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Directed fibrations and covering projections

By IOAN POP (Iasi)

Abstract. In this note a notion of Hurewicz fibration in the category d**Top** of directed spaces in the sense of M. Grandis ([8], [9]) is defined. The directed homotopy lifting property is characterized by means of lifting pairs. The unique lifting property for directed paths and loops is studied. Relations with the fundamental category and the fundamental monoid are established. In the context a notion of a directed covering projection is also studied and some relations of this notion with the dicovering spaces of L. Fajstrup ([2], [3]) are established.

1. Introduction

Directed Algebraic Topology is a recent subject which arose from the study of some phenomena in the analysis of concurrency, traffic networks, space-time models, etc. ([1], [4], [5], [7]). It was systematically developed by Marco Grandis ([8], [9]). The directed spaces have privileged directions and directed paths therein do not need to be reversible. M. Grandis introduced and studied its 'non-reversible' homotopical tools corresponding to ordinary homotopies, fundamental group and fundamental n-groupoids: directed homotopies, fundamental monoids and fundamental n-categories. Also some directed homotopy constructions were considered: pushouts and pullbacks, mapping cones and homotopy fibres, suspensions and loops, cofibre and fibre sequences. As for directed fibrations, M. Grandis refers to these (more precisely to the so-called lower and upper d-fibrations) only in relation with the directed h-pullbacks.

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Therefore we have considered the study of lifting properties of directed homotopies (simple, double and 2-homotopy) to be of interest. With this aim we introduce the notion of a directed Hurewicz fibration (called bilateral d-fibration in [9]) for which some unique lifting properties are studied. It is interesting that for a directed map the properties of unique path lifting and unique loop lifting are distinct, while in the usual case they coincide. This is related to the fact that a directed space can have non-constant directed paths but all its loops are trivial (see the ordered circle $\uparrow\mathbb{O}^1 \subset \mathbb{R}\times\uparrow\mathbb{R}$, [8]). Moreover the lifting property for directed homotopies (paths) itself imposes two types of lifts corresponding to both faces of the direct interval $\uparrow[0,1]$, since it is necessary to preserve some results from the usual case, such as the homotopical invariance of the lifting property. Other interesting properties appear in relation with the fundamental category and the fundamental monoid functors $(\uparrow\Pi_1, \uparrow\pi_1)$ applied to a direct Hurewicz fibration.

Directed covering projections constitute important examples of directed maps with the directed (unique) homotopy lifting property. These permit to get interpretations of some d-structures (such as the symmetric d-structure for the sphere $(\mathbb{S}^n)^\sim$ as covering d-structure of directed projective space, [8]) and to find new interesting d-structures (such as the directed n-fold covering of the ordered circle $\uparrow\mathbb{O}^1$).

The basics of Directed Algebraic Topology which we will use are taken from the 2003 paper of Grandis [8]:

A *directed space*, or a *d-space*, is a topological space X equipped with a set $\mathrm{d}X$ of continuous maps $a : \mathbf{I} = [0,1] \longrightarrow X$, called *directed paths*, or *d-paths*, satisfying the following three axioms:

(i) (constant paths) every constant map $\mathbf{I} \longrightarrow X$ is a directed path,

(ii) (reparametrisation) $\mathrm{d}X$ is closed under composition with (weakly) increasing maps $\mathbf{I} \longrightarrow \mathbf{I}$,

(iii) (concatenation) $\mathrm{d}X$ is closed under concatenation (the product of consecutive paths, which will be denoted by $*$ or by $+$).

We use the notations $\underline{X}$ or $\uparrow X$ if X is the underlying topological space; if $\underline{X}$ (or $\uparrow X$) is given, then the set of directed paths is denoted by $\mathrm{d}\underline{X}$ (resp. $\mathrm{d}\uparrow X$) and the underlying space by $|\underline{X}|$) (resp. $|\uparrow X|$).

The standard d-interval with the directed paths given by increasing (weakly) maps $\mathbf{I} \longrightarrow \mathbf{I}$ is denoted by $\uparrow\mathbf{I} = \uparrow[0,1]$.

A *directed map*, or *d-map* $f : \underline{X} \longrightarrow \underline{Y}$, is a continuous mapping between d-spaces which preserves the directed paths: if $a \in \mathrm{d}\underline{X}$ then $f \circ a \in \mathrm{d}\underline{Y}$. The category of directed spaces and directed maps is denoted by $\mathrm{d}\mathbf{Top}$ (or $\uparrow\mathbf{Top}$). A directed path $a \in \mathrm{d}\underline{X}$ defines a directed map $a : \uparrow \mathbf{I} \longrightarrow \underline{X}$ which is also a *path* of $\underline{X}$.

For two points $x, x' \in \underline{X}$ we write $x \preceq x'$ if there exists a directed path from x to x'. The equivalence relation $\simeq$ spanned by $\preceq$ yields the partition of a d-space $\underline{X}$ in its directed path components and a functor $\uparrow \Pi_0 : \mathrm{d}\mathbf{Top} \longrightarrow \mathbf{Set}$, $\uparrow \Pi_0(\underline{X}) = |\underline{X}|/\simeq$. A non-empty d-space $\underline{X}$ is *directed path connected* if $\uparrow \Pi_0(\underline{X})$ contains only one element.

The *directed cylinder* of a d-space $\underline{X}$ is the d-space $\uparrow (\underline{X} \times \mathbf{I})$, denoted by $\underline{X}\times \uparrow \mathbf{I}$ or $\uparrow \mathbf{I}\underline{X}$, for which a path $\mathbf{I} \longrightarrow |\underline{X}| \times \mathbf{I}$ is directed if and only if its components $\mathbf{I} \longrightarrow |\underline{X}|, \mathbf{I} \longrightarrow \mathbf{I}$ are directed. The directed maps $\partial^\alpha : \underline{X} \longrightarrow \underline{X} \times \uparrow \mathbf{I}$, $\alpha = 0, 1$, defined by $\partial^\alpha(x) = (x, \alpha)$, are called the *faces* of the cylinder.

If $f, g : \underline{X} \longrightarrow \underline{Y}$ are directed maps, a *directed homotopy* φ from f to g, denoted by $\varphi : f \longrightarrow g$, or $\varphi : f \preceq g$, is a d-map $\varphi : \underline{X}\times \uparrow \mathbf{I} \longrightarrow \underline{Y}$ such that $\partial^0 \circ \varphi = f$ and $\partial^1 \circ \varphi = g$. The equivalence relation defined by the d-homotopy preorder $\preceq$ is denoted by $f \simeq g$. This means that there exists a finite sequence $f \preceq f_1 \succeq f_2 \preceq f_3 \succeq \ldots g$.

2. The directed homotopy lifting property

Definition 2.1. Let $p : \underline{E} \longrightarrow \underline{B}$, $f : \underline{X} \longrightarrow \underline{B}$ be directed maps. A d-map $f' : \underline{X} \longrightarrow \underline{E}$ is called a directed lift of f with respect to p if $p \circ f' = f$.

Definition 2.2. A directed map $p : \underline{E} \longrightarrow \underline{B}$ is said to have the directed homotopy lifting property with respect to a d-space $\underline{\mathrm{X}}$ if, given d-maps $f' : \underline{X} \longrightarrow \underline{E}$ and $\varphi : \underline{X}\times \uparrow \mathbf{I} \longrightarrow \underline{B}$, and $\alpha \in \{0, 1\}$, such that $\varphi \circ \partial^\alpha = p \circ f'$, there is a directed lift of φ, $\varphi' : \underline{X}\times \uparrow \mathbf{I} \longrightarrow \underline{E}$, with respect to p, $p \circ \varphi' = \varphi$, such that $\varphi' \circ \partial^\alpha = f'$.

$$\begin{tikzcd}
\underline{E} \arrow[r, "p"] & \underline{B} \\
\underline{X} \arrow[u, "f'"] \arrow[r, "\partial^\alpha"'] & \underline{X}\times \uparrow \mathbf{I} \arrow[u, "\varphi"'] \arrow[ul, "\varphi'"]
\end{tikzcd}$$

Theorem 2.3. *If $p : \underline{E} \longrightarrow \underline{B}$ has the directed homotopy lifting property with respect to $\underline{X}$ and $f_0, f_1 : \underline{X} \longrightarrow \underline{B}$ are directed homotopic, $f_0 \simeq f_1$, then f_0 has a directed lift with respect to p if and only if f_1 has this property.*

Proof. Let $f_0' : \underline{X} \longrightarrow \underline{E}$ be a directed lifting of f_0, $p \circ f_0' = f_0$. If $\varphi f_0 \preceq f_1$ or $\varphi f_1 \preceq f_0$, let $\varphi \circ \partial^\alpha = f_0, \alpha \in \{0, 1\}$. Then, if φ' is a lifting of φ, $p \circ \varphi' = \varphi$, we define $f_1' = \varphi' \circ \partial^{1-\alpha}$. For this we have $p \circ f_1' = p \circ \varphi' \circ \partial^{1-\alpha} = \varphi \circ \partial^{1-\alpha} = f_1$.

In the general case, $f_0 \preceq g_1 \succeq g_2 \preceq g_3 \succeq \ldots f_1$, we recurrently apply the consequences of the individual d-homotopies. $\square$

Definition 2.4. A directed map $p : \underline{E} \longrightarrow \underline{B}$ is called a directed (Hurewicz) fibration if p has the directed homotopy lifting property with respect to every directed space (dHLP).

Remark 2.5. In [9] the notions of lower d-fibration and upper d-fibration corresponding to the unilateral lifting properties by taking $\alpha = 1$ and $\alpha = 0$ respectively, are given. A directed fibration is thus both a lower and an upper d-fibration.

Corollary 2.6. *Let $p : \underline{E} \longrightarrow \underline{B}$ be a directed fibration and $a : \uparrow \boldsymbol{I} \longrightarrow \underline{B}$ a directed path in $\underline{B}$ with $a(\alpha) = p(e_\alpha)$, for a point $e_\alpha \in |E|$ and $\alpha \in \{0,1\}$. Then there exists a directed path $a_\alpha : \uparrow \boldsymbol{I} \longrightarrow \underline{B}$ which is a lift of a, $p \circ a_\alpha = a$, with the α-endpoint e_α, $a_\alpha(\alpha) = e_\alpha$. We will refer to such a path a_α as $l_\alpha(a)$ in e_α.*

Example 2.7. Let $\underline{F}$ be any directed space and let $p : \underline{B} \times \underline{F} \longrightarrow \underline{B}$ be the projection on the first factor. Then p is a directed fibration.

Example 2.8. Let $p : E \longrightarrow B$ be a Hurewicz fibration and $\underline{B}$ a d-structure for B. For E we consider the maximal d-structure compatible with p, i.e. $a \in \mathrm{d}(\uparrow E)$ if and only if $p{\circ}a \in \mathrm{d}\underline{B}$. Then $\uparrow p : \uparrow E \longrightarrow \underline{B}$ is a directed fibration. If $f' : \underline{X} \longrightarrow \uparrow E$ and $F : \underline{X} \times \uparrow \mathbf{I} \longrightarrow \underline{B}$, $\alpha \in \{0,1\}$ are given such that $F \circ \partial^\alpha = p \circ f'$, then there exists a homotopy $F' : |\underline{X}| \times \mathbf{I} \longrightarrow |\underline{E}|$ with $F' \circ \partial^\alpha = f'$ and $p \circ F' = F$. Under the given hypothesis on $\uparrow E$, F' is a d-map. Indeed, if $u \circ \in \mathrm{d}(\underline{X} \times \uparrow \mathbf{I})$, then $p \circ (F' \circ u) = F \circ u \in d\underline{B}$, which implies $F' \circ u \in \mathrm{d}(\uparrow E)$. A concrete example: $\uparrow \exp : \uparrow \mathbb{R} \longrightarrow \uparrow \mathbb{S}^1$, defined by $\exp(t) = e^{2\pi i t}$.

Remark 2.9. Obviously, the conditions from the example above are not necessary conditions for a directed fibration. At first, trivially, any directed map from an arbitrary d-space into a d-discrete space (only constant d-paths) is a directed fibration, even if the underlying map is not a Hurewicz fibration. Secondly, even if $p : \underline{E} \longrightarrow \underline{B}$ is a directed fibration with a Hurewicz fibration for the underlying map, we can have a nondirected path in $\underline{E}$ which is projected onto a directed path in $\underline{B}$. Such an example is a trivial directed projection $p : \underline{B} \times \underline{F} \longrightarrow \underline{B}$.

Example 2.10. The following example shows that there exist d-maps such that the underlying maps are Hurewicz fibrations and lower d-fibrations without being directed fibrations. Consider in $\uparrow \mathbb{R}^2$ the subspace $\underline{B} = \uparrow [0,1] \times \{0\}$ and the full triangle $\underline{E}$ with the vertices in the points $(1,0)$, $(0,1)$ and $(1,1)$. Let $p : \underline{E} \longrightarrow \underline{B}$ denote the vertical projection, $p(x,y) = x$, which is a d-map. Then it is immediate that p verifies the desired conditions. But we can see that it is not an upper d-fibration. Indeed, if we consider the map $a : \uparrow \mathbf{I} \longrightarrow \underline{B}$, $a(t) = (t, 0)$, and the point $e_1 = (1, 0) \in p^{-1}(a(1))$, then there is no directed lift of a ending at e_1.

Example 2.11. If $p : \underline{E} \longrightarrow \underline{B}$ is a directed fibration and if for a d-space $\underline{X}$, $\underline{X}^{op}$ denotes the opposite d-space ([8]), ($a \in \mathrm{d}(\underline{X}^{op}) \Leftrightarrow a^{op} = a \circ r \in \mathrm{d}\underline{X}$, where $r(t) = 1 - t$), then $p : \underline{E}^{op} \longrightarrow \underline{B}^{op}$ is also a directed fibration.

Example 2.12. The composite and the product of directed fibrations are directed fibrations.

3. The characterisation of a directed fibration by a directed lifting pair

Given a d-map $p : \underline{E} \longrightarrow \underline{B}$ and $\alpha \in 0, 1$, we consider the following d-subspace of the product $\underline{E} \times \underline{B}^{\uparrow\mathbf{I}}$

$$\underline{B}_\alpha = \{(e, \omega) \in \underline{E} \times \underline{B}^{\uparrow\mathbf{I}} \mid \omega(\alpha) = p(e)\}. \tag{3.1}$$

(The d-structure of $\underline{B}^{\uparrow\mathbf{I}}$ is given by the exponential law, $\mathrm{d}\mathbf{Top}(\uparrow \mathbf{I}, \mathrm{d}\mathbf{Top}(\uparrow \mathbf{I}, \underline{B})) \approx \mathrm{d}\mathbf{Top}(\uparrow \mathbf{I} \times \uparrow \mathbf{I}, \underline{B})$, [8]).

Definition 3.1. A directed lifting pair for a directed map $p : \underline{E} \longrightarrow \underline{B}$ is a pair of d-maps

$$\lambda_\alpha : \underline{B}_\alpha \longrightarrow \underline{E}^{\uparrow\mathbf{I}}, \quad \alpha = 0, 1, \tag{3.2}$$

satisfying the following conditions:

$$\lambda_\alpha(e, \omega)(\alpha) = e, \tag{3.3}$$

$$p \circ \lambda_\alpha(e, \omega) = \omega, \tag{3.4}$$

for each $(e, \omega) \in \underline{B}_\alpha$.

Theorem 3.2. *A directed map $p : \underline{E} \longrightarrow \underline{B}$ is a directed fibration if and only if there exists a directed lifting pair for p.*

Proof. If p is a directed fibration and $\alpha \in 0, 1$, let $f'_\alpha : \underline{B}_\alpha \longrightarrow \underline{E}$ and $F_\alpha : \underline{B}_\alpha \times \uparrow \mathbf{I} \longrightarrow \underline{B}$ be the maps defined by $f'_\alpha(e, \omega) = e$ and $F_\alpha((e, \omega), t) = \omega(t)$. It is obvious that f'_α is continuous and a directed map as a projection. Regarding F_α, this is continuous since if D is an open set in $\underline{B}$, then $F_\alpha^{-1}(D) = \{((e, \omega), t) \in \underline{B}_\alpha \times \uparrow \mathbf{I} \mid \omega(t) \in D\}$, which is an open subset of $\underline{B}_\alpha \times \uparrow \mathbf{I}$. Then if $c : \uparrow \mathbf{I} \longrightarrow \underline{B}_\alpha \times \uparrow \mathbf{I}$ is a directed path, write $c(t') = ((e(t'), \omega(t'), t(t'))$, with some directed paths $e : \uparrow \mathbf{I} \longrightarrow \underline{E}$, $\omega : \uparrow \mathbf{I} \longrightarrow \underline{B}^{\uparrow\mathbf{I}}$, and $t : \uparrow \mathbf{I} \longrightarrow \uparrow \mathbf{I}$. By the exponential d-structure of $\underline{B}^{\uparrow\mathbf{I}}$

([8], p. 291), the map $\widetilde{\omega} : \uparrow\mathbf{I}\times\uparrow\mathbf{I} \longrightarrow \underline{B}$ defined by $\widetilde{\omega}(t', t'') = \omega(t')(t'')$ is directed. This implies that $(F_\alpha \circ c)(t') = \omega(t')(t(t')) = \widetilde{\omega}(t', t(t'))$ is directed. Hence F_α is directed. Then by the equality $F_\alpha((e,\omega),\alpha) = \omega(\alpha) = p(e) = (p \circ f'_\alpha)(e,\omega)$ and by the dHLP for p, there exists $F'_\alpha : \underline{B}_\alpha \times \uparrow\mathbf{I} \longrightarrow \underline{E}$ such that $F'_\alpha((e,\omega),\alpha)) = f'_\alpha)(e,\omega) = e$ and $p \circ F'_\alpha = F_\alpha$. Hence we can define $\lambda_\alpha : \underline{B}_\alpha \longrightarrow \underline{E}^{\uparrow\mathbf{I}}$ by $\lambda_\alpha(e,\omega)(t) = F_\alpha((e,\omega),t)$, which is directed since $\widetilde{F_\alpha} : \underline{B}_\alpha \longrightarrow \underline{E}^{\uparrow\mathbf{I}}$, given by $\widetilde{F_\alpha}((e,\omega),t) = F'_\alpha((e,\omega),t))$, is directed. We have thus obtained a directed lifting pair for p, $(\lambda_\alpha)_{\alpha=0,1}$.

Conversely, if the pair $(\lambda_\alpha)_{\alpha=0,1}$ is given, $\alpha \in \{0,1\}$, let $f' : \underline{X} \longrightarrow \underline{E}$, and $F : \underline{X}\times\uparrow\mathbf{I} \longrightarrow \underline{B}$ be such that $F \circ \partial^\alpha = p \circ f'$. Consider the directed map $g : \underline{X} \longrightarrow \underline{B}^{\uparrow\mathbf{I}}$, defined by $g(x)(t) = F(x,t)$. This permits to define $F' : \underline{X}\times\uparrow\mathbf{I} \longrightarrow \underline{E}$, by $F'(x,t) = \lambda_\alpha(f'(x), g(x))(t)$. This is a directed lift of F, and $F' \circ \partial^\alpha = f'$. Thus p has the dHLP. $\square$

Example 3.3. Let $p : \underline{E} \longrightarrow \underline{B}$ be a directed fibration and $f : \underline{B'} \longrightarrow \underline{B}$ a directed map. Consider $\underline{E'} = \{(b',e) \in \underline{B'} \times \underline{E} \mid f(b') = p(e)\}$, and the projection $p' : \underline{E'} \longrightarrow \underline{B'}$. We can verify that p' is also a directed fibration. For this we define a directed lifting pair starting from a pair $(\lambda_\alpha)_{\alpha=0,1}$ for p. With the above convention of notation, we have $\underline{B'}_\alpha = \{((b',e),\omega') \in \underline{E'} \times \underline{B'}^{\uparrow\mathbf{I}} \mid \omega'(\alpha) = b'\}$. Define $\lambda'_\alpha : \underline{B'}_\alpha \longrightarrow \underline{E'}^{\uparrow\mathbf{I}}$ by $\lambda'_\alpha((b',e),\omega'))(t) = (\omega'(t), \lambda_\alpha(e, f\circ\omega')(t))$, $\alpha = 0,1$. These are d-maps and $\lambda'_\alpha((b',e),\omega'))(\alpha) = (\omega'(\alpha), \lambda_\alpha(e, f\circ\omega')(\alpha)) = (b',e)$ and $(p' \circ \lambda'_\alpha((b',e),\omega')) = \omega'$. Thus $(\lambda'_\alpha)_\alpha = 0,1$ is a directed lifting pair for p'.

Example 3.4. In $\uparrow\mathbb{R}\times\uparrow\mathbb{R}^{op}$ consider the subspaces $\underline{B} = \uparrow[0,1]$ and $\underline{E} = \uparrow([MN] \cup [MP])$, for $M(0,1)$, $N(1,1)$, $P(1,0)$, and let $p : \underline{E} \longrightarrow \underline{B}$ be the vertical projection $p((x,y)) = x$. Then it is immediate that p does not have a lower lifting function, but an upper lifting function $\lambda_1 : \underline{B}_1 \longrightarrow \underline{E}^{\uparrow\mathbf{I}}$. If $(e,\underline{\omega}) \in \underline{B}_1$, with $\underline{\omega}(t) = (\omega(t),0)$, then $\lambda_1(e,\underline{\omega}) = \varepsilon_M$, if $\underline{\omega}$ is the constant path $\varepsilon_{(0,0)}$; $\lambda_1(e,\underline{\omega})(t) = (\omega(t),1)$, if $e \in [MN] \cap p^{-1}(\omega(1))$, and $\lambda_1(e,\underline{\omega})(t) = (\omega(t), 1-\omega(t))$, if $e \in [MP] \cap p^{-1}(\omega(1))$.

Example 3.5. Let $p : \uparrow\mathbf{I}^{\uparrow\mathbf{I}} \to \uparrow\mathbf{I}$ be the map defined by $p(\omega) = \omega(1)$. This is a directed map. Indeed, if $c \in \mathrm{d}(\uparrow\mathbf{I}^{\uparrow\mathbf{I}})$ then the map $\widehat{c} : \uparrow\mathbf{I}\times\uparrow\mathbf{I} \to \uparrow\mathbf{I}$, with $\widehat{c}(t,t') = c(t)(t')$, is directed. Then we have $(p \circ c)(t) = c(t)(1) = \widehat{c}(t,1)$, i.e. $p \circ c = \widehat{c}(-,1)$, which is a directed path. Moreover, we can prove that p is a directed fibration. With the notations from Definition 3.1, we have $(\uparrow\mathbf{I})_\alpha = \{(\omega,\theta) \in \uparrow\mathbf{I}^{\uparrow\mathbf{I}}\times\uparrow\mathbf{I}^{\uparrow\mathbf{I}} \mid \omega(1) = \theta(\alpha)\}, \alpha \in \{0,1\}$.

For $\alpha = 0$ we define the map $\lambda_0 : (\uparrow\mathbf{I})_0 \to (\uparrow\mathbf{I}^{\uparrow\mathbf{I}})^{\uparrow\mathbf{I}}$, by taking $\lambda_0(\omega,\theta) = (\omega * \theta)\left(\frac{t'(t+1)}{2}\right)$. The proof of the continuity of this map is the usual one (see for example [10], Chapter II, §7, p. 159). The property to be directed is also easily verified by the method applied above for p.

Finally, for λ_0 we have: $\lambda_0(\omega,\theta)(0)(t')=(\omega*\theta)(\frac{t'}{2})=\omega(t')$ i.e., $\lambda_0((\omega,\theta))(0)=\omega$, and $\lambda_0(\omega,\theta)(t)(1) = (\omega*\theta)(\frac{t+1}{2}) = \theta(t)$, i.e. $p\circ\lambda_0(\omega,\theta) = \theta$. So, we have a lower lifting function for p.

For $\alpha = 1$, we can define $\lambda_1 : (\uparrow \mathbf{I})_1 \to (\uparrow \mathbf{I}^{\uparrow\mathbf{I}})^{\uparrow\mathbf{I}}$ by $\lambda_1(\omega,\theta)(t)(t') = \frac{\omega(t')\cdot\theta(t)}{\omega(1)}$, if $\omega(1) = \theta(1) \neq 0$ and $\lambda_1(\omega,\theta)(t)(t') = 0$ if $\omega(1) = \theta(1) = 0$. This is a well defined map since $\omega(1) = \theta(1) = 0$ implies that ω and θ are constant paths.

Then we can immediately see that λ_1 is continuous and directed.

Finally, $\lambda_1(\omega,\theta)(1)(t') = \omega(t')$, i.e. $\lambda_1(\omega,\theta)(1) = \omega$ and $\lambda_1(\omega,\theta)(t)(1) = \theta(t)$, i.e. $p\circ\lambda_1(\omega,\theta) = \theta$. So, λ_1 is an upper lifting function for p.

By Theorem 3.2 we conclude that p is a directed fibration.

Remark 3.6. In the theory of undirected fibrations, a series of very important examples are obtained starting from path spaces. Such fibrations are: $p : Y^{\mathbf{I}} \to Y\times Y$, $p(\omega) = (\omega(0),\omega(1))$, $p_\alpha : Y^{\mathbf{I}} \to Y$, $p_\alpha(\omega) = \omega(\alpha)$, $\alpha\in\{0,1\}$, $p_{y_0} : P(Y,y_0) := \{\omega\in Y^{\mathbf{I}} \mid \omega(0) = y_0\} \to Y$, $p_{y_0}(\omega) = \omega(1)$, and $p_f : E_f := \{(e,\beta)\in E\times B^{\mathbf{I}} \mid f(e) = \beta(0)\}$, $p_f((e,\beta)) = \beta(1)$, for $f : E\to B$ an arbitrary map (see [12], Chapter 2, Section 8).

In the directed case, these examples have no counterpart.

Generalizing Example 3.5, consider the directed map $p : \underline{Y}^{\uparrow\mathbf{I}} \to \underline{Y}$, $p(\omega) = \omega(1)$, for an arbitrary space Y.

With the notations from Definition 3.1, we have $\underline{Y}_\alpha = \{(\omega,\theta)\in \underline{Y}^{\uparrow\mathbf{I}}\times\underline{Y}^{\uparrow\mathbf{I}} \mid \omega(1) = \theta(\alpha)\}$, $\alpha\in\{0,1\}$.

For $\alpha = 0$ we can define a directed map $\lambda_0 : \underline{Y}_0 \to (\underline{Y}^{\uparrow\mathbf{I}})^{\uparrow\mathbf{I}}$, by taking $\lambda_0(\omega,\theta) = (\omega*\theta)\big(\frac{t'(t+1)}{2}\big)$. For this we have: $\lambda_0(\omega,\theta)(0)(t') = (\omega*\theta)(\frac{t'}{2})\omega(t')$ i.e., $\lambda_0((\omega,\theta))(0) = \omega$, and $\lambda_0(\omega,\theta)(t)(1) = (\omega*\theta)\big(\frac{t+1}{2}\big) = \theta(t)$ i.e., $p\circ\lambda_0(\omega,\theta) = \theta$. So, we have a lower lifting function for p.

The obvious candidate, in the general case, for an upper lifting function is the map $\lambda_1 : \underline{Y}_1 \to (\underline{Y}^{\uparrow\mathbf{I}})^{\uparrow\mathbf{I}}$, defined by $\lambda_1(\omega,\theta)(t)(t') = (\omega*\theta^{-1})\big(\frac{t'(2-t)}{2}\big)$. For this map we have $\lambda_1(\omega,\theta)(1)(t') = (\omega*\theta^{-1})(\frac{t'}{2}) = \omega(t')$ i.e., $\lambda_1(\omega,\theta)(1) = \omega$, and $\lambda_1(\omega,\theta)(t)(1) = (\omega*\theta^{-1})(\frac{2-t}{2}) = \theta(t)$ i.e., $p\circ\lambda_1(\omega,\theta) = \theta$. But this map is not directed. It seems unlikely that the map p is a directed fibration for a general d-space Y. If $\underline{Y}$ is symmetric or only reflexive (see[8], p. 285) then p is a directed fibration. Similar remarks hold for the other maps from the beginning of this remark.

Remark 3.7. In the general case, for an arbitrary d-map $p : \underline{E} \longrightarrow \underline{B}$, the d-spaces $\underline{B}_0$ and $\underline{B}_1$ are distinct. But if $\underline{B}$ is symmetric then these d-spaces are

d-homeomorphic. In this case we can define $F : \underline{B}_0 \longrightarrow \underline{B}_1$ and $G : \underline{B}_1 \longrightarrow \underline{B}_0$, by $F(e,\omega)) = (e, \omega^{op})$ and $G((e',\omega')) = (e', \omega'^{op})$. We verify that F is a directed map. First it is continuous: if U is an open set in $|\underline{E}|$ and $\mathfrak{B}(K,D)$ is an element of the subbase of the compact-open topology in $\underline{B}^{\uparrow\mathbf{I}}$, then $F^{-1}((U \times \mathfrak{B}(K,D)) \cap |\underline{B}_1|) = (U \times \mathfrak{B}(1-K,D)) \cap |\underline{B}_0|$. Then, if $u : \uparrow \mathbf{I} \longrightarrow \underline{B}_0$ is a directed path, let $u(t) = (e(t), \omega(t))$, with directed paths $e : \uparrow \mathbf{I} \longrightarrow \underline{E}$ and $\omega : \uparrow \mathbf{I} \longrightarrow \underline{B}^{\uparrow\mathbf{I}}$, (satisfying $p(e(t)) = \omega(t)(0)$), the path $(F \circ u)(t) = (e(t), \omega(t)^{op})$ is directed since $\underline{B}^{\uparrow\mathbf{I}}$ is symmetric. Now it is clear that F is an d-homeomorphism with inverse G.

But even in this case the functions λ_0 and λ_1 are both necessary. However if $\underline{E}$ is also symmetric, we can define $\lambda_1((e',\omega'))=(\lambda_0(e',\omega'^{op}))^{op}$, for any $(e',\omega') \in \underline{B}_1$.

Theorem 3.8. *If $p : \underline{E} \longrightarrow \underline{B}$ is a directed fibration, then the d-spaces $\underline{B}_0$ and $\underline{B}_1$ are d-homotopy equivalent.*

Proof. Consider a directed lifting pair $(\lambda_\alpha)_{\alpha=0,1}$ for p. Then we can define the maps $f : \underline{B}_0 \longrightarrow \underline{B}_1$ and $g : \underline{B}_1 \longrightarrow \underline{B}_0$, by $f((e,\omega)) = (\lambda_0(e,\omega)(1), \omega)$ and $g((e',\omega')) = (\lambda_1(e',\omega')(0), \omega')$. We will prove that these define a directed homotopy equivalence. At first we we can see that these maps are continuous and directed since the maps λ_α and ∂^α, $\alpha = 0,1$, have these properties.

Now $(g \circ f)(e,\omega) = (\lambda_1(\lambda_0(e,\omega)(1),\omega)(0),\omega)$. Define $H : \underline{B}_0 \times \uparrow \mathbf{I} \longrightarrow \underline{B}_0$ by $H((e,\omega),t) = (\lambda_1(\lambda_0(e,\omega)(t),\omega_t)(0),\omega)$, where $\omega_t(t') = \omega(tt')$. By the same simple arguments as above we deduce that H is a continuous directed map. Moreover, this map satisfies the relation $H : H_0 \preceq g \circ f$, where $H_0((e,\omega)) = (\lambda_1(e,\omega_0)(0),\omega)$. Then, for $K : \underline{B}_0 \times \uparrow \mathbf{I} \longrightarrow \underline{B}_0$, defined by $K((e,\omega),t) = (\lambda_1(e,\omega_0)(t),\omega)$, we have $K : H_0 \preceq \mathrm{id}_{\underline{B}_0}$. Therefore we obtain $g \circ f \simeq_d \mathrm{id}_{\underline{B}_0}$. Similarly one verifies the relation $f \circ g \simeq_d \mathrm{id}_{\underline{B}_1}$. $\square$

4. The unique directed lifting properties

Definition 4.1. We say that a directed map $p : \underline{E} \longrightarrow \underline{B}$ has unique directed path lifting if, given $\alpha \in \{0,1\}$ and directed paths $\omega, \omega' : \uparrow \mathbf{I} \longrightarrow \underline{E}$, such that $\omega \circ \partial^\alpha = \omega' \circ \partial^\alpha$ and $p \circ \omega = p \circ \omega'$, it follows that $\omega = \omega'$.

Definition 4.2. We say that a directed map $p : \underline{E} \longrightarrow \underline{B}$ has unique directed loop lifting if, given directed loops $\theta, \theta' : \uparrow \mathbf{I} \longrightarrow \underline{E}$, with the same endpoints and $p \circ \theta = p \circ \theta'$, it follows that $\theta = \theta'$.

Remark 4.3. In the usual case (natural d-structures) these unique lifting properties coincide.

Remark 4.4. It is obvious that the property of unique directed path lifting implies the property of unique directed loop lifting.

Example 4.5. If $\uparrow p : \uparrow E \longrightarrow \uparrow B$ is a direct map such that $p : E \longrightarrow B$ is a Hurewicz fibration with unique path lifting then $\uparrow p$ has unique directed path lifting.

Example 4.6. Any d-map $p : \uparrow \mathbb{O}^1 \longrightarrow \underline{B}$ has unique directed loop lifting since the ordered circle $\uparrow \mathbb{O}^1$ does not have non-constant loops.

Example 4.7. Consider in the complex plane $\mathbb{C}$ the standard directed circle $\uparrow \mathbb{S}^1$ and the ordered circle $\uparrow \mathbb{O}^1$ as a directed subspace of $\uparrow \mathbb{R}^{op} \times \mathbb{R}$, and let $p : \uparrow \mathbb{O}^1 \longrightarrow \uparrow \mathbb{S}^1$ be the map defined by

$$p(z) = \begin{cases} z^2, & \text{if } 0 \leq \arg z \leq \pi, \\ \bar{z}^2, & \text{if } \pi \leq \arg z \leq 2\pi. \end{cases}$$

Then p is a d-map with unique directed loop lifting (as in the precedent example). But the distinct paths $\omega, \omega' : \uparrow \mathbf{I} \longrightarrow \uparrow \mathbb{O}^1$ defined by $\omega(t) = e^{\pi i t/4}$ and $\omega'(t) = e^{-\pi i t/4}$, have the same origin (in 1) and the same projection. Hence p does not have unique directed path lifting.

Theorem 4.8. *Let $p : \underline{E} \longrightarrow \underline{B}$ be a directed fibration with unique directed path lifting. Then if $\underline{Y}$ is a directed path connected d-space and $f, g : \underline{Y} \longrightarrow \underline{E}$ are d-maps such that $p \circ f = p \circ g$ and $f(y_0) = g(y_0)$, for some $y_0 \in \underline{Y}$, then $f = g$.*

Proof. Let $y \in \underline{Y}$ be an arbitrary point. If $\omega : y_0 \preceq y$ or $\omega : y \preceq y_0$, then we consider the directed paths $f \circ \omega$, $g \circ \omega$, with $p \circ (f \circ \omega) = p \circ (g \circ \omega)$ and $f \circ \omega \circ \partial^\alpha = g \circ \omega \circ \partial^\alpha$, for $\alpha = 0$ or $\alpha = 1$. This implies $f \circ \omega = g \circ \omega$. Particularly, $f \circ \omega \circ \partial^{1-\alpha} = g \circ \omega \circ \partial^{1-\alpha}$, which implies $f(y) = g(y)$.

In the general case, $y_0 \preceq y_1 \succeq y_2 \preceq \ldots y$, we recurrently apply the consequences of the immediate d-connection relations. □

Theorem 4.9. *If a directed fibration $p : \underline{E} \longrightarrow \underline{B}$ has unique directed path (loop) lifting, then every fiber of p has no nonconstant directed paths (loops).*

Proof. Let $\omega : \uparrow \mathbf{I} \longrightarrow \uparrow p^{-1}(b)$ be an arbitrary directed path (loop) in the fiber $p^{-1}(b)$, $b \in \underline{B}$, with the d-structure of d-subspace of $\underline{E}$. Then if ω' is the constant path $\omega'(t) = \omega(0), \forall t \in [0, 1]$, we have $p \circ \omega = p \circ \omega'$ and this implies $\omega = \omega'$. □

Remark 4.10. It is known that for usual fibrations the converse of 4.9 is also true ([12], Theorem 5, p. 68). But for the directed case we have obtained only a particular result: If $p : \underline{E} \longrightarrow \underline{B}$ is a directed fibration with the total space $\underline{E}$ a symmetric d-space, then p has unique directed path lifting if and only if every fiber has no nonconstant directed paths. The proof is the same as in the usual case and we omit it.

Theorem 4.11. *A directed fibration with unique directed path lifting has the directed* 2*-homotopy lifting property.*

Proof. Let $p : \underline{E} \longrightarrow \underline{B}$ be a directed fibration with unique directed path lifting. Suppose that $\alpha \in \{0,1\}$, $\varphi : \underline{X} \times \uparrow \mathbf{I} \longrightarrow \underline{E}$ is a directed homotopy, and $\Phi : \underline{X} \times \uparrow \mathbf{I}^2 \longrightarrow \underline{B}$ is a 2-homotopy with $\Phi(x,t,\alpha) = (p \circ \varphi)(x,t), \forall (x,t) \in \underline{X} \times \uparrow \mathbf{I}$, and for $\beta \in \{0,1\}, \Phi(x,\beta,t') = \Phi(x,\beta,\alpha) = (p \circ \varphi)(x,\beta)$ (see[8], p. 297). By the dHLP there exists a double homotopy $\Phi' : \underline{X} \times \uparrow \mathbf{I}^2 \longrightarrow \underline{B}$, with $\Phi'(x,t,\alpha) = \varphi(x,t)$ and $p \circ \Phi' = \Phi$. We verify that under the condition of the unique directed path lifting property for p, Φ' is a 2-homotopy. For $x \in \underline{X}$ consider the directed paths $\Phi'_x, \varphi_x : \uparrow \mathbf{I} \longrightarrow \underline{E}$, defined by $\Phi'_x(t') = \Phi'(x,\beta,t')$ and $\varphi_x(t') = \varphi(x,\beta)$. For these paths we have $p \circ \Phi'_x = p \circ \varphi_x$ and $\Phi'_x(\alpha) = \Phi'(x,\beta,\alpha) = \varphi(x,\beta) = \varphi_x(\alpha)$. It follows that $\Phi'_x = \varphi_x$ and therefore $\Phi'(x,\beta,t') = \varphi(x,\beta), \forall t' \in [0,1]$. $\square$

Theorem 4.12. *If $p : \underline{E} \longrightarrow \underline{B}$ is a directed fibration with unique directed path lifting, the spaces $\underline{B}_0$ and $\underline{B}_1$ are d-isomorphic.*

Proof. We use the notations from the proof of Theorem 3.8. By the unique directed path lifting property we have that $\lambda_1(\lambda_0(e,\omega)(1),\omega) = \lambda_0(e,\omega)$ since $\lambda_1(\lambda_0(e,\omega)(1),\omega)(1) = \lambda_0(e,\omega)(1)$ and $p \circ \lambda_1(\lambda_0(e,\omega)(1),\omega) = \omega = p \circ \lambda_0$. It follows that $(g \circ f)((e,\omega)) = \lambda_1(\lambda_0(e,\omega)(1),\omega)(0),\omega) = (\lambda_0(e,\omega)(0),\omega) = (e,\omega)$. Hence $g \circ f = \mathrm{id}_{\underline{B}_0}$. Similarly we have $f \circ g = \mathrm{id}_{\underline{B}_1}$. $\square$

Remark 4.13. In the general case, for a directed fibration there might exist more directed lifting pairs. But if the fibration has unique directed path lifting there exists only one.

5. Relations with the fundamental category and the fundamental monoid

For a d-space $\underline{X}$ denote by $\uparrow \Pi_1(\underline{X})$ the *fundamental category*([8], p. 301). This has the points of $|\underline{X}|$ as objects and $[a] : x \to x'$ the 2-homotopy classes of paths from x to x' as arrows. More precisely, two directed paths $a, a' : \uparrow \mathbf{I} \to \underline{X}$ from x to x' are in relation $a \preceq_2 a'$ if there is a 2-path $A : \uparrow \mathbf{I}^2 \to \underline{X}$ with $A \circ \partial_2^0 = a, A \circ \partial_2^1 = a'$ and with the faces ∂_1^α degenerate. This means: $A(t,0) =$

$a(t)$, $A(t,1) = a'(t)$, $A(\alpha,t') = A(\alpha,0) = a(\alpha) = A(\alpha,1) = a'(\alpha)$, $\alpha \in \{0,1\}$. The equivalence relation spanned by the preorder $\preceq_2$ is denoted by $\simeq_2$ and $[a]$ is the equivalence class of the directed path a with respect to this relation. The composition of arrows, written additively, is induced by the concatenation of consecutive paths, $[a]+[b] = [a+b]$, and the identities derive from degenerate paths, $0_x = [e(x)] = [0_x]$.

In a natural way, a covariant functor $\uparrow\Pi_1 : \mathrm{d}\mathbf{Top} \longrightarrow \mathbf{Cat}$ (small categories) is defined by correspondences $\underline{X} \Rightarrow \uparrow\Pi_1(\underline{X}), (f : \underline{X} \to \underline{Y}) \Rightarrow \uparrow\Pi_1(f) : \uparrow\Pi_1(\underline{X}) \longrightarrow \uparrow\Pi_1(\underline{Y})$, with $\uparrow\Pi_1(f)(x) = f(x)$ and $\uparrow\Pi_1(f)([a]) = [f\circ a]$. This functor also preserves d-homotopy, d-homotopy equivalences and deformation retracts. But for a d-homotopy equivalence $f : \underline{X} \longrightarrow \underline{Y}$, the induced functor $\uparrow\Pi_1(f)$ does not have to be full nor faithful.
For a pointed d-space $(\underline{X}, x)$ the *fundamental monoid* $\uparrow\pi_1(\underline{X}, x) = \uparrow\Pi_1(\underline{X})(x,x)$ is defined. Its elements are the 2-homotopy classes of loops at x. It gives a functor from the category $\mathrm{d}\mathbf{Top}_*$ of pointed d-spaces, to the category of monoids, $\uparrow\pi_1 : \mathrm{d}\mathbf{Top}_* \longrightarrow \mathbf{Mon}, (\underline{X}, x) \Rightarrow \uparrow\pi_1(\underline{X}, x)$, $(f : \underline{X} \to \underline{Y}) \Rightarrow f_{*1} : \uparrow\pi_1(\underline{X}, x) \to \uparrow\pi_1(\underline{Y}, y), f_{*1}([a]) = [f\circ a])$.

Theorem 5.1. *If $p : \underline{E} \longrightarrow \underline{B}$ is a directed fibration with unique directed path lifting then the functor $\uparrow\Pi_1(p) : \uparrow\Pi_1(\underline{E}) \longrightarrow \uparrow\Pi_1(\underline{B})$ is faithful.*

Proof. Let $e, e' \in \underline{E}$ and $[a], [a'] \in \uparrow\Pi_1(e, e')$ be such that $p_{*1}([a]) = p_{*1}([a'])$, i.e., $p\circ a \simeq_2 p\circ a'$. We need to prove that $a \simeq_2 a'$. First we suppose that $p\circ a \preceq_2 p\circ a'$ or $p\circ a' \preceq_2 a$ by a 2-homotopy $A : \uparrow \mathbf{I}^2 \to \underline{B}$. By the dHLP there exists $A' : \uparrow \mathbf{I}^2 \to \underline{B}$ with $p\circ A' = A$ and $A'(t,\alpha) = a(t)$. Then since $p(A'(0,t')) = A(0,t') = A(0,0) = p(a(0) = p(e)$ and $p(A'(1,t')) = A(1,t') = A(1,0) = p(a(1)) = p(e')$, we have that $A'(0,t') \in p^{-1}(p(e))$ and $A'(1,t') \in p^{-1}(p(e')), \forall t' \in [0,1]$, and by Theorem 4.9 it follows that these paths are constant. Hence $A'(0,t') = A'(0,0) = e$, and $A'(1,t') = e'$. Moreover, $p(A'(t, 1-\alpha)) = A(t, 1-\alpha) = p(a'(t))$ and $A'(0, 1-\alpha) = e = a'(0)$, which by the unique directed path lifting implies $A'(t, 1-\alpha) = a'(t)$. Thus we have verified that $A" : a \preceq_2 a'$ or $A' : a' \preceq_2 a$. The general case can be reduced to the situation $p\circ a \preceq_2 b_1 \succeq_2 p\circ a'$. Let $H : p\circ a \preceq_2 b_1$. Then if H' is the lift of H such that $H'(t,0) = a(t)$, we define $a_1(t) = H'(t,1)$. As above we deduce that $a_1(0) = e, a_1(1) = e'$ and $H' : a \preceq_2 a_1$. Then by $p\circ a_1 = b_1$ and $b_1 \succeq_2 p\circ a'$ we have also the relation $a_1 \succeq_2 a'$. Now the general implication $p\circ a \simeq_2 a' \Rightarrow a \simeq_2 a'$ is clear and this finishes the proof. $\square$

Remark 5.2. By Theorem 5.1 we can deduce that if $p : \underline{E} \longrightarrow \underline{B}$ is a directed fibration with unique directed path lifting then $\uparrow\pi_1(p) : \uparrow\pi_1(\underline{E}, e) \longrightarrow$

$\uparrow \pi_1(\underline{B}, p(e))$, $e \in \underline{E}$ has a trivial kernel. But we will prove below that this conclusion holds under the weaker condition of the unique directed loop lifting.

Theorem 5.3. *Let $p : \underline{E} \longrightarrow \underline{B}$ be a directed fibration, with $\underline{B}$ symmetric. Let $\uparrow F = \uparrow p^{-1}(b)$ be the fiber over b with the d-structure of d-subspace of $\underline{E}$ and $e \in F$. Let $i : (\uparrow F, e) \to (\underline{E}, e)$ be the d-inclusion. Then the following sequence in the the category of monoids*

$$\uparrow \pi_1(\uparrow F, e) \xrightarrow{\uparrow \pi_1(i)} \uparrow \pi_1(\underline{E}, e) \xrightarrow{\uparrow \pi_1(p)} \uparrow \pi_1(\underline{B}, b)$$

is exact.

Proof. (According to [13], p. 242). Denote $\uparrow \pi_1(p) = p_{*1}$ and $\uparrow \pi_1(i) = i_{*1}$. The inclusion $\operatorname{Im} i_{*1} \subseteq \ker p_{*1}$ is immediate since $p \circ i$ is a constant map.

Now suppose that $[a] \in \ker p_{*1}$, i.e., $p \circ a \simeq_2 0_b$. We need to prove that $[a] \in \operatorname{Im} i_{*1}$. It is sufficient to suppose that $A : p \circ a \preceq_2 0_b$ or $A : 0_b \preceq_2 p \circ a$ is given. Let $A \circ \partial_2^\alpha = p \circ a$. If $\uparrow \mathbb{S}^1$ is the standard directed circle, we can define the directed maps $f : \uparrow \mathbb{S}^1 \longrightarrow \underline{B}$ and $\Phi : \uparrow \mathbb{S}^1 \times \uparrow \mathbf{I} \longrightarrow \underline{B}$ by $f(e^{2\pi i t} = a(t)$ and $\Phi(e^{2\pi i t}, \tau) = A(t, \tau)$. Then we have $\Phi \circ \partial_2^\alpha = p \circ f$. Then there is a directed homotopy Φ' : $\uparrow \mathbb{S}^1 \times \uparrow \mathbf{I} \longrightarrow \underline{E}$, with $p \circ \Phi' = \Phi$ and $\Phi \circ \partial_2^\alpha = f$. This defines $A' : \uparrow \mathbf{I} \times \uparrow \mathbf{I} \longrightarrow \underline{E}$, by $A'(t, \tau) = \Phi'(e^{2\pi i t}, \tau)$. For this we have $A' \circ \partial_2^\alpha = a$, $A'(0, \tau) = A'(1, \tau) = e$ and $p \circ A' = A$. By the last relation it follows $p \circ A' \circ \partial_2^{1-\alpha} = A \circ \partial_2^{1-\alpha} = 0_b$, so that $A' \circ \partial_2^{1-\alpha}$ defines an element of $\uparrow \pi_1(\uparrow F, e)$ 2-homotopy equivalent in $(\underline{E}, e)$ to a. Hence we have $[a] = [A' \circ \partial_2^{1-\alpha}] = i_{*1}[A' \circ \partial_2^{1-\alpha}]$. thus we have proved the inclusion $\ker p_{*1} \subseteq \operatorname{Im} i_{*1}$ and this finishes the proof. □

Now by this theorem and by Theorem 4.9 we obtain:

Corollary 5.4. *If $p : \underline{E} \longrightarrow \underline{B}$ is a directed fibration with unique directed loop lifting and $\underline{B}$ symmetric, then $\uparrow \pi_1(p) : \uparrow \pi_1(\underline{E}, e) \longrightarrow \uparrow \pi_1(\underline{B}, p(e)), e \in \underline{E}$ has a trivial kernel.*

Remark 5.5. We make the remark the fact that the condition of trivial kernel for a morphism in the category of monoids is only necessary but not sufficient for the monomorphism property in this category.

6. Directed covering projections

Definition 6.1. A directed map $p : \underline{\widetilde{X}} \longrightarrow \underline{X}$ is called a directed covering projection if there exists an open cover $\mathcal{U} = \{U\}$ of the underlying space $|\underline{X}|$ satisfying the following conditions:

(i) For any subset $U \in \mathcal{U}$, $p^{-1}(U)$ is the disjoint union of open subsets of $|\underline{\widetilde{X}}|$ each of which is mapped homeomorphically onto U by p.

(ii) If $U \in \mathcal{U}$ and $p^{-1}(U) = \cup \widetilde{U}$ is as in (i), then considering U and $\widetilde{U}$ with the directed structures $\uparrow U$ and $\uparrow \widetilde{U}$ of subspaces of $\underline{X}$ and $\underline{\widetilde{X}}$ respectively, the homeomorphism $p/\widetilde{U} : \widetilde{U} \to U$ becomes a directed isomorphism (homeomorphism) $p/\widetilde{U} : \uparrow \widetilde{U} \to \uparrow U$. We will say (after [12], p. 62) that $\mathcal{U}$ consists of open subsets directed evenly covered by p.

If a direct covering projection $p : \underline{\widetilde{X}} \longrightarrow \underline{X}$ exists, then we say that $\underline{\widetilde{X}}$ is a directed covering space of $\underline{X}$.

Remark 6.2. If $p : \underline{\widetilde{X}} \longrightarrow \underline{X}$ is a directed covering projection then the underlying map $p : |\underline{\widetilde{X}}| \longrightarrow |\underline{X}|$ is a covering projection.

Remark 6.3. As in the classical case, in order to obtain some special properties of directed covering projections, it is necessary to impose some conditions on the base space, such as directed path connectedness, locally directed path-connectedness etc. These conditions are not especially mentioned in this paper, but they are implied by the corresponding classical results.

Theorem 6.4. *A directed covering projection is a directed fibration with unique directed path lifting.*

Proof. We reproduce the more important steps from the proof in the usual case ([12], p. 67–68), emphasizing only the directed aspects.

Let $p : \underline{\widetilde{X}} \longrightarrow \underline{X}$ be a directed covering projection and let $f' : \underline{Y} \longrightarrow \underline{\widetilde{X}}$ and $F : \underline{Y} \times \uparrow \mathbf{I} \longrightarrow \underline{X}$ be directed maps such that $F \circ \partial^{\alpha} = p \circ f', \alpha \in \{0,1\}$. First we suppose $\alpha = 0$. For a point $y \in \underline{Y}$ there is an open neighborhood N_y of y and a sequence $0 = t_0 < t_1 < \cdots < t_m = 1$ of points of $\mathbf{I}$ such that for $i = 1, \ldots, m$, $F(N_y \times [t_{i-1}, t_i]$ is contained in some open subset of $|\underline{X}|$ directed evenly covered by p. Then there is a directed map $F'_y : \uparrow N_y \times \uparrow \mathbf{I} \longrightarrow \underline{\widetilde{X}}$, where vertical arrows indicate the directed subspace structures, such that $F'_y \circ \partial^0 = f'/\uparrow N_y$ and $p \circ F'_y = F/\uparrow N_y \times \uparrow \mathbf{I}$. We prove this. Assume that $F(N_y \times [t_0, t_1]) \subset U$, where U is an open set directed evenly covered by p, with $p^{-1}(U) = \cup \widetilde{U}_j$ and p maps $\uparrow \widetilde{U}_j$ isomorphically directed onto $\uparrow U$ for each j. Let $V_j = f'^{-1}(\widetilde{U}_j)$. Then V_j is a collection of disjoint open sets covering N_y, and we define $G_1 : N_y \times [t_0, t_1] \longrightarrow |\underline{\widetilde{X}}|$ such that for each j, $G_1/V_j \times [t_0, t_1] = (p/U_j)^{-1} \circ F/V_j \times [t_0, t_1]$. Since $G_1/V_j \times [t_0, t_1]$ is a directed map, as a composition of directed maps, and $\uparrow N_y \times \uparrow [t_0, t_1]$ coincide with the sum $\Sigma_j \uparrow V_j \times \uparrow [t_0, t_1]$, we deduce a directed map $G_1 : \uparrow N_y \times \uparrow [t_0, t_1] \longrightarrow |\underline{\widetilde{X}}|$. This verifies $G_1 \circ \partial_0 = f'/N_y$ and $p \circ G_1 = F/\uparrow N_y \times \uparrow [t_0, t_1]$. Then, by induction on i, one can define directed maps $G_i \uparrow$

$N_y\times\uparrow[t_{i-1},t_i]\longrightarrow \underline{\widetilde{X}}, i=2,\ldots,m|$ such that $p\circ G_i=F/\uparrow N_y\times\uparrow[t_{i-1},t_i]$ and $G_{i-1}/N_y\times t_{i-1}=G_i/N_y\times t_{i-1}$. Assume G_{i-1} defined for $1<i\leqslant m$. Let U' be an open subset of $|\underline{X}|$ directed evenly covered by p such that $F(N_y\times[t_{i-1},t_i]))\subset U'$.

Let $\{\widetilde{U'_k}\}$ be a collection of disjoint open subsets of $|\underline{\widetilde{X}}|$ such that $p^{-1}(U')=\cup\widetilde{U'_k}$ and p maps $\uparrow\widetilde{U'_k}$ directed isomorphically onto U' for each k. Let $V'_k=\{y'\in N_y\mid G_{i-1}(y',t_{i-1})\in\widetilde{U'_k}\}$. Then $\{V'_k\}$ is a collection of disjoint open sets covering N_y, and define G_i such that $G_i/V'_k\times[t_{i-1},t_i]=(p/\uparrow\widetilde{U'_k})^{-1}\circ F/V'_k\times[t_{i-1},t_i]$. This is a directed map and verifies the desired properties.

Now we can define F'_y by $F'_y/N_y\times[t_{i-1},t_i]=G_i$. This is well defined and verifies the desired properties. In order to see that it is also a directed map, let $a:\uparrow\mathbf{I}\to\uparrow N_y\times\uparrow\mathbf{I}$ be a directed path written as $a(t)=(n(t),u(t)$. Define $a_i:\uparrow\mathbf{I}\to\uparrow N_y\times\uparrow[t_{i-1},t_i], i=1,\ldots,m$ by $a_i(t)=(n(t),u(t_{i-1}+t(t_i-t_{i-1}))$. Then $G_i\circ a_i$ are consecutive directed paths and $F'_y\circ a$ is the concatenation $G_1\circ a_1+G_2\circ a_2+\cdots+G_m\circ a_m$. Thus F'_y is a directed map.

It is proved in ([12], p. 67) that these neighborhoods N_y and the (directed) maps F'_y verify $F'_y\mid(N_y\cap N_{y'})\times\mathbf{I}=F'_{y'}\mid(N_y\cap N_{y'})\times\mathbf{I}$ such that there is a continuous map $F':|\underline{Y}|\times\mathbf{I}\longrightarrow|\underline{\widetilde{X}}|$ such that $F'\mid N_y\times\mathbf{I}=F'_y$, and F' is a lift of F such that $F'\circ\partial^0=f'$. Moreover, if a is a directed path in $\underline{Y}\times\uparrow\mathbf{I}$, then this can be written as a finite concatenation of directed paths in some $\uparrow N_y\times\uparrow\mathbf{I}$ and this implies that $F'\circ a$ is a finite concatenation of directed pats, hence is itself a directed path in $|\underline{\widetilde{X}}|$. By this we have proved that p has the lower dHLP.

In the case $F\circ\partial^1=p\circ f'$, one defines the collection of directed maps $\{G_i\}_{i=1,\ldots,m}$ starting from $G_m:\uparrow N_y\times\uparrow[t_{m-1},t_m]\longrightarrow\underline{\widetilde{X}}$ and we proceed as in the case $\alpha=0$.

The unique directed path lifting property followed by the fact that $p:|\underline{\widetilde{X}}|\longrightarrow|\underline{X}|$ is a covering projection. $\square$

Theorem 6.5. *Let $p:\underline{E}\longrightarrow\underline{B}$ be a directed map with the underlying map $p:|\underline{E}|\longrightarrow|\underline{B}|$ a covering projection. Then the following conditions are equivalent:*

- (i) *p is a directed covering projection.*
- (ii) *p is a directed fibration with unique directed path lifting.*
- (iii) *A path in $|\underline{E}|$ is a directed path of $\underline{E}$ if and only if its projection is a directed path of $\underline{B}$.*

Proof. (i) $\Longrightarrow$ (ii) is from Theorem 6.4.

(ii) $\Longrightarrow$ (iii). Let $a':\mathbf{I}\longrightarrow|\underline{E}|$ be a path with $p\circ a'=:a\in \mathrm{d}\underline{B}$. By the dHLP there exists a directed path $\widetilde{a}\in\mathrm{d}\underline{E}$ with $p\circ\widetilde{a}=a$ and $\widetilde{a}(0)=a'(0)$ (or with $\widetilde{a}(1) = a'(1)$). Then the unique path lifting (for all paths) property implies $a' = \widetilde{a}$.

(iii) $\Longleftrightarrow$ (i). Consider $\{U\}$ a cover of $|\underline{B}|$ consisting of open sets evenly covered by p. If $\widetilde{U}$ is a connected component of $p^{-1}(U)$, it is sufficient to prove that $p/\uparrow\widetilde{U} : \uparrow\widetilde{U} \to \uparrow U$ is a directed isomorphism. This means that $(p/\widetilde{U})^{-1}$ is a d-map. But if $a \in \mathrm{d}\uparrow U$, then $a \in d\underline{B}$ and this implies $(p/\widetilde{U})^{-1}(a) \in \mathrm{d}\underline{E}$ and since $\mathrm{Im}\, a \subset \widetilde{U}$ we conclude that $a \in \mathrm{d}\uparrow\widetilde{U}$. $\square$

Corollary 6.6. *If $p : \underline{\widetilde{X}} \longrightarrow \underline{X}$ is a directed covering projection then the lifting of an reversible directed path of $\underline{X}$ is an reversible directed path of $\underline{\widetilde{X}}$.*

Example 6.7. Any isomorphism in the category d**Top** is a directed covering projection.

Example 6.8. If $\underline{\widetilde{X}}$ is the product in d**Top** of a directed pace $\underline{X}$ with a discrete space, then the product projection $\underline{\widetilde{X}} \longrightarrow \underline{X}$ is a directed covering projection.

Example 6.9. The map $\exp : \uparrow\mathbb{R} \longrightarrow \uparrow\mathbb{S}^1$, defined by $\exp(t) = e^{2\pi i t}$, is a directed covering projection.

Example 6.10. For any positive integer n, the map $p : \uparrow\mathbb{S}^1 \longrightarrow \uparrow\mathbb{S}^1$, defined by $p(z) = z^n$, is a directed covering projection.

Example 6.11. The standard circle $\mathbb{S}^1$ is not a directed covering space of the standard directed circle $\uparrow\mathbb{S}^1$, nor of the ordered circle $\uparrow\mathbb{O}^1$, since a directed map $\mathbb{S}^1 \longrightarrow \uparrow\mathbb{S}^1$ or $\mathbb{S}^1 \longrightarrow \uparrow\mathbb{O}^1$ is necessarily constant.

Example 6.12. Neither of the directed spaces $\mathbb{S}^1$ and $\uparrow\mathbb{S}^1$ is a directed covering space of the standard circle $\mathbb{S}^1$. (see also Example 6.16 below).

Example 6.13. The ordered circle $\uparrow\mathbb{O}^1$ is not a directed covering space of the standard directed circle $\uparrow\mathbb{S}^1$ since it is known that a covering projection $p : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is of the form $p(z) = z^n$, for a positive integer n,which is not a directed map $\uparrow\mathbb{O}^1 \longrightarrow \uparrow\mathbb{S}^1$.

Example 6.14. Consider the standard n-sphere $\underline{\mathbb{S}^n}$, $n \geqslant 1$, d-isomorphic with $(\uparrow\mathbf{I}^n)/(\partial\mathbf{I}^n)$, ([8], p. 286) and the corresponding directed projective space $\underline{\mathbb{P}^n}$, obtained as a quotient of $\underline{\mathbb{S}^n}$ by identifying the antipodal points. Then the quotient map $\pi : \underline{\mathbb{S}^n} \longrightarrow \underline{\mathbb{P}^n}$ is not a directed covering projection since the condition (iii) from Theorem 6.5 is not satisfied. If $a \in \mathrm{d}\underline{\mathbb{S}^n}$ then the antipodal path $(-a)(t) = -a(t)$ is not a directed path, while its projection $\pi \circ (-a) = \pi \circ a$ is directed.

But if we replace $\underline{\mathbb{S}^n}$ by $(\mathbb{S})^{\sim}$, i.e. the join of $\underline{\mathbb{S}^n}$ and $(\underline{\mathbb{S}^n})^{op}$ ([8], p. 287), then π becomes a directed covering projection.

Example 6.15. If $p : \underline{\widetilde{X}} \longrightarrow \underline{X}$ is a directed covering projection then $R(p) = p : \underline{\widetilde{X}}^{op} \longrightarrow \underline{X}^{op}$ has the same property.

Example 6.16. If $\uparrow X$ has the natural d-structure of all paths of X, then any directed covering space of $\uparrow X$ has the natural d-structure.

Example 6.17. If in Example 3.3,$p : \underline{E} \longrightarrow \underline{B}$ is a directed covering projection, then for any d-map $f : \underline{B'} \to \underline{B}$ the pullback projection $p' : \underline{E'} \longrightarrow \underline{B'}$, $p'((b', e)) = b'$, is a directed covering projection (see [6], p. 113).

Particularly, if $|\underline{B'}| = |\underline{B}|$ and $f = \mathrm{id}_{\underline{B}}$, then p' is a 'restriction' of p.

Example 6.18. If $\uparrow X$ is a d-space and $p : \widetilde{X} \longrightarrow X$ is a covering projection, then for $\widetilde{X}$ there exists a unique d-structure $\uparrow \widetilde{X}$ such that p becomes a directed covering projection $p : \uparrow \widetilde{X} \longrightarrow \uparrow X$. Define $a \in \mathrm{d}(\uparrow \widetilde{X})$ if and only if $p \circ a \in \mathrm{d}(\uparrow X)$ and it is immediate that this is a d-structure and we can apply Theorem 6.5.

For example, if we consider $p : \mathbb{S}^1 \longrightarrow \uparrow \mathbb{O}^1$, defined by $p(z) = z^n$, $n \geqslant 1$, we obtain a d-structure for $\mathbb{S}^1$ generalizing that of $\uparrow \mathbb{O}^1$. Thus, if we consider $\uparrow \mathbb{O}^1$ as a subspace of $\uparrow \mathbb{R}^{op} \times \mathbb{R}$ then the 2-folding covering projection has the d-structure that of directed subspace of $(\uparrow(-\infty, 0] \cup \uparrow [0, \infty)^{op}) \times \mathbb{R}$.

The map $\exp : \mathbb{R} \longrightarrow \uparrow \mathbb{O}^1$ becomes a directed projection if we consider for $\mathbb{R}$ the following d-structure: $\mathrm{d}\mathbb{R} := \cup_{n \in \mathbb{Z}} (\uparrow [n, n + \frac{1}{2}] \cup \uparrow [n - \frac{1}{2}, n]^{op})$.

7. Some properties of directed covering spaces

In this section we present some results which are nontrivial generalizations of the usual case.

First we generalize the situation from Example 6.14.

Theorem 7.1. *Let G be a properly discontinuous group of d-isomorphisms of a d-space $\underline{Y}$. Then the projection of $\underline{Y}$ onto the directed orbit space $\underline{Y}/G$ is a directed covering projection.*

Proof. Let $p : \underline{Y} \longrightarrow \underline{Y}/G$ be the orbit projection. From the quotient d-structure ([8], p. 285) we have that p is a directed map. Then since G is properly discontinuous, there is a cover $\{U\}$ of $|\underline{Y}|$ with open sets U, such that if $g, g' \in G$ and gU meets $g'U$, then $g = g'$. In [12](Theorem 7,Chapter 2, Section 6) it is proved that $p(U)$ is evenly covered by p, namely, $p^{-1}(p(U)) = \cup_{g \in G} gU$, and $p/gU : gU \to p(U)$ is a homeomorphism. We can show that in fact this is even a d-isomorphism $p/gU : \uparrow gU \to \uparrow p(U)$, for the subspace d-structures. For this it is necessary and sufficient to prove that in $p(U)$ a path is directed if and only if it is a projection of a directed path in gU. Let $a : \mathbf{I} \to gU$ be a path such that $p \circ a$ is a directed path in $\uparrow p(U)$. First if we have the simple situation $p \circ a = p \circ a'$, for a directed path a' of $\uparrow U$, then $g \circ a' \in \mathrm{d}(g \uparrow U) = \mathrm{d}(\uparrow gU)$ and $p \circ a = p \circ (g \circ a')$, which implies $a = g \circ a'$. This shows that $a \in \mathrm{d}(\uparrow gU)$. If $p \circ a = p \circ a_1 + p \circ a_2$, with $a_1, a_2 \in \mathrm{d}(\uparrow U)$, let $g' \in G$ be such that $g'(a_1(1)) = a_2(0)$. Then we have $a_2(0) \in U \cap g'U$ which implies $g' = e$, such that $p \circ a = p \circ (a_1 + a_2)$, and by the 'simple situation' we have $a = g \circ (a_1 + a_2) \in \mathrm{d}(\uparrow gU)$. The general case, $p \circ a = p \circ a_1 + \ldots. + p \circ a_n$, inductively implies that $a_1, \ldots, a_n$ are consecutive paths in $\mathrm{d}(\uparrow U)$ and then $a = g \circ (a_1 + \cdots + a_n) \in \mathrm{d}(\uparrow U)$. This finishes the proof. $\square$

Remark 7.2. In Example 6.14, where $G = \mathbb{Z}_2$, the antipodal map -1 is not a d-isomorphism for $\underline{\mathbb{S}^n}$ but it is for $(\mathbb{S})^{\sim}$.

From Theorem 6.5, Theorem 5.1, Remark 5.2, we have the following result:

Theorem 7.3. *For a directed covering projection, $p : \widetilde{\underline{X}} \longrightarrow \underline{X}$, the functor*

$$\uparrow \Pi_1(p) : \uparrow \Pi_1(\widetilde{\underline{X}}) \longrightarrow \uparrow \Pi_1(\underline{X}) \tag{7.1}$$

is faithful and, for a point $\widetilde{x} \in \widetilde{\underline{X}}$, the monoid morphism

$$\uparrow \pi_1(p) : \uparrow \pi_1(\widetilde{\underline{X}}, \widetilde{x}) \longrightarrow \uparrow \pi_1(\underline{X}, p(\widetilde{x})) \tag{7.2}$$

has trivial kernel.

Lemma 7.4. *If $p : \widetilde{\underline{X}} \longrightarrow \underline{X}$ is a directed covering projection then any lift of a directed map $f : \underline{Y} \longrightarrow \underline{X}$ with respect to the underlying map $p : |\widetilde{\underline{X}}| \longrightarrow |\underline{X}|$ is a directed map.*

Proof. Let $\widetilde{f} : |\underline{Y}| \longrightarrow |\widetilde{\underline{X}}|$ be a continuous map such that $p \circ \widetilde{f} = f$. If $a \in \mathrm{d}\underline{Y}$, then $p \circ (\widetilde{f} \circ a) = f \circ a \in \mathrm{d}\underline{X}$. By Theorem 6.5 it follows that $\widetilde{f} \circ a \in \mathrm{d}\widetilde{\underline{X}}$, which proves that $\widetilde{f}$ is a directed map. $\square$

Proposition 7.5 (see [12], p. 79)**.** *Let a commutative triangle in the category* d***Top*** *be given.*

$$\begin{tikzcd} \tilde{\underline{X}}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{\underline{X}}_2 \arrow[dl, "p_2"] \\ & \underline{X} & \end{tikzcd}$$

with $|\widetilde{\underline{X}}_1|$, $|\widetilde{\underline{X}}_2|$ connected spaces, $|\underline{X}|$ a locally path-connected space, and p_1, p_1 directed covering projections. Then f is a directed covering projection.

Corollary 7.6. *Two directed covering projections $p_i : \underline{\widetilde{X}}_i \longrightarrow \underline{X}$, $i = 1, 2$, are equivalent if and only if the underlying maps $p_i : \widetilde{X}_i \longrightarrow X$, $i = 1, 2$, are equivalent covering projections.*

Corollary 7.7. *A directed covering projection is a universal directed covering projection if and only if its underlying map is a universal covering projection.*

Corollary 7.8. *For a directed covering projection the group of directed covering transformations is isomorphic to the group of covering transformations of the underlying covering projection.*

Theorem 7.9. *A directed covering projection exhibits its base as a directed quotient space of its directed covering space.*

Proof. Let $p : \underline{\widetilde{X}} \longrightarrow \underline{X}$ be a directed fibration. On the directed space $\underline{\widetilde{X}}$ consider the equivalence relation $\widetilde{x}_1 \sim \widetilde{x}_2 \Leftrightarrow p(\widetilde{x}_1) = p(\widetilde{x}_2)$, and let $\widetilde{p} : \underline{\widetilde{X}}/_\sim \longrightarrow \underline{X}$ be the induced map, $\widetilde{p}([\widetilde{x}]) = p(\widetilde{x})$, for $[\widetilde{x}] \in \underline{\widetilde{X}}/_\sim$. Then the underlying map of $\widetilde{p}$ is a homeomorphism (cf. [12], p. 63). Therefore it is sufficient to verify that via this map the two structures coincide.

Consider a directed path of the directed quotient $\underline{\widetilde{X}}/_\sim$. This is of the form $\varpi := \pi \circ \widetilde{\omega}_1 + \cdots + \pi \circ \widetilde{\omega}_n$, for $\pi : \underline{\widetilde{X}} \to \underline{\widetilde{X}}/_\sim$, the quotient projection and $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_n$ directed paths of $\underline{\widetilde{X}}$ (see [8], p. 285). Since $\widetilde{p} \circ \pi = p$, we deduce $\widetilde{p} \circ \varpi = \widetilde{p} \circ \pi \circ \widetilde{\omega}_1 + \cdots + \widetilde{p} \circ \pi \circ \widetilde{\omega}_n = p \circ \widetilde{\omega}_1 + \cdots + p \circ \widetilde{\omega}_n$ and because p is a d-map, we conclude that $\widetilde{p} \circ \varpi \in \mathrm{d}\underline{X}$.

Conversely, suppose that θ is a path of $|\underline{\widetilde{X}}|/_\sim$ such that $p \circ \theta$ is a directed path of $\underline{X}$. Consider a point $\widetilde{x}_0 \in \pi^{-1}(\theta(0))$. Then, since p is a directed covering projection, there exists a directed path $\widetilde{\omega} \in d\underline{\widetilde{X}}$ such that $p \in \widetilde{\omega} = \widetilde{p} \circ \theta$ (and $\widetilde{\omega}(0) = \widetilde{x}_0$). It follows that $\widetilde{p} \circ \pi \circ \widetilde{\omega} = \widetilde{p} \circ \theta$, and since $\widetilde{p}$ is a homeomorphism, we obtain $\vartheta = \pi \in \widetilde{\omega} \in \mathrm{d}\underline{\widetilde{X}}/_\sim$. $\square$

Definition 7.10. We will say that a directed covering projection $p : \underline{\widetilde{X}} \longrightarrow \underline{X}$ is directed regular if, given any directed loop ω in $\underline{X}$, either every lifting of ω is closed or none is closed.

Formally following the proof of Theorem 11 from [12], p. 74, we can state:

Theorem 7.11. *A directed covering projection $p : \underline{\widetilde{X}} \longrightarrow \underline{X}$ is directed regular if and only if $p_{*1}(\uparrow \pi_1(\underline{\widetilde{X}}, \widetilde{x}_0) = p_{*1}(\uparrow \pi_1(\underline{\widetilde{X}}, \widetilde{x}_1)$ whenever $p(\widetilde{x}_0) = p(\widetilde{x}_1)$.*

Theorem 7.12. *A directed covering spaces of a directed topological group is a directed topological group with the corresponding covering projection a directed homomorphism.*

PROOF. Let $\underline{G}$ be a directed topological group and $p : \underline{\widetilde{X}} \longrightarrow \underline{G}$ a directed covering projection. Then $|\underline{\widetilde{X}}|$ is a topological group and p is a homomorphism (see [6], p. 155). It is sufficient to verify that the group operations are directed maps: $\mu : \underline{\widetilde{X}} \times \underline{\widetilde{X}} \to \underline{\widetilde{X}}$, $\iota : \underline{\widetilde{X}} \longrightarrow \underline{\widetilde{X}}$. But this is implied by Lemma 7.4 applied for the following commutative diagrams:

$$\begin{tikzcd}
\underline{\widetilde{X}} \times \underline{\widetilde{X}} \arrow[r, "p\times p"] \arrow[d, "\mu"'] & \underline{G} \times \underline{G} \arrow[d, "\mu"] & & \underline{\widetilde{X}} \arrow[r, "p"] \arrow[d, "\iota"'] & \underline{G} \arrow[d, "\iota"] \\
\underline{\widetilde{X}} \arrow[r, "p"'] & \underline{G} & & \underline{\widetilde{X}} \arrow[r, "p"'] & \underline{G}
\end{tikzcd} \tag{7.3}$$

□

Now we want to give a result on directed covering spaces of cellular complexes. It is known that any covering space of a cellular complex has a cellular structure such that the covering projection becomes a cellular map (see for example [6], p. 153). Here we consider the definition of cellular complexes and the correlated notations from [6], Chapter III.

Denote by $\uparrow \mathbb{S}^m$ the directed n-dimensional sphere ($\uparrow \mathbb{S}^m = (\uparrow \mathbf{I}^m)/(\partial \uparrow \mathbf{I}^m)$ $(m > 0)$, $\uparrow \mathbb{S}^0 = \{-1, 1\}$), and $\uparrow \mathbb{B}^m$ denotes one of the directed spaces $\uparrow C^+$ $(\uparrow \mathbb{S}^{m-1})$ or $\uparrow C^+(\uparrow \mathbb{S}^{m-1})$, if $m > 0$, and $\mathbb{B}^0$ is a single point.

Definition 7.13. We will say that a directed space $\uparrow K$ is a directed cellular complex if K is a cellular complex for which the following conditions are satisfied:

(i) Any skeleton K^m of K is endowed with the directed subspace structure and will be denoted by $\uparrow K^m$.

(ii) Any m-cell e^m of K is endowed with the directed subspace structure and will be denoted by $\uparrow e^m$.

(iii) Any attaching map of a m-cell e^m is a directed map $f^m : \uparrow \mathbb{B}^m \to \uparrow K^m$ and the restriction $f^m/\mathbb{B}^m - \mathbb{S}^{m-1}$ is a directed isomorphism $f^m/\mathbb{B}^m - \mathbb{S}^{m-1} : \uparrow (\mathbb{B}^m - \mathbb{S}^{m-1}) \to \uparrow e^m$.

Example 7.14. 1. $\uparrow \mathbb{S}^n$ is a directed cellular complex with one 0-cell and one directed n-cell.

2. The ordered circle $\uparrow \mathbb{O}^1$ is a directed cellular complex with two 0-cells and two directed 1-cells.

3. $\uparrow C^+(\uparrow \mathbb{S}^n)$ and $\uparrow C^-(\uparrow \mathbb{S}^n)$ are directed cellular complexes each with one 0-cell, one directed $(n-1)$-cell and one directed n-cell.

4. $\uparrow C^+(\uparrow \mathbb{O}^1)$ and $\uparrow C^-(\uparrow \mathbb{O}^1)$ are not directed cellular complexes with respect to the usual cellular structure of $\mathbb{B}^2$. But we can obtain a new directed space

with the same boundary if we consider a directed cellular complex with two 0-cells, two 1-cells and two 2-cells.

Remark 7.15. Our definition for directed cellular complex differs from the notion of "globular CW-complex" introduced in [5] to be used in the study of higher dimensional automata.

Theorem 7.16. *A directed covering space of a directed cellular complex is a directed cellular complex such that the corresponding covering projection is a directed cellular map.*

Proof. Let $p :\uparrow \widetilde{K} \to\uparrow K$ be a directed covering projection with $\uparrow K$ a directed cellular complex. Then since the underlying map $p : \widetilde{K} \to K$ is a covering projection, by [6], p. 153, $\widetilde{K}$ has a cellular decomposition with respect to which p is a cellular map. Since the covering projection p is trivial above any cell of K the connected components of the inverse image by p of cells of K determine a cellular partition of $\widetilde{K}$ for which $\widetilde{K}^m = p^{-1}(K^m)$. If e^m is a directed m-cell of $\uparrow K$, $f^m :\uparrow \mathbb{B}^m \to\uparrow K^m$, and ε^m is a m-cell of $p^{-1}(e^m)$ then there exists a directed lift $g :\uparrow \mathbb{B}^m \to\uparrow L^m$ with the following properties:

(a) $g(\uparrow(\mathbb{S}^{m-1}) \subset\uparrow \widetilde{K}^{m-1}$;

(b) $g/\uparrow(\mathbb{B}^m - \mathbb{S}^{m-1})$ is a directed isomorphism of $\uparrow(\mathbb{B}^m - \mathbb{S}^{m-1})$ on $\uparrow \varepsilon^m$ (see [6], p. 153, Theorem 6.5 and Lemma 7.4).

These conditions ensure Definition 7.13 for $\uparrow \widetilde{K}$ and this ends the proof. □

8. On the dicovering spaces of L. Fajstrup [2], [3]

The first notion of covering space in connection with d-spaces was considered by Lisbeth Fajstrup in [2] and [3]. This author does not start from a definition based on (directed) evenly covered subsets as in the usual case (see for example [12], Chapter 2, Section 5) and the way we approached the present paper. But in order to define a universal covering space she takes as model the construction of covering spaces of connected locally path-connected spaces (see [12], Chapter 2, Section 5, Theorem 13), and then for defining arbitrary covering spaces only the unique lifting properties for *dipaths* and *dihomotopies* are used. Among the two quoted papers, [3] refers to d-spaces in the sense which is used in this paper. In this section we briefly present this paper with a view to compare the two approaches.

If $\underline{X}$ is a d-space, for subsets $W, U \subset |\underline{X}|$, $\overrightarrow{P}(\underline{X}, W, U)$ denotes the set of d-paths of $\mathrm{d}\underline{X}$ with initial point in W and final point in U.

Definition 8.1 ([3], Definition 2. 2). Let $\underline{X}$ be a d-space. On a subset $V \subset |\underline{X}|$ a relation $x \leq_V y$ if $\overrightarrow{P}(V,x,y) \neq \emptyset$ is defined.

The future of x in V is $\uparrow_V x := \{y \in V \mid x \leq_V y\}$.

The d-space $\underline{X}$ is locally ordered if there is a cover $\mathcal{U}$ of X, which is a basis for the topology and if on each $U \in \mathcal{U}$ the relation '$\leq_U$' is a partial order.

Definition 8.2 ([3], Definition 2.5). Let us consider for a d-space $\underline{X}$ a point $x \in X$, a subset $U \subset X$ and let $\gamma_1, \gamma_2 \in \overrightarrow{P}(\underline{X}, x, U)$. Then $\gamma_1 \sim_U \gamma_2$ if there is a d-map (dihomotopy), $H : I \times \uparrow I \to \underline{X}$ such that $H(0,t) = \gamma_1(t)$, $H(1,t) = \gamma_2(t)$, $H(s,0) = x$, $H(s,1) \in U$. If $U = \{y\}$, a single point, one defines $\overrightarrow{\pi}_1(\underline{X},x,y) = \overrightarrow{P}(\underline{X},x,y)/\sim$, and $[\gamma]$ denotes the equivalence class of a d-path γ.

Remark 8.3. This relation '$\sim$' is different from the relation '$\simeq_2$' used by us in section 5 (see [11], Remark 1.2, p. 260).

Definition 8.4 ([3], Definition 2.5). A locally ordered d-space $(\underline{X},\mathcal{U})$ is locally relatively diconnected with respect to $x_0 \in X$ if

- For all $U \in \mathcal{U}$ and all $x, y \in U, |\overrightarrow{\pi}_1(U,x,y)| \leq 1$.
- For all $x \in X$, there is a $U \in \mathcal{U}$ such that for $\gamma_i \in \overrightarrow{P}(\underline{X}, x_0, y)$, $i = 1,2$, $\gamma_1 \sim_U \gamma_2$ if and only if $[\gamma_1] = [\gamma_2]$.

Definition 8.5 ([3], Definition 2.6). For a locally ordered d-space $(\underline{X},\mathcal{U})$, locally relatively connected with respect to $x_0 \in X$, the universal dicovering with respect to x_0 is

$$\widetilde{X}_{x_0} := \{[\gamma] \in \overrightarrow{\pi}_1(\underline{X}, x_0, -)\} \tag{8.1}$$

with topology $\mathcal{U}_{[\gamma]}$ generated by the sets

$$U_{[\gamma]} := \{[\mu] \mid \mu \in \overrightarrow{P}(\underline{X}, x_0, U),\ \mu \sim_U \gamma\}, \tag{8.2}$$

for $U \in \mathcal{U}$ and $\gamma \in \overrightarrow{P}(\underline{X}, x_0, U)$.

The d-structure is

$$\overrightarrow{P}(\widetilde{X}_{x_0}, [\gamma], -) := \{\eta : \mathbf{I} \to \widetilde{X}_{x_0} \mid \eta(t) =: [\gamma * \mu_t] \mid \mu \in \overrightarrow{P}(\widetilde{X}_\gamma(1), -),$$
$$\mu_t(t') = \mu(tt')\} \tag{8.3}$$

The projection

$$\Pi : \underline{\widetilde{X}}_{x_0} \longrightarrow \underline{X} \tag{8.4}$$

is defined by

$$\Pi([\gamma]) := \gamma(1) \tag{8.5}$$

Remark 8.6. If in the usual case of the undirected topological spaces, the two well known approaches on the covering spaces, either by definition using neighborhoods evenly covered ([12], Chapter 2, Section 1) or by a construction of them starting from a path space $P(X, x_0)$ ([12], Chapter 2, Section 5), coincide (under some natural conditions), in the case of the directed spaces these approaches are different, as we emphasize in some examples and comments.

Example 8.7. For $\underline{X} = \uparrow \mathbb{S}^1$, the standard directed circle, and the point $x_0 = 1$, the universal dicovering $\widetilde{\underline{X}}_{x_0}$ can be identified with $\uparrow [0, \infty)$ and then the projection Π is $\exp /[0, \infty) : \uparrow [0, \infty) \to \uparrow \mathbb{S}^1$.

Example 8.8. For $\underline{X} = \uparrow \mathbb{O}^1$, the ordered circle, as a directed subspace of $\uparrow \mathbb{R}^{op} \times \mathbb{R}$ and $x_0 = -1$, the universal dicovering $\widetilde{\underline{X}}_{x_0}$ is a singleton.

If $x_0 = 1$, then $\widetilde{\underline{X}}_{x_0}$ can be identified with the directed space $\uparrow \big[-\frac{1}{2}, 0\big]^{op} \cup \uparrow \big[0, \frac{1}{2}\big]$, and then the projection Π becomes the restriction of the exponential map.

The above examples obviously show that the universal dicoverings are not directed covering spaces.

However we can emphasize some common elements. At first it is proved in [3](Proposition 2.8) that dipaths and dihomotopies initiating in $y \in \uparrow x_0$ lift uniquely given an initial point in $\Pi^{-1}(y)$. Then it is proved (see the proof of Proposition 3.9 in [2]) that $\Pi^{-1}(U) = \cup_{\{\gamma mid \gamma(1) \in U\}} U_{[\gamma]}$ and $[\gamma_1] \neq [\gamma_2]$ implies $U_{[\gamma_1]} \cap U_{[\gamma_2]} = \emptyset$ (Proposition 3.8 in [2]). There also exist other similarities which we mention below.

We begin with the following proposition which shows that the d-structure of a directed covering space is obtained by using dihomotopies, similar with the case of universal dicoverings.

Proposition 8.9. *Let $p : \uparrow \widetilde{X} \to \uparrow X$ be a directed covering projection. Suppose that $\widetilde{X}$ is a quotient space $P(X, x_0)/ \sim = \{\langle \omega \rangle \mid \omega(0) = x_0\}$ ([12], Theorem 13, p. 82) and $p(\langle \omega \rangle) = \omega(1)$. Then a directed path of $\uparrow \widetilde{X}$ can be identified with a dihomotopy in $\uparrow X$ initiating in x_0.*

Proof. Let $\widetilde{\alpha} : \uparrow \mathbf{I} \to \uparrow \widetilde{X}$ be a directed path with $\widetilde{\alpha}(0) = \langle \omega \rangle$. If we write

$$\widetilde{\alpha}(t) = \langle \omega_t \rangle \tag{8.6}$$

then $\omega_t(0) = x_0$, $\omega_0 = \omega$. Moreover since $\widetilde{\alpha}$ is a directed path if and only if $p \circ \widetilde{\alpha} \in d(\uparrow X)$, we have that the map

$$\alpha(t) := \omega_t(1) \tag{8.7}$$

is a directed path $\alpha : \uparrow \mathbf{I} \to \uparrow X$.

Now we can define

$$H : \mathbf{I} \times \uparrow \mathbf{I} \to \uparrow X. \tag{8.8}$$

$$H(t', t) = \omega_t(t'). \tag{8.9}$$

This is a dimap satisfying the conditions $H(0,t) = x_0$, $H(t',0) = \omega(t')$, and $H(1,t) = \alpha(t)$ is a directed path of $\uparrow X$. Therefore it is a dihomotopy in $\uparrow X$ initiating in x_0.

Conversely it is obvious that such a dihomotopy defines a directed path of $\uparrow \widetilde{X}$. □

Moreover we can see now that the *d*-structure of $\uparrow \widetilde{X}$ in Proposition 8.9 is itself analogous with that of the universal dicovering (see 8.3).

Proposition 8.10. *Let α be a directed path of $\uparrow X$, $\langle\omega\rangle \in p^{-1}(\alpha(i)), i \in 0,1$, and $\widetilde{\alpha}$ the directed lift of α with $\widetilde{\alpha}(i) = \langle\omega\rangle$.*

Then $\widetilde{\alpha}$ is defined by one of the following formulas:

$$\widetilde{\alpha}(t) = \langle\omega * \alpha_t\rangle, \qquad \alpha_t(t') = \alpha(tt'), \qquad \textit{if } i = 0 \tag{8.10}$$

$$\widetilde{\alpha}(t) = \langle\omega * \widehat{\alpha}_{1-t}\rangle, \quad \widehat{\alpha}_{1-t}(t') = \alpha(1-t'+tt'), \quad \textit{if } i = 1 \tag{8.11}$$

Particularly, if $\widetilde{\alpha}$ is a directed loop in $\uparrow \widetilde{X}$, with base point $\langle\omega\rangle$, then α is a directed loop in $\uparrow X$, with base point $\omega(1)$, and satisfying $\langle\omega\rangle = \langle\omega * \alpha\rangle$.

Proof. If we write $\widetilde{\alpha}(t) = \langle\widetilde{\alpha}_t\rangle$, using the existence and the uniqueness of $\widetilde{\alpha}$ we can search for it by requiring the following conditions: $\widetilde{\alpha}_0 = \omega$, $\widetilde{\alpha}_t(0) = x_0$ and $\widetilde{\alpha}_t(1) = \alpha(t)$. These conditions are satisfied if we define $\widetilde{\alpha}$ by 8.10 or 8.11. These formulas can be written since the condition $\langle\omega\rangle \in p^{-1}(\alpha(i)), i \in 0,1$, i.e. $p(\langle\omega\rangle) = \alpha(i)$ implies $\omega(1) = \alpha(i)$. Continuity can be proved by analogy with [12] (p. 83), where the case $i = 0$ and ω only the constant path in x_0 is considered. But the differences are unessential.

In the case $i = 0$, $\widetilde{\alpha}$ is a path beginning at $\langle\omega\rangle$ and ending at $\langle\omega * \alpha\rangle$ and in the case $i = 1$, $\widetilde{\alpha}$ begins at $\langle\omega * \alpha^{-1}\rangle$ and ends at $\langle\omega\rangle$. □

Remark 8.11. If we consider directed paths for $\widetilde{X}$ only given by the formulas 8.10 and 8.11, with α a directed path of $\uparrow X$, then the projection $p : \uparrow \widetilde{X} \longrightarrow \uparrow X$, $p(\langle\omega\rangle) = \omega(1)$, is a directed covering projections.

Proposition 8.12. *Let $p : \underline{\widetilde{X}} \longrightarrow \underline{X}$ be a directed covering projection and $\mathcal{U} = \{U\}$ a cover of X by open subsets directed evenly covered by p.*

Suppose that the pair $(\underline{X}, \mathcal{U})$ is a locally ordered d-space, locally relatively connected with respect to a point $x_0 \in X$.

Then there exists a directed map $\phi : \underline{\widetilde{X}}_{x_0} \longrightarrow \underline{\widetilde{X}}$ satisfying the following *commutative diagram.*

$$\begin{tikzcd} \underline{\tilde{X}}_{x_0} \arrow[rr, "\phi"] \arrow[dr, "\Pi"'] & & \underline{\tilde{X}} \arrow[dl, "p"] \\ & \underline{X} & \end{tikzcd}$$

Proof. As above we consider the space $\widetilde{X}$ constructed as in [12] (Theorem 13, p. 82), corresponding to the cover $\mathcal{U}$ and a subgroup H of $\pi_1(X, x_0)$. The elements of $\widetilde{X}$ are denoted by $\langle\omega\rangle$ and those of $\underline{\widetilde{X}}_{x_0}$ by $[\gamma]$. We define ϕ define by $\phi([\gamma]) = \langle\gamma\rangle$, since if $[\gamma] = [\gamma']$ then $\gamma \sim \gamma'$ *rel* $\partial\mathbf{I}$, such that $[\gamma * \gamma'^{-1}] = 0 \in H$ and therefore $\langle\gamma\rangle = \langle\gamma'\rangle$.

The commutativity of the diagram is obvious since $\Pi([\gamma]) = \gamma(1)$ and $p(\langle\omega\rangle) = \omega(1)$.

Now we verify continuity at a point $[\gamma] \in \underline{\widetilde{X}}_{x_0}$. At first we observe that if $\langle\omega, U\rangle$ is an element of the base of the used topology of $\widetilde{X}$ which contains $\langle\gamma\rangle$ then $\langle\omega, U\rangle = \langle\gamma, U\rangle$. Thus, it is sufficient to prove that $\phi^{-1}(\langle\gamma\rangle, U)$ is a neighborhood of $\langle\gamma\rangle$. This will result if we verify that $\phi(U_{[\gamma]}) \subseteq \langle\gamma, U\rangle$. Let $[\mu]$ be an arbitrary element of $U_{[\gamma]}$. We need to prove that $\langle\mu\rangle$ is an element of $\langle\gamma, U\rangle$. We have $\mu \in \overrightarrow{P}(X, x_0, U)$ and $\mu \sim_U \gamma$, i.e. there exists a dihomotopy $H : \mathbf{I}\times \uparrow \mathbf{I} \to X$ such that $H(0,t) = \gamma(t), H(1,t) = \mu(t), H(s,0) = x_0$ and $H(s,1) \in U$. Then we consider the path $\omega' : \mathbf{I} \to U$ given by $\omega'(s) = H(s,1)$. For this we have $\omega'(0) = \gamma(1)$ and $\omega'(1) = \mu(1)$. Moreover, gluing the maps H and ω' we obtain a homotopy $H' : \mathbf{I} \times \mathbf{I} \to U$ given by

$$H'(s,t) = \begin{cases} H\left(s, \dfrac{2t}{1+s}\right), & \text{if } 0 \le t \le \dfrac{s+1}{2}, \\[2ex] \omega'\left(\dfrac{2t-s-1}{1-s}\right), & \text{if } \dfrac{s+1}{2} \le t \le 1. \end{cases}$$

For this we have: $H'(0,t) = (\gamma * \omega')(t)$, $H'(1,t) = \mu(t)$, $H'(s,0) = x_0$, $H'(s,1) = (\gamma * \omega')(1) = \mu(1)$. Thus we have $\mu \in \langle\gamma, U\rangle$ and this proves the continuity of ϕ.

Finally the property of directed map of ϕ results from Definition 8.5 and Proposition 8.10 since the directed paths of both directed spaces $\underline{\widetilde{X}}_{x_0}$ and $\underline{\widetilde{X}}$ are the lifts of the directed paths of $\underline{X}$. This finishes the proof. □

Remark 8.13. An essential part of the proof of Proposition 8.12, that concerning the continuity of the map ϕ, can be recovered using Remark 3.4 from [2].

Remark 8.14. Under the conditions of Proposition 8.12 if we suppose the future of x_0 is $\underline{X}$ then $\Pi/U_{[\gamma]} : U_{[\gamma]} \to U$ is a directed bijection (but not a directed isomorphism).

Example 8.15 (L. Fajstrup). This example (called "The box with a lid but no bottom") was communicated to the author by Professor Lisbeth Fajstrup (see also [2], p. 12). It is destined to show that the directed map ϕ in Proposition 8.12 is not generally injective, as the Examples 8.7 and 8.8 may suggest.

Let $\underline{X} =\uparrow (\partial \mathbf{I}^3 \backslash (0,1) \times (0,1) \times \{0\})$, as a directed subspace of $\uparrow \mathbb{R}^3$, and let $x_0 = (0,0,0)$. Consider the universal dicovering projection $\Pi : \widetilde{\underline{X}}_{x_0} \to \underline{X}$, $\Pi([\gamma]) = \gamma(1)$. Let us consider the points $B(1,1,0)$ and $B'(1,1,1)$. Then the fibre of Π over a point $M(1,1,z)$, with $0 \le z < 1$, of the segment $[AB)$, contains two points corresponding to the distinct elements of $\overrightarrow{\pi}_1(\underline{X}, x_0, M)$ represented by dipaths of $\underline{X}$ from x_0 to M placed in the semi-spaces $x \ge y$ and $x \le y$ respectively. For all other points of X the fibres contain only one point. Thus we deduce that the universal dicovering $\widetilde{\underline{X}}_{x_0}$ is the directed subspace of $\uparrow \mathbb{R}^3$ obtained from $\underline{X}$ by splitting it along the semi-open edge $[AB)$.

Now since for a covering projection of a path connected space any two fibers are homeomorphic, by the commutativity of the diagram from Proposition 8.12, we can conclude that this $\widetilde{\underline{X}}_{x_0}$ cannot be a directed subspace of a directed covering space of $\underline{X}$. In fact the underlying space of a connected directed covering space is X and, obviously, $|\widetilde{\underline{X}}_{x_0}| \nsubseteq X$.

Regarding arbitrary dicoverings, the following definition is given in [3].

Definition 8.16 ([3], Definition 2.9). A dimap $p : Y \to X$ is a dicovering with respect to $x_0 \in X$ if $\uparrow_X x_0 = X, \uparrow p^{-1}(x_0) = Y$, p is surjective and dipaths and dihomotopies initiating in x_0 lift uniquely, given an initial point in $p^{-1}(x_0)$.

If we take into account Definition 2.2, Definition 2.4, Corollary 2.6 and Theorem 4.8, we can state the following corollary.

Corollary 8.17. *A directed fibration with unique path lifting $p : Y \to X$ which is surjective and for a point $x_0 \in X$ satisfies $\uparrow_X x_0 = X, \uparrow p^{-1}(x_0) = Y$ is a dicovering with respect to x_0.*

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IOAN POP
FACULTY OF MATHEMATICS
AL. I. CUZA UNIVERSITY
700506 IASI
ROMANIA

E-mail: ioanpop@uaic.ro

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