

## Projective changes of Finsler spaces of constant curvature

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*To the memory of Professor András Rapcsák*

We have many interesting papers concerned with the theory of projective changes of Finsler spaces. In particular, projective changes between Finsler spaces of constant curvature have been studied by some authors, for instance, [1], [6] and [7]. The most essential result seems to be FUKUI–YAMADA's [1]: A projective map of a Finsler space  $F^n$  of constant curvature is also a space  $\bar{F}^n$  of constant curvature if and only if the projective factor  $P(x, y)$  satisfies  $Q_{ij} = P_{.ij} - P_{.j.i} = 0$ . Precisely speaking, the meaning of this “if and only if” is as follows:

Let  $F^n \rightarrow \bar{F}^n$  be a projective map between Finsler spaces and let  $F^n$  be of constant curvature. Then  $\bar{F}^n$  is also of constant curvature if and only if the projective factor  $P$  satisfies  $Q_{ij} = 0$ .

It is, however, not sure that the map  $\bar{F}^n$  has also a Finsler metric, and the “if and only if” above may cause misunderstanding.

The purpose of the present paper is to discuss the metrizability of the projective change of Finsler spaces of constant curvature based on A. RAPCSÁK's remarkable results [5]. The main results are stated as Theorems 1 and 2.

### §1. Rapcsák's theorem on projective changes

Let  $F^n = (M^n, L(x, y))$  be a Finsler space on a differentiable  $n$ -manifold  $M^n$  equipped with the fundamental function  $L(x, y)$ . We denote by  $g_{ij}(x, y)$  and  $g^{ij}(x, y)$  the fundamental tensor  $g_{ij} = \partial^2 F / \partial y^i \partial y^j$  ( $F = L^2/2$ ) and its reciprocal. If we put

$$G_i(x, y) = \{y^r (\partial^2 F / \partial x^r \partial y^i) - \partial F / \partial x^i\} / 2,$$

and  $G^i = g^{ij} G_j$ , then the Berwald connection  $B\Gamma = (G_j^i(x, y), G_j^i(x, y))$  of  $F^n$  is defined by  $G_j^i = \partial G^i / \partial y^j$  and  $G_j^i = \partial G_j^i / \partial y^k$ , and a geodesic

of  $F^n$  is given by the differential equations

$$d^2x^i/ds^2 + 2G^i(x, dx/ds) = 0,$$

where we put  $ds = L(x, dx)$ . The  $h$ - and  $v$ -covariant differentiations with respect to  $B\Gamma$  are denoted by  $(;)$  and  $(\dot{\phantom{a}})$  respectively; if we are concerned, for instance, with a covariant vector field  $X_i(x, y)$ , we have

$$X_{i;j} = \partial_j X_i - (\dot{\partial}_r X_i) G^r_j - X_r G_i^r{}_j, \quad X_{i;j} = \dot{\partial}_j X_i,$$

where  $\partial_j = \partial/\partial x^j$  and  $\dot{\partial}_j = \partial/\partial y^j$ .

Let  $\bar{F}^n = (M^n, \bar{L}(x, y))$  be another Finsler space on the same manifold  $M^n$  equipped with the fundamental function  $\bar{L}(x, y)$  and the Berwald connection  $B\bar{\Gamma} = (\bar{G}^i{}_k, \bar{G}^i{}_j)$ . As A. RAPCSÁK [5] has shown, we obtain the following general relation between  $G^i$  and  $\bar{G}^i$ :

$$(1.1) \quad 2\bar{G}^i = 2G^i + \bar{L}_{;0} y^i / \bar{L} - \bar{L} \bar{g}^{ij} \Delta_j(\bar{L}),$$

where the subscript 0 denotes transvection by  $y^i$ , and  $\Delta_j(S)$  [4] for a scalar field  $S$  stands for

$$(1.2) \quad \Delta_j(S) = S_{;j} - S_{;r} y^r{}_j.$$

Now the change  $p : L(x, y) \rightarrow \bar{L}(x, y)$  of the metric is called a *projective change* and  $F^n$  is called *projective* to  $\bar{F}^n$ , if any geodesic of  $F^n$  is also a geodesic of  $\bar{F}^n$  as a point set and vice versa. As is well-known,  $p$  is projective if and only if there exists a positively homogeneous function  $P(x, y)$  of degree one in  $y^i$ , called the *projective factor*, satisfying

$$(1.3) \quad \bar{G}^i(x, y) = G^i(x, y) + P(x, y) y^i.$$

From (1.3) we get

$$(1.4) \quad \bar{G}^i{}_j = G^i{}_j + P_{;j} y^i + P \delta_j^i.$$

Then the metricity condition  $\bar{L}_{;\bar{i}} = \partial_i \bar{L} - (\dot{\partial}_r \bar{L}) \bar{G}^r{}_i = 0$  for  $B\bar{\Gamma}$  of  $\bar{F}^n$  is written in the form

$$\bar{L}_{;\bar{i}} = \partial_i \bar{L} - \dot{\partial}_r \bar{L} (G^r{}_i + P_{;i} y^r + P \delta_i^r) = \bar{L}_{;i} - (\bar{L}P)_{;i} = 0.$$

Thus we have  $\Delta_j(\bar{L}) = (\bar{L}P)_{;i} - (\bar{L}P)_{;r} y^r{}_i$ , which is equal to zero from the homogeneity of  $\bar{L}P$ . Conversely, if we have  $\Delta_j(\bar{L}) = 0$ , then (1.1) has the form of (1.3) where  $P = \bar{L}_{;0}/2\bar{L}$ .

Therefore we obtain Rapcsák's fundamental theorem [4], [5] on projective changes as follows:

**Theorem** (Rapcsák). *If a Finsler space  $\bar{F}^n = (M^n, \bar{L}(x, y))$  is projective to a Finsler space  $F^n = (M^n, L(x, y))$ , then  $\bar{L}$  satisfies the differential equations*

$$\Delta_i(\bar{L}) = \bar{L}_{;i} - \bar{L}_{;r.i}y^r = 0,$$

and the projective factor  $P(x, y)$  is given by  $P = \bar{L}_{;0}/2\bar{L}$ . Conversely, if we have a scalar field  $\bar{L}$  satisfying the differential equations  $\Delta_i(\bar{L}) = 0$  and the well-known conditions for a fundamental function of a Finsler space, then we get a Finsler space  $\bar{F}^n = (M^n, \bar{L}(x, y))$  which is projective to  $F^n$ .

As RAPCSÁK has shown, the equations  $\Delta_i(\bar{L}) = 0$  are equivalent to one of the following two conditions:

$$(1.5) \quad (1) \quad \bar{L}_{;i.j} - \bar{L}_{;j.i} = 0, \quad (2) \quad \bar{L}_{;i;j} - \bar{L}_{;j;i} = 0.$$

For later use we shall express  $\Delta_i(\bar{L}) = 0$ ,  $P$  and (2) of (1.5) in terms of  $\bar{F} = \bar{L}^2/2$ :

$$(1.6) \quad 2\bar{F}(\bar{F}_{;i} - \bar{F}_{;r.i}y^r) + \bar{F}_{;0}\bar{y}_i = 0,$$

$$(1.7) \quad P = \bar{F}_{;0}/4\bar{F},$$

$$(1.8) \quad 2\bar{F}(\bar{F}_{;i.j} - \bar{F}_{;j.i}) = \bar{F}_{;i}\bar{y}_j - \bar{F}_{;j}\bar{y}_i.$$

## §2. Projective change between Finsler spaces of scalar curvature

We consider a projective change  $p : F^n \rightarrow \bar{F}^n$  with the projective factor  $P(x, y)$ . According to Z. SZABÓ's theorem [2], a projectively invariant Weyl tensor vanishes if and only if the space is of scalar curvature. Thus, if  $F^n$  be assumed to be of scalar curvature  $K(x, y)$ , then  $\bar{F}^n$  too must be of scalar curvature  $\bar{K}(x, y)$ .

The Berwald connection  $B\Gamma$  of  $F^n$  has two surviving curvatures and one surviving torsion [3]:

$$\begin{aligned} h\text{-curvature } H_h^i{}_{jk} &= R^i{}_{jk.h}, & hv\text{-curvature } G_h^i{}_{jk} &= \dot{\partial}_k G_h^i{}_{j}, \\ (v)h\text{-torsion } R^i{}_{jk} &= \partial_k G^i{}_j - \partial_j G^i{}_k - (\partial_r G^i{}_j)G^r{}_k + (\partial_r G^i{}_k)G^r{}_j. \end{aligned}$$

From (1.4) we get the relation [2]

$$(2.1) \quad \bar{R}^i{}_{0k} = R^i{}_{0k} + y^i(Q_{0k} + Q_k) - \delta_k^i Q_0,$$

where we put

$$Q_k = P_{;k} - PP_{;k}, \quad Q_{ik} = P_{;i;k} - P_{;k;i} = -(Q_{i.k} - Q_{k.i}).$$

It is easy to show the relation

$$(2.2) \quad Q_{0.k} = (Q_r y^r)_{.k} = 2Q_k - Q_{0k}.$$

As is well-known [3], the space  $F^n$  is of scalar curvature  $K$ , if and only if we have  $R^i_{0k} = L^2 K h^i_k$  where  $h^i_k = \delta^i_k - l^i l_k$  is the angular metric tensor. Since we have also the similar equation  $\bar{R}^i_{0k} = \bar{L}^2 \bar{K} \bar{h}^i_k$  for  $\bar{F}^n$ , (2.1) implies

$$(2.3) \quad \bar{L}^2 \bar{K} \bar{h}^i_k = L^2 K h^i_k + y^i (Q_{0k} + Q_k) - \delta^i_k Q_0.$$

Contracting (2.3) by  $i = k$ , we get

$$(2.4) \quad \bar{L}^2 \bar{K} = L^2 K - Q_0.$$

Eliminating  $Q_0$  from (2.3) and (2.4) and paying attention to  $Ll^i = y^i$ , (2.3) reduces to the form

$$(2.5) \quad \bar{L} \bar{K} \bar{l}_k = L K l_k - (Q_{0k} + Q_k).$$

Differentiating (2.4) by  $y^k$  and substituting from (2.2) and (2.5), we obtain

$$(2.6) \quad \bar{L}^2 \bar{K}_{.k} = L^2 K_{.k} + 3Q_{0k}.$$

Summarizing the above, we have

**Proposition 1.** *Let  $p : F^n = (M^n, L(x, y)) \rightarrow \bar{F}^n = (M^n, \bar{L}(x, y))$  be a projective change from a Finsler space  $F^n$  of scalar curvature  $K(x, y)$  to a Finsler space  $\bar{F}^n$ . Then  $\bar{F}^n$  is of scalar curvature  $\bar{K}(x, y)$ , where  $\bar{L}$  and  $\bar{K}$  satisfy (2.4) and (2.6).*

### §3. Projective change between Finsler spaces of constant curvature

Let  $p : F^n = (M^n, L(x, y)) \rightarrow \bar{F}^n = (M^n, \bar{L}(x, y))$  be a projective change from a Finsler space  $F^n$  of constant curvature  $K$  to a Finsler space  $\bar{F}^n$  of constant curvature  $\bar{K}$ . Then (2.6) reduces to  $Q_{0k} = 0$  and (2.2) implies  $2Q_k = (Q_r y^r)_{.k}$ , that is,  $Q_k = Q_{r.k} y^r$ . Differentiating this by  $y^i$ , we get  $Q_{k.i} - Q_{i.k} = Q_{r.k.i} y^r$ . Consequently we have

$$(3.1) \quad Q_{r.k.i} y^r = 0,$$

$$(3.2) \quad Q_{i.k} - Q_{k.i} = P_{.k;i} - P_{.i;k} = 0.$$

The former is, however, only a consequence of the latter, because the latter shows that  $Q_{r.k.i} y^r = Q_{k.r.i} y^r = Q_{k.i.r} y^r$ , which is equal to zero from the homogeneity.

Now (2.5) reduces to

$$(3.3) \quad \bar{K} \bar{y}_k = K y_k - Q_k.$$

Differentiating this by  $y^i$ , we get

$$(3.4) \quad \bar{K}\bar{g}_{ik} = Kg_{ik} - Q_{k.i}.$$

If we put

$$(3.5) \quad Q = Q_r y^r / 2 = (P_{;0} - P^2) / 2,$$

then we have  $Q_{.i} = (Q_i + Q_{r.i} y^r) / 2$ , which is equal to  $(Q_i + Q_{i.r} y^r) / 2 = Q_i$  from (3.2); that is,

$$(3.6) \quad Q_i = Q_{.i}.$$

Thus (3.4) is written in the form

$$(3.7) \quad \bar{K}\bar{g}_{ij} = Kg_{ij} - Q_{.i.j},$$

and (2.4) is also written as

$$(3.8) \quad \bar{L}^2 \bar{K} = L^2 K - 2Q.$$

Summarizing the above we get

**Proposition 2.** *Let  $p : F^n = (M^n, L(x, y)) \rightarrow \bar{F}^n = (M^n, \bar{L}(x, y))$  be a projective change from a Finsler space  $F^n$  of constant curvature  $K$  to a Finsler space  $\bar{F}^n$  of constant curvature  $\bar{K}$  with the projective factor  $P(x, y)$ . Then  $P$  must satisfy (3.2) and we have (3.8) where  $Q(x, y)$  is defined by (3.5).*

#### §4. Metrizability condition

We consider RAPCSÁK's metrizability condition (1.8). Since we have (3.8), that is,  $\bar{F}\bar{K} = FK - Q$ , (1.8) is written as

$$(4.1) \quad 2(FK - Q)(Q_{.i;j} - Q_{.j;i}) = (Ky_i - Q_{.i})Q_{;j} - (i/j),$$

provided that  $\bar{K}$  is assumed not to be equal to zero, where  $(i/j)$  stands for the terms obtained from the preceding terms by interchanging indices  $i, j$ . We have easily from (3.6), (3.2) and the Ricci identity

$$\begin{aligned} Q_{.i;j} - Q_{.j;i} &= P_{;i.j} - P_{.i}P_{;j} - (i/j) = \\ &= -P_{.r}R^r_{ij} - P_{.i}(Q_j + PP_{.j}) + P_{.j}(Q_i + PP_{.i}) = \\ &= -P_{.r}K(y_i\delta_j^r - y_j\delta_i^r) - P_{.i}Q_j + P_{.j}Q_i = P_{.i}(Ky_j - Q_j) - (i/j). \end{aligned}$$

Thus (4.1) is rewritten as

$$(Ky_i - Q_i)\{Q_{;j} + 2(FK - Q)P_{.j}\} - (i/j) = 0,$$

which implies that we must have a scalar field  $v(x, y)$  satisfying

$$(4.2) \quad Q_{;i} + 2(FK - Q)P_{;i} = v(Ky_i - Q_i).$$

Then, in virtue of (3.8), (1.7) is written as  $P = \bar{F}_{;0}\bar{K}/4\bar{F}\bar{K} = -Q_{;0}/4(FK - Q)$ , and (4.2) gives  $Q_{;0} = 2(FK - Q)(v - P)$ . Consequently we have  $v = -P$  and (4.2) is rewritten in the form

$$(4.3) \quad Q_{;i} + P(Ky_i - Q_i) + 2(FK - Q)P_{;i} = 0.$$

Summarizing all the above, we have

**Theorem 1.** *Let  $p : F^n = (M^n, L(x, y)) \rightarrow \bar{F}^n = (M^n, \bar{L}(x, y))$  be a projective change from a Finsler space  $F^n$  of constant curvature  $K$  to a Finsler space  $\bar{F}^n$  of non-zero constant curvature  $\bar{K}$  with the projective factor  $P(x, y)$ . Then  $P$  must satisfy (3.2) and (4.3), where  $Q_i = P_{;i} - PP_{;i}$ ,  $Q = Q_0/2$  and  $F = L^2/2$ . Then we get the relation (3.8) between  $(L, K)$  and  $(\bar{L}, \bar{K})$ .*

The metrizable condition (1.8) has been considered satisfactorily, and the necessity of the following conditions has been stated above. We can conclude the sufficiency as follows:

**Theorem 2.** *Let  $F^n = (M^n, L(x, y))$  be a Finsler space of constant curvature  $K$  and assume that there exists in  $F^n$  a positively homogeneous function  $P(x, y)$  of degree one in  $y^i$  such that  $L^2K - 2Q \neq 0$  and (3.2) and (4.3) are satisfied, where  $Q_i = P_{;i} - PP_{;i}$ ,  $Q = Q_0/2$  and  $F = L^2/2$ . Then  $F^n$  is projective to a Finsler space  $\bar{F}^n = (M^n, \bar{L}(x, y))$  of non-zero constant curvature  $\bar{K}$ , where  $\bar{L}$  and  $\bar{K}$  are given by (3.8).*

*Remark.* For a non-zero quantity  $L^2K - 2Q$ , we may choose any constant  $\bar{K}$  of the same sign with  $L^2K - 2Q$  and determine  $\bar{L}$  by (3.8). A different choice of the constant  $\bar{K}$  corresponds to a homothetic change of the metric  $\bar{L}$ .

## §5. Projective change between Riemannian spaces of constant curvature

We shall apply our result to Riemannian spaces of constant curvature. Then (3.2) shows that  $P_i$  of  $P = P_i(x)y^i$  is locally a gradient vector field. We have

$$Q_i = (P_{i;r} - P_i P_r)y^r, \quad Q = (P_{r;s} - P_r P_s)y^r y^s / 2.$$

Thus, putting

$$(5.1) \quad p_{ij}(x) = P_{i;j} - P_i P_j,$$

it defines a symmetric tensor field and  $Q = p_{ij}y^i y^j / 2$ ,  $Q_i = p_{i0}$  and  $Q_{i,j} = p_{ij}$ . Therefore (3.7) gives

$$(5.2) \quad \bar{K} \bar{g}_{ij} = K g_{ij} - p_{ij}.$$

Further we consider (4.3); it is easy to show that the left-hand side of (4.3) is a quadratic form in  $y^i$ . If we consider the coefficients, then (4.3) is written in the form

$$(5.3) \quad p_{jk;i} = P_j(p_{ik} - K g_{ik}) + P_k(p_{ij} - K g_{ij}) + 2P_i(p_{jk} - K g_{jk}).$$

Consequently we get the following result from Theorem 1:

**Theorem 3.** *Let  $p : R^n = (M^n, g_{ij}(x)) \rightarrow \bar{R}^n = (M^n, \bar{g}_{ij}(x))$  be a projective change from a Riemannian space  $R^n$  of constant curvature  $K$  to a Riemannian space  $\bar{R}^n$  of non-zero constant curvature  $\bar{K}$ . Then we have the projective factor  $P = P_i(x)y^i$  with a locally gradient vector field  $P_i$  and (5.2) where  $p_{ij}(x)$  is defined by (5.1) and must satisfy (5.3).*

The theorem corresponding to Theorem 2 is easily stated.

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