Publ. Math. Debrecen
75/3-4 (2009), 327–338

Multiple fractional integrals through Gamma-mixed Ornstein–Uhlenbeck process

By CONSTANTIN TUDOR (Bucharest) and MARIA TUDOR (Bucharest)

Abstract. We prove the mean square convergence of multiple Riemann–Stieltjes integrals based on the integral process defined by the Gamma-mixed Ornstein–Uhlenbeck process to multiple fractional Stratonovich integrals. The integrands belong to the subclass of the Schwartz space $\mathcal{S}(R^n)$ of rapidly decreasing functions whose fractional integrals remain rapidly decreasing. In particular the result applies for integrands in the Lizorkin space, i.e., the subspace of $\mathcal{S}(R^n)$ which is orthogonal to all polynomials.

1. Introduction

Processes with properties like self-similarity and long-range dependence have attracted much attention both for their applications and their intrinsic mathematical interest ([6], [24]).

A well known example of a process which enjoys such properties (long-range dependence for Hurst parameter greater than $\frac{1}{2}$) is the fractional Brownian motion (*fBm* for short), which is a suitable generalization of the standard Brownian motion (*Bm* for short). This process was first introduced by Kolmogorov (1940) and later studied by Mandelbrot and Van Ness (1968).

In fact fBm is the most important example of Gaussian process which is self-similar and has stationary increments.

It is known that functionals of Bm or fBm have orthogonal decomposition in terms of corresponding multiple integrals. Then, the approximation of such

Mathematics Subject Classification: 60H05.
Key words and phrases: fractional Brownian motion, multiple Stratonovich integral, multiple Wiener–Itô integral, trace of a function, square mean approximation.

functionals in strong or weak sense could be obtained by approximating the corresponding multiple integrals from their orthogonal decomposition. We mention also that multiple integrals appear in problems like asymptotic statistics and nonlinear filtering are given in [1], [2].

In the present paper we consider such multiple integrals for fBm. Let us mention that the case of multiple integrals with respect to Bm was first considered in [11] (see also [18] for basic properties).

Multiple Stratonovich integrals with respect to Bm are introduced in [7] and later studied in [13], [23], [26].

The case of fBm is considered in [4], [5] for the case of finite time interval, by using the reproducing kernel Hilbert space theory and in [19] by using a transfer idea from multiple integrals with respect to Bm.

Strong convergence results have been obtained in [3], [8], [12] for the case of multiple Stratonovich integrals with respect to the Bm and recently [25] for the case multiple Stratonovich integrals with respect to fractional Bm.

Mainly the above mentioned results are obtained for Wong–Zakai or mollifier approximations.

In the present paper we introduce another strong approximation based on the Gamma-mixed Ornstein–Uhlenbeck process (ΓMOU) (such a process was introduced in [10]). The ΓMOU process has various interesting properties and it is a possible candidate for the modelling in the presence of stationarity and long-range dependence properties, and also for a construction of the stochastic calculus with respect to fBm.

The main result (Theorem 3.1) shows the mean square convergence of multiple improper Riemann–Stieltjes integrals driven by the integral process associated to ΓMOU to the multiple improper Stratonovich fractional integral. The deterministic integrands are rapidly decreasing functions (Schwartz space) whose fractional integrals remain rapidly decreasing. In particular the result applies for integrands in the Lizorkin space, i.e., the subspace of the Schwartz space which is orthogonal to all polynomials.

2. Preliminaries

Let $h \in \left(0, \frac{1}{2}\right)$, $\lambda > 0$ and let $(W_t)_{t \in R}$ be a real Bm defined on a probability space $(\Omega, \mathcal{F}, P)$.

Recall that a fBm is a continuous centered Gaussian process $(B_t^{(h)})_{t\in R}$, starting from 0, with covariance

$$C^{(h)}(s,t)=\frac{1}{2}\left[\,|s|^{2h+1}+|t|^{2h+1}-|t-s|^{2h+1}\right],\quad s,t\in R.$$

The constant $H=h+\frac{1}{2}$ is called the Hurst parameter.

Recall that for $h\in\left(0,\frac{1}{2}\right)$ the fBm has long-range dependence.

Following Mandelbrot and Van Ness [17] (see also Samorodnitsky and Taqqu [22]) $B^{(h)}$ has the following representation as a Wiener integral

$$\begin{aligned}B_t^{(h)}&=\int_R r^{(h)}(t,s)dW_s,\quad t\in R,\\ r^{(h)}(t,s)&=\frac{1}{c_1(h)}\left[(t-s)_+^h-(-s)_+^h\right],\quad s,t\in R,\\ c_1^2(h)&=\int_0^\infty\left[(1+s)^h-s^h\right]^2ds+\frac{1}{2h+1}.\end{aligned}\tag{2.1}$$

Recall that the Riemann–Liouville fractional integral is defined by

$$I_-^{h,p}f(t_1,\dots,t_p)=\frac{1}{\Gamma^p(h)}\int_{R^p}f(s_1,\dots,s_p)\prod_{j=1}^p(s_j-t_j)_+^{h-1}ds_1\dots ds_p,$$

(for the theory of fractional integrals and derivatives see [21]).

Remark 2.1 (Hardy–Litlewood). The operator $I_-^{h,p}:L^{\frac{2}{2h+1}}(R^p)\longrightarrow L^2(R^p)$ is continuous (see [21, Theorem 24.1]).

The kernel (2.1) represents a particular value of the following continuous operator

$$\begin{aligned}\Lambda_p^{(h)}&:L^{\frac{2}{2h+1}}\left(R^p\right)\longrightarrow L^2\left(R^p\right),\\ \left(\Lambda_p^{(h)}f\right)(t_1,\dots,t_p)&=\frac{\Gamma^p(h+1)}{[c_1(h)]^p}I_-^{h,p}f(t_1,\dots,t_p),\end{aligned}$$

in fact $(\Lambda_1^{(h)}1_{(0,t)})(s)=r^{(h)}(t,s)$.

We assume that the reader is familiar with elementary facts about multiple Wiener–Itô integrals with respect to the Bm (see for example [11], [18]).

We shall denote by $I_p^0(f_p)$ the multiple Wiener–Itô integral of $f_p\in L_s^2(R^p)$ (i.e., f is symmetric and square integrable) with respect to W.

Now we introduce the multiple Wiener–Itô fractional integrals by using a transfer idea (see [19]).

Definition 2.2. We define the *multiple Wiener–Itô fractional integral of f_p with respect to $B^{(h)}$* by

$$I_p^h(f_p) = I_p^0(\Lambda_p^{(h)} f_p).$$

Remark 2.3. The multiple Wiener–Itô fractional integral is well defined if $\Lambda_p^{(h)} f_p \in L_s^2(R^p)$.

In particular if $f_p \in L^{\frac{2}{2h+1}}(R^p)$ we have $\Lambda_p^{(h)} f_p \in L_s^2(R^p)$ (Remark 2.1) and therefore the multiple fractional integral $I_p^h(f_p)$ is well defined.

Moreover it can be shown ([19], [20]) that if

$$\int_{R^{2p}} |f_p(u)||f_p(v)| \prod_{i=1}^{p} |u_i - v_i|^{2h-1} dudv < \infty, \tag{2.2}$$

then

$$\|\Lambda_p^{(h)} f_p)\|_{L^2(R^p)}^2 = h(2h+1) \int_{R^{2p}} f_p(u) f_p(v) \prod_{i=1}^{p} |u_i - v_i|^{2h-1} dudv,$$

and thus if (2.2) holds, then $\Lambda_p^{(h)} f_p \in L_s^2(R^p)$ and consequently the multiple Wiener–Itô fractional integral $I_p^h(f_p)$ is meaningful.

Next, we address the case of multiple Stratonovich fractional integrals. Since we consider an infinite time interval (R in our case) it is convenient to adopt a Hilbertian approach for multiple Stratonovich integrals with respect to the Bm and again a transfer idea (the same is possible for multiple Wiener–Itô fractional integrals) to pass to the case of multiple fractional integrals.

Let $(e_i)_{i \in N} \subset L^2(R)$ be a CONS (complete orthonormal system), $f_p \in L_s^2(R^p)$ with the decomposition

$$f_p^e(t_1, \ldots, t_p) = \sum_{i_1, \ldots, i_p = 1}^{\infty} a_{i_1, i_2, \ldots, i_p} e_{i_1}(t_1) \ldots e_{i_p}(t_p),$$

$$a_{i_1, i_2, \ldots, i_p} = \left\langle f_p, e_{i_1} \otimes \cdots \otimes e_{i_p} \right\rangle_{L^2(R^p)}.$$

Denote

$$f_p^{N,e}(t_1, \ldots, t_p) = \sum_{i_1, \ldots, i_p = 1}^{N} a_{i_1, i_2, \ldots, i_p} e_{i_1}(t_1) \ldots e_{i_p}(t_p).$$

Definition 2.4. (a) We define the multiple Stratonovich integral of $f_p^{N,e}$ with respect to the Bm W by

$$I_p^0 \circ \left(f_p^{N,e}\right) = \sum_{i_1, \ldots, i_p = 1}^{N} a_{i_1, i_2, \ldots, i_p} I_1^0(e_{i_1}) \ldots I_1^0(e_{i_p})$$

(b) If for every CONS $(e_i)_{i\in N}$, the sequence $I_p^0\circ(f_p^{N,e})$ converges in $L^2(\Omega,\mathcal{F},P)$ as $N\to\infty$ to the same limit $I_p^0\circ(f_p)$, then we say that f_p is Stratonovich integrable and the limit $I_p^0\circ(f_p)$ is called the *multiple Stratonovich integral of* f_p *with respect to* W.

In order to obtain criteria for the existence of multiple Stratonovich integrals and to connect them with multiple Wiener–Itô integrals the concept of trace of a several variables deterministic function is useful.

We take the approach from [13], [14] (for a different approach see [23]).

For $1\le j\le\left[\frac{p}{2}\right]$ we introduce the *j-trace* of $f_p^{N,e}$ by

$$\mathrm{Tr}_j\ f_p^{N,e}(s_1,\dots,s_{p-2j})=\sum_{i_1,\dots,i_p=1}^{N}a_{i_1,i_1,i_2,i_2,\dots i_j,i_j,i_{2j+1},\dots,i_p}e_{i_{2j+1}}(s_1)\dots e_{i_p}(s_{p-2j}).$$

If for every CONS $(e_i)_{i\in N}$, the sequence $\mathrm{Tr}_j\ f_p^{N,e}$ converges in $L^2(R^{p-2j})$ as $N\to\infty$ to the same limit $\mathrm{Tr}_j\ f_p$, then we call the limit the *j-trace* of f_p.

Next, $\mathcal{S}(R^p)$ denotes the Schwartz space of rapidly decreasing functions, i.e.,

$$\mathcal{S}(R^p)=\left\{\varphi\in C^\infty(R^p):\sup_{t\in R^p}(1+\|t\|^2)^{\frac{\alpha}{2}}\left|\frac{\partial^{\beta_1+\dots+\beta_p}}{\partial t_1^{\beta_1}\dots\partial t_p^{\beta_p}}\varphi(t)\right|<\infty,\ \forall\alpha,\beta_i\in Z_+\right\}.$$

Consider the Hermite polynomials

$$h_j(t)=\frac{(-1)^j}{\sqrt{j!}}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\right)^{-1}\frac{d^n}{dt^n}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\right),\ j\ge 0.$$

Recall that the operator $-\frac{d^2}{dt^2}+\frac{t^2}{4}+1$, with dense domain in $L^2(R)$, has the normalized Hermite functions

$$\varphi_j(t)=\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\right)^{\frac{1}{2}}h_{j-1}(t),\ j\ge 1,$$

as its eigenfunctions, $\lambda_j=j+\frac{1}{2}$ as the corresponding eigenvalues and the sequence $(\varphi_j)_{j\ge1}\subset\mathcal{S}(R)$ is a CONS in $L^2(R)$.

The following criterion on the existence of traces is useful.

Theorem 2.5 ([13, Theorems 9.3 and 10.1], [14, Theorem 3.12])**.** *Let* $f_p\in\mathcal{S}(R^p)$ *be symmetric with the orthogonal decomposition*

$$f_p(t_1,\dots,t_p)=\sum_{i_1,\dots,i_p=1}^{\infty}a_{i_1,i_2,\dots,i_p}\varphi_{i_1}(t_1)\dots\varphi_{i_p}(t_p).\tag{2.3}$$

Then every $\mathrm{Tr}_j \ f_p$ *exists and it is given by the relation*

$$\mathrm{Tr}_j \ f_p(s_1, \ldots, s_{p-2j})$$

$$= \sum_{i_{2j+1},\ldots,i_p=1}^{\infty} \left(\sum_{i_1,\ldots,i_j=1}^{\infty} a_{i_1,i_1,i_2,i_2,\ldots i_j,i_j,i_{2j+1},\ldots,i_p} \right) \varphi_{i_{2j+1}}(s_1) \ldots \varphi_{i_p}(s_{p-2j}).$$

The following important result holds (see [13], [14], [26]).

Theorem 2.6 (Hu–Meyer formula)**.** *Let* $f_p \in L_s^2(R^p)$*. Then* $I_p^0 \circ (f_p)$ *exists if and only if* f_p *admits all* $\mathrm{Tr}_j \ f_p$, $1 \le j \le \left[\frac{p}{2}\right]$, *and the the following Hu–Meyer formula holds*

$$I_p^0 \circ (f_p) = I_p^0(f_p) + \sum_{j=1}^{\left[\frac{p}{2}\right]} \frac{p!}{2^j h!(p-2j)!} I_{p-2j}^0(\mathrm{Tr}_j \ f_p).$$

As in [19] we introduce the following definition.

Definition 2.7. We define the multiple Stratonovich fractional integral of f_p *with respect to* $B^{(h)}$ by

$$I_p^h \circ (f_p) = I_p^0 \circ (\Lambda_p^{(h)} f_p),$$

provided that $I_p^0 \circ (\Lambda_p^{(h)} f_p)$ exists.

Finally we recall another process with long-range dependence introduced in [10].

Definition 2.8. The *Gamma-mixed Ornstein–Uhlenbeck* process $(Y_t^{\lambda,h})_{t \in R}$ (ΓMOU for short) is defined as the Wiener integral

$$Y_t^{\lambda,h} = \int_{-\infty}^{t} \left(\frac{\lambda}{\lambda + t - s} \right)^{1-h} dW_s.$$

Remark 2.9 (see [10])*.* The ΓMOU is a centered continuous Gaussian process which is stationary and has long-range dependence.

Moreover ΓMOU is asymptotically self-similar, it is a semimartingale and its scaled integral process converges as $\lambda \to 0$, a.s. and uniformly on compact time intervals, to the fBm and $Y_t^{\lambda,h} - Y_0^{\lambda,h}$ converges as $\lambda \to \infty$, in $L^2(\Omega, \mathcal{F}, P)$ and uniformly on compact time intervals, to the Bm.

3. Main result

Consider the scaled integral process associated to $Y^{\lambda,h}$

$$Z_t^{\lambda,h} = \frac{h\lambda^{h-1}}{c_1(h)} \int_0^t Y_s^{\lambda,h} ds, \quad t \in R,$$

and for appropriate functions $f_p : R^p \longrightarrow R$ consider the multiple improper Riemann–Stieltjes integral

$$\begin{aligned} J_p^{\lambda,h}(f_p) &= \int_{R^p} f_p(t_1, \ldots, t_p) dZ_{t_1}^{\lambda,h} \ldots dZ_{t_p}^{\lambda,h} \\ &= \frac{h^p \lambda^{p(h-1)}}{[c_1(h)]^p} \int_{R^p} f_p(t_1, \ldots, t_p) Y_{t_1}^{\lambda,h} \ldots Y_{t_p}^{\lambda,h} dt_1 \ldots dt_p. \end{aligned}$$

Theorem 3.1. *Let $f_p \in \mathcal{S}(R^p)$ be symmetric such that $I_-^{h,p} f_p \in \mathcal{S}(R^p)$. Then $J_p^{\lambda,h}(f_p)$ converges as $\lambda \to 0$ to $I_p^h \circ (f_p)$ in $L^2(\Omega, \mathcal{F}, P)$.*

Proof. First, we prove that for $g \in \mathcal{S}(R^p)$ every $\mathrm{Tr}_j\ g$ is given by the relation

$$\mathrm{Tr}_j\ g(s_1, \ldots, s_{p-2j}) = \int_{R^j} g\,(t_1, t_1, \ldots, t_j, t_j, s_1, \ldots, s_{p-2j})\, dt_1 \ldots dt_j. \tag{3.1}$$

Recall the following characterization of $\mathcal{S}(R^p)$ (see [15], [16])

$$\mathcal{S}(R^p) = \Big\{ f_p \in L^2(R^p) : \sum_{i_1,\ldots,i_p=1}^{\infty} (\lambda_{i_1} \ldots \lambda_{i_p})^{2r} a^2_{i_1,i_2,\ldots,i_p} < \infty, \ \forall r = 0, 1, \ldots \Big\},$$

$$a_{i_1,i_2,\ldots,i_{;p}} = \big\langle f_p, \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_p} \big\rangle_{L^2(R^p)}. \tag{3.2}$$

For $r > 2$ we have from (3.2)

$$|a_{i_1,i_2,\ldots,i_p}| \le \frac{c_{r,p}}{(\lambda_{i_1} \ldots \lambda_{i_p})^r}, \ \forall i_1, \ldots, i_p, \tag{3.3}$$

for some constant c_r. □

From (3.3) and the inequality $|\varphi_k| \le c(2k+1)$, we obtain for every $0 \le l \le \left[\frac{p}{2}\right]$,

$$\begin{aligned} &\sum_{i_1,\ldots,i_p=1}^{\infty} \left| a_{i_1,i_2,\ldots,i_p} \varphi_{i_{2l+1}} \otimes \cdots \otimes \varphi_{i_p} \right| \\ &\qquad \le d_{r,p} \sum_{i_1,\ldots,i_p=1}^{\infty} \frac{1}{(2i_1+1)^{r-1} \ldots (2i_p+1)^{r-1}} < \infty, \end{aligned} \tag{3.4}$$

and consequently (choosing $l = 0$), the series (2.3) converges also uniformly to f_p.

From (3.4) with $l = j$ and using orthonormality of $(\varphi_k)_k$ and Fubini's theorem, we have

$$\sum_{i_1,\dots,i_p=1}^{\infty} \int_{R^j} \left| a_{i_1,i_2,\dots,i_{;p}} \varphi_{i_1}(t_1)\varphi_{i_2}(t_1)\dots\varphi_{i_{2j-1}}(t_j)\varphi_{i_{2j}}(t_j) \right.$$

$$\left. \times\, \varphi_{i_{2j+1}}(s_1)\dots\varphi_{i_p}(s_{p-2j}) \right| dt_1\dots dt_j \le d_{r,p} \sum_{i_1,\dots,i_p=1}^{\infty} \frac{1}{(2i_1+1)^{r-1}\dots(2i_p+1)^{r-1}}$$

$$\times \int_{R^j} \left| \varphi_{i_1}(t_1)\varphi_{i_2}(t_1)\dots\varphi_{i_{2j-1}}(t_j)\varphi_{i_{2j}}(t_j) \right| dt_1\dots dt_j$$

$$\le d_{r,p} \sum_{i_1,\dots,i_p=1}^{\infty} \frac{1}{(2i_1+1)^{r-1}\dots(2i_p+1)^{r-1}} \int_R \left| \varphi_{i_1}(t_1)\varphi_{i_2}(t_1) \right| dt_1 \dots$$

$$\dots \int_R \left| \varphi_{i_{2j-1}}(t_j)\varphi_{i_{2j}}(t_j) \right| dt_{1j} \le d_{r,p} \sum_{i_1,\dots,i_p=1}^{\infty} \frac{1}{(2i_1+1)^{r-1}\dots(2i_p+1)^{r-1}}$$

$$\times \left(\int_R \left| \varphi_{i_1}(t_1) \right|^2 dt_1 \right)^{\frac{1}{2}} \left(\int_R \left| \varphi_{i_2}(t_1) \right|^2 dt_2 \right)^{\frac{1}{2}} \dots$$

$$\dots \left(\int_R \left| \varphi_{i_{2j-1}}(t_j) \right|^2 dt_j \right)^{\frac{1}{2}} \left(\int_R \left| \varphi_{i_{2j}}(t_j) \right|^2 dt_j \right)^{\frac{1}{2}}$$

$$\le d_{r,p} \sum_{i_1,\dots,i_p=1}^{\infty} \frac{1}{(2i_1+1)^{r-1}\dots(2i_p+1)^{r-1}} < \infty.$$

Then, by the dominated convergence theorem, Fubini's theorem, orthonormality of $(\varphi_k)_k$ and Theorem 2.5, we can write

$$\int_{R^j} g(t_1, t_1, \dots, t_j, t_j, s_1, \dots, s_{p-2j}) dt_1 \dots dt_j$$

$$= \int_{R^j} \left[\sum_{i_1,\dots,i_p=1}^{\infty} \langle g, \varphi_{i_1} \otimes \dots \otimes \varphi_{i_p} \rangle_{L^2(R^p)} \varphi_{i_1}(t_1)\varphi_{i_2}(t_1)\dots\varphi_{i_{2j-1}}(t_j)\varphi_{i_{2j}}(t_j) \right.$$

$$\left. \times\, \varphi_{i_{2j+1}}(s_1)\dots\varphi_{i_p}(s_{p-2j}) \right] dt_1 \dots dt_j = \sum_{i_1,\dots,i_p=1}^{\infty} \langle g, \varphi_{i_1} \otimes \dots \otimes \varphi_{i_p} \rangle_{L^2(R^p)}$$

$$\times \left(\int_{R^j} \varphi_{i_1}(t_1)\varphi_{i_2}(t_1)\dots\varphi_{i_{2j-1}}(t_j)\varphi_{i_{2j}}(t_j) dt_1 \dots dt_j \varphi_{i_{2j+1}}(s_1)\dots\varphi_{i_p}(s_{p-2j}) \right)$$

$$= \sum_{i_1,\dots,i_p=1}^{\infty} \langle g, \varphi_{i_1} \otimes \dots \otimes \varphi_{i_p} \rangle_{L^2(R^p)} \langle \varphi_{i_1}, \varphi_{i_2} \rangle \dots \langle \varphi_{i_{2j-1}}, \varphi_{i_{2j}} \rangle \varphi_{i_{2j+1}}(s_1)\dots\varphi_{i_p}(s_{p-2j})$$

$$= \sum_{k_1,\dots,k_j,i_{2j+1},\dots,i_p=1}^{\infty} \langle g, \varphi_{k_1} \otimes \varphi_{k_1} \cdots \otimes \varphi_{k_j} \otimes \varphi_{k_j} \otimes \varphi_{i_{2j+1}} \otimes \cdots \otimes \varphi_{i_p} \rangle_{L^2(R^p)}$$

$$\times \varphi_{i_{2j+1}}(s_1) \dots \varphi_{i_p}(s_{p-2j})$$

$$= \sum_{i_{2j+1},\dots,i_p=1}^{\infty} \left(\sum_{k_1,\dots,k_j=0}^{\infty} \langle g, \varphi_{k_1} \otimes \varphi_{k_1} \cdots \otimes \varphi_{k_j} \otimes \varphi_{k_j} \otimes \varphi_{i_{2j+1}} \otimes \cdots \otimes \varphi_{i_p} \rangle_{L^2(R^p)} \right)$$

$$\times \varphi_{i_{2j+1}}(s_1) \dots \varphi_{i_p}(s_{p-2j}) = \mathrm{Tr}_j \;\; g(s_1, \dots, s_{p-2j}),$$

and thus (3.1) is satisfied.

By Theorem 2.6 and (3.1) the fractional Stratonovich integral $I_p^h \circ (f_p)$ exists and is given by the Hu–Meyer formula

$$I_p^h \circ (f_p) = I_p^0(\Lambda_p^{(h)} f_p)$$

$$+ \sum_{j=1}^{\left[\frac{p}{2}\right]} \frac{p!}{2^j h!(p-2j)!} I_{p-2j}^0 \left(\int_{R^j} (\Lambda_p^h f_p)(t_1, t_1, \dots, t_j, t_j, .) dt_1 \dots dt_j \right).$$

Define

$$(\Lambda_p^{\lambda,h} f_p)(t_1, \dots, t_p)$$

$$= \frac{h^p}{[c_1(h)]^p} \int_{t_1}^{\infty} \dots \int_{t_p}^{\infty} f_p(s_1, \dots, s_p) \prod_{j=1}^{p} (\lambda + s_j - t_j)^{h-1} ds_1 \dots ds_p.$$

It is easily seen that $\Lambda_p^{\lambda,h} f_p \in \mathcal{S}(R^p)$ and consequently from (3.1)

$$\mathrm{Tr}_j(\Lambda_p^{\lambda,h} f_p)(s_1, \dots, s_{p-2j})$$

$$= \int_{R^j} (\Lambda_p^{\lambda,h} f_p)(t_1, t_1 \dots, t_j, t_j, s_1, \dots, s_{p-2j}) dt_1 \dots dt_j. \tag{3.5}$$

Now, we show the relations

$$J_p^{\lambda,h}(f_p) = I_p^0 \circ (\Lambda_p^{\lambda,h} f_p) = I_p^0(\Lambda_p^{\lambda,h} f_p)$$

$$+ \sum_{j=1}^{\left[\frac{p}{2}\right]} \frac{p!}{2^j h!(p-2j)!} I_{p-2j}^0 \left(\int_{R^j} (\Lambda_p^{\lambda,h} f_p)(t_1, t_1, \dots, t_j, t_j, .) dt_1 \dots dt_j \right). \tag{3.6}$$

First assume that $f_p = f^{\otimes p}$, $f \in \mathcal{S}(R)$. By Fubini's theorem (deterministic and stochastic) we have

$$J_p^{\lambda,h}(f_p) = \frac{h^p \lambda^{p(h-1)}}{[c_1(h)]^p} \left[\int_R f(t) Y_t^{\lambda,h} dt \right]^p$$

$$= \left[I_1^0 \left(\frac{h}{c_1(h)} \int_s^\infty f(t)(\lambda + t - s)^{h-1} dt \right) \right]^p$$

$$= \left[I_1^0 \circ \left(\frac{h}{c_1(h)} \int_s^\infty f(t)(\lambda + t - s)^{h-1} dt \right) \right]^p$$

$$= I_p^0 \circ \left(\frac{h^p}{[c_1(h)]^p} \int_{s_1}^\infty \dots \int_{s_p}^\infty f^{\otimes p}(t_1, \dots, t_p) \prod_{j=1}^p (\lambda + t_j - s_j)^{h-1} dt_1 \dots d_p \right)$$

$$= I_p^0 \circ (\Lambda_p^{\lambda,h} f^{\otimes p}).$$

The second equality in (3.6) is a consequence of the Hu–Meyer's formula and (3.5).

The general case follows by a density argument by taking into account the continuity of the operators

$$\Lambda_p^{\lambda,h} : \mathcal{S}(R^p) \longrightarrow L^2(R^p),$$
$$\mathrm{Tr}_j \ \Lambda_p^{\lambda,h}. : \mathcal{S}(R^p) \longrightarrow L^2(R^{p-2j}).$$

By Remark 2.1 and the dominated convergence theorem we obtain that

$$\Lambda_p^{\lambda,h} f_p \xrightarrow[\lambda \to 0]{L^2(R^p)} \Lambda_p^{(h)} f_p,$$

$$\int_{R^j} (\Lambda_p^{\lambda,h} f_p)(t_1, t_1, \dots, t_j, t_j, .) dt_1 \dots dt_j$$

$$\xrightarrow[\lambda \to 0]{L^2(R^{p-2j})} \int_{R^j} (\Lambda_p^{(h)} f_p)(t_1, t_1, \dots, t_j, t_j, .) dt_1 \dots dt_j,$$

and therefore, passing to the limit in (3.6), we obtain the convergence

$$J_p^{\lambda,h}(f_p) \xrightarrow[\lambda \to 0]{L^2(\Omega,\mathcal{F},P)} I_p^h \circ (f_p).$$

Remark 3.2. The existence of traces for a deterministic function $f_p(t_1, \dots, t_p)$ requires smoothness properties of f_p. The space $\mathcal{S}(R^p)$ is a an important example of a space of functions for which there exist traces (see Theorem 2.5).

On the other hand, multiple fractional integrals are defined via a transfer principle from multiple Wiener–Itô integrals. The transfer operator is defined in terms of deterministic fractional integrals $I_-^{h,p}$ and it is known that even the well known spaces $\mathcal{S}(R^p)$ and $C_0^\infty(R^p)$ are poorly adapted to $I_-^{h,p}$.

These are some reasons why a rather restrictive condition on f_p is required in Theorem 3.1.

Remark 3.3. Recall that the Lizorkin space Φ is defined by

$$\Phi = \left\{ f_p \in \mathcal{S}(R^p) : \frac{\partial^{|\alpha|}}{\partial t^\alpha} \hat{f}_p(0) = 0,\ \forall \alpha = (\alpha_1, \ldots, \alpha_p),\ \alpha_i \geq 0 \right\},$$

i.e., Φ is the subspace of $\mathcal{S}(R^p)$ orthogonal to all polynomials (the hat accent denotes the Fourier transform).

It is not difficult to see that $I_-^{h,p}\Phi \subset \Phi$, and consequently the above theorem applies for $f_p \in \Phi$.

ACKNOWLEDGMENT. The authors thank the referees for careful reading and constructive suggestions.

References

[1] A. BUDHIRAJA and G. KALLIANPUR, Hilbert space valued traces and multiple Stratonovich integrals with statistical applications, *Probab. Math. Statist.* **15** (1995), 127–163.

[2] A. BUDHIRAJA and G. KALLIANPUR, Approximation to the solution of the Zakai equation using multiple Wiener and Stratonovich integral expansions, *Stoch. Stoch. Rep.* **56** (1996), 271–315.

[3] A. BUDHIRAJA and G. KALLIANPUR, Two results on multiple Stratonovich integrals, *Statistica Sinica* **7** (1997), 907–922.

[4] A. DASGUPTA and G. KALLIANPUR, Chaos decomposition of multiple fractional integrals and applications, *Probab. Theory Related Fields* **115** (1999), 505–525.

[5] A. DASGUPTA and G. KALLIANPUR, Multiple fractional integrals, *Probab. Theory Related Fields* **115** (1999), 527–548.

[6] P. DOUKHAN, G. OPPENHEIM and M. TAQQU, Theory and Applications of Long Range Dependence, *Birkhäuser, Boston*, 2003.

[7] Y. HU and P. A. MEYER, Sur les integrales multiples de Stratonovich, Séminaire de Probabilités XXII. Lecture Notes in Mathematics 1321, *Springer-Verlag*, 1988, 72–81.

[8] Y. HU and P. A. MEYER, On the approximation of Stratonovich multiple integrals, In: Stochastic Processes, A festschrift in honor of G. Kallianpur, (S. Cambanis, J. K. Ghosh, R. L. Karandikar, P. K. Sen, eds.), *Springer-Verlag*, 1993, 141–147.

[9] Y. HU, Integral transformations and anticipative calculus for fractional Brownian motions, *Mem. Amer. Math. Soc.* **175**, no. 825 (2005).

[10] E. IGLÓI and G. TERDIK, Long-range dependence through Gamma-mixed Ornstein–Uhlenbeck process, *Electronic J. Probab.* **4** (1999), 1–13.

[11] K. ITÔ, Multiple Wiener integral, *J. Math. Soc. Japan* **3** (1951), 157–169.

[12] J. W. JEON, K. S. LEE and Y. T. KIM, A Wong–Zakai type approximation for multiple Wiener–Stratonovich integrals, *Stoch. Anal. Appl.* **22** (2004), 131–145.

[13] G. JOHNSON and G. KALLIANPUR, Homogeneous chaos, p-forms, scaling and the Feynman integral, *Trans. Amer. Math. Soc.* **340** (1993), 503–548.

[14] G. JOHNSON and G. KALLIANPUR, Remarks on the existence of k-traces book In: Chaos Expansions, Multiple Wiener–Itô Integrals and Their Applications, (Christian Houdré, Victor Pérez-Abreu, eds.), *CRC Press*, 1994, 47–71.

[15] G. Kallianpur and J. Xiong, Stochastic Differential Equations in Infinite Dimensional Spaces, Vol. 26, Institute of Mathematical Statistics, *Lecture Notes-Monograph Series*, 1995.

[16] Y-J. Lee, Analytic version of test functionals, Fourier transform and a characterization of measures in white noise calculus, *J. Funct. Anal.* **1000** (1991), 359–380.

[17] B. B. Mandelbrot and J. W Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.* **10** (1968), 422–437.

[18] D. Nualart, The Malliavin Calculus and Related Topics, Second Edition, *Springer-Verlag, Berlin*, 2006.

[19] V. Pérez-Abreu and C. Tudor, Multiple stochastic fractional integrals: A transfer principle for multiple stochastic fractional integrals, *Bol. Soc. Mat. Mex.* **8**(3) (2002), 187–203.

[20] V. Pipiras and M. Taqqu, Integraion questions related to fractional Brownian motion, *Probab. Theory Related Fields* **118** (2) (2000), 251–291.

[21] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives, *Gordon and Breach Science, Yverdon*, 1993.

[22] G. Samorodnitsky and M. Taqqu, Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance, *Chapman and Hall, New York, London*, 1994.

[23] J. L. Solé and F. Utzet, Stratonovich integral and trace, *Stoch. Stoch. Reports* **29** (1990), 203–220.

[24] M. Taqqu, Self-similarity and long-range dependence, In: 10th Brazilian Achool of Probability and 2006. Annual Meeting of the Institute of Mathematical Statistics, *IMPA, Rio de Janeiro*, 2006.

[25] C. Tudor and M. Tudor, Approximation of multiple Stratonovich fractional integrals, *Stoch. Anal. Appl.* **25** (2007), 781–799.

[26] M. Zakai, Stochastic integration, trace and the skeleton of Wiener functionals, *Stoch. Stoch. Reports* **32** (1990), 93–108.

CONSTANTIN TUDOR
FACULTY OF MATHEMATICS
AND COMPUTER SCIENCE
UNIVERSITY OF BUCHAREST
14 ACADEMIEI ST.
BUCHAREST 010014
ROMANIA

E-mail: ctudor@fmi.unibuc.ro

MARIA TUDOR
DEPARTMENT OF MATHEMATICS
ACADEMY OF ECONOMICAL STUDIES
BUCHAREST
ROMANIA

E-mail: ctudor50@rdslink.ro

(Received July 2, 2008; revised April 16, 2009)