.2 one concludes that , for each flag P at x containing y of this Einstein metric, the flag curvature $K(P, y)$ is a multiple of its Ricci scalar. If F is Ricci flat then it is of zero flag curvature. According to AKBAR-ZADEH [2] (every Finsler metric of zero flag curvature is locally Minkowskian), the proof is completed. $\square$

The following question still remains open.
Is there any R-flat Landsberg metric which is not locally Minkowskian?
The above theorem is answered to this question in dimension 3. This states that if there is R-flat Landsberg metric in dimension 3 which is not locally Minkowski, it can not be of SCR type.

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NASRIN SADEGHZADEH
DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCES
QOM UNIVERSITY
QOM, IRAN

E-mail: *nasrin_sadeghzadeh@yahoo.com*

ASADOLLAH RAZAVI
DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCES
AMIRKABIR UNIVERSITY
OF TECHNOLOGY (TEHRAN POLYTECHNIC)
TEHRAN, IRAN

E-mail: *arazavi@aut.ac.ir*

BAHMAN REZAEI
DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCES
URMIA UNIVERSITY
URMIA, IRAN

E-mail: *b.rezaei@urmia.ac.ir*

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