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On the weighted ℓ^p−space of a discrete group

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Abstract. Let G be a locally compact group and $1 < p < \infty$. The L^p-conjecture asserts that $L^p(G)$ is closed under the convolution if and only if G is compact. For $2 < p < \infty$, we have recently shown that $f * g$ exists and belongs to $L^\infty(G)$ for all $f, g \in L^p(G)$ if and only if G is compact. Here, we consider the weighted case of this result for a discrete group G and a weight function ω on G; we prove that $f * g$ exists and belongs to $\ell^\infty(G, 1/\widetilde{\omega})$ for all $f, g \in \ell^p(G, \omega)$ if and only if $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\widetilde{\omega})$, the dual of $\ell^p(G, \widetilde{\omega})$.

1. Introduction

Let G be a locally compact group with a fixed left Haar measure λ. For $1 \le p < \infty$ and a measurable function $\alpha : G \to (0, \infty)$, we denote by $L^p(G, \alpha)$ the space of all measurable complex-valued functions f on G such that $f\,\alpha \in L^p(G)$, the usual Lebesgue space on G with respect to λ; see [7] for more details. Then $L^p(G, \alpha)$ with the norm $\|.\|_{p,\alpha}$ defined by

$$\|f\|_{p,\alpha} := \|f\,\alpha\|_p$$

for all $f \in L^p(G, \alpha)$ is a Banach space. Let us remark that when G is discrete, $L^p(G, \alpha)$ is the space $\ell^p(G, \alpha)$ of all complex-valued functions f on G such that

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$f\,\alpha \in \ell^p(G)$, the space of all complex-valued functions g on G with

$$\|g\|_p := \left(\sum_{x\in G} |g(x)|^p\right)^{1/p} < \infty.$$

We also denote by $L^\infty(G, 1/\alpha)$ the space of all measurable complex valued functions f on G such that $f/\alpha \in L^\infty(G)$, the space as defined in [7]. Then $L^\infty(G, 1/\alpha)$ with the norm $\|.\|_{\infty,1/\alpha}$ defined by

$$\|f\|_{\infty,1/\alpha} := \|f/\alpha\|_\infty$$

for all $f \in L^\infty(G, 1/\alpha)$. Furthermore, for $1 \leq p < \infty$, the continuous dual of $L^p(G, \alpha)$ coincides with $L^q(G, 1/\alpha)$, where q is the conjugate number $p/(p-1)$ of p; that is, $1 < q < \infty$ and $p + q = pq$. In fact, the mapping τ from $L^q(G, 1/\alpha)$ to $L^p(G, \alpha)^*$ defined by

$$\langle \tau(f), \varphi\rangle = \int_G f(x)\ \varphi(x)\ d\lambda(x) \tag{1}$$

is an isometric isomorphism. For measurable functions f and g on G, the convolution multiplication

$$(f * g)(x) = \int_G f(y)\ g(y^{-1}x)\ d\lambda(y).$$

is defined at each point $x \in G$ for which this makes sense; i.e. the function $y \mapsto f(y)\ g(y^{-1}x)$ is λ-integrable. Then $f * g$ is said to exist if $(f * g)(x)$ exists for almost all $x \in G$.

For $1 < p < \infty$, the L^p-conjecture states that $L^p(G)$ is closed under the convolution if and only if G is compact. The first result related to this conjecture is due to Zelazko [21] and Urbanik [20] in 1961; they proved that the conjecture is true for all locally compact abelian groups. However, this conjecture was first formulated for a locally compact group G by Rajagopalan in his Ph.D. thesis in 1963.

The truth of the conjecture has been established for $p > 2$ by Zelazko [22] and Rajagopalan [14] independently; see also Rajagopalan's works [13] for the case where $p \geq 2$ and G is discrete, [14] for the case where $p = 2$ and G is totally disconnected, and [15] for the case where $p > 1$ and G is either nilpotent or a semidirect product of two locally compact groups. In the joint work [16], they showed that the conjecture is true for $p > 1$ and amenable groups; this result can be also found in Greenleaf's book [6]. Rickert [18] confirmed the conjecture for $p = 2$. For related results on the subject see also Crombez [2] and [3], Gaudet and Gamlen [5], Johnson [8], Kunze and Stein [9], Lohoue [11], Milnes [12], Rickert [17], and Zelazko [23]. Finally, in 1990, Saeki [19] gave an affirmative answer to the conjecture by a completely self-contained proof; see also Kuznetsova [10] and El Kinani and Benazzouz [4] for some results on the weighted L^p−spaces of locally compact groups.

Motivated by the L^p-conjecture, in [1], we have considered only the property that $f * g$ exists for all $f, g \in L^p(G)$ and proved the following result which was indeed our purpose of that work.

Theorem 1.1. *Let G be a locally compact group and $2 < p < \infty$. If $f * g$ exists for all $f, g \in L^p(G)$, then G is compact.*

On the other hand, it is well-known from [13] that for $2 < p < \infty$, $L^p(G) \subseteq L^q(G)$ if and only if G is compact. As a consequence of these observations, we have the following result.

Theorem 1.2. *Let G be a locally compact group and $2 < p < \infty$. Then the following assertions are equivalent.*

(a) $L^p(G) \subseteq L^q(G)$.

(b) *$f * g$ exists and belongs to $L^\infty(G)$ for all $f, g \in L^p(G)$.*

In the case where G is discrete and $1 \leq p \leq 2$ we have $\ell^p(G) \subseteq \ell^2(G)$, and hence $f * g$ exists and belongs to $\ell^\infty(G)$ for all $f, g \in \ell^p(G)$ by the Holder inequality; moreover, $p \leq q$, and so $\ell^p(G) \subseteq \ell^q(G)$. These facts together with 1.2 led us to the following result.

Corollary 1.1. *Let G be a discrete group and $1 \leq p < \infty$. Then the following assertions are equivalent.*

(a) $\ell^p(G) \subseteq \ell^q(G)$.

(b) *$f * g$ exists and belongs to $\ell^\infty(G)$ for all $f, g \in \ell^p(G)$.*

In fact, this result gives a necessary and sufficient condition for that

$$\ell^p(G) * \ell^p(G) \subseteq \ell^\infty(G).$$

It states that the assertions (a) and (b) are always true if $1 < p \leq 2$, and that they are equivalent to finiteness of G if $2 < p < \infty$.

In this paper, we investigate this equivalence in the weighted case and prove an analogue of that result in terms of a weight function ω on G. We also observe that ω plays a significant role in this respect; in fact, we show that there is an infinite group for which the weighted forms of these equivalent assertions hold.

2. The results

Throughout this section, let G be a discrete group and ω be a weight function on G; that is, a measurable function with $\omega(x) > 0$ and $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. Also, let $\widetilde{\omega}$ be the weight function defined by $\widetilde{\omega}(x) = \omega(x^{-1})$ for all $x \in G$. We commence with the following lemma.

Lemma 2.1. *Let G be a discrete group, ω be a weight function on G and $1 \leq p < \infty$. If $f*g \in \ell^\infty(G, 1/\widetilde{\omega})$ for all $f, g \in \ell^p(G, \omega)$, then the map $(f, g) \mapsto f*g$ from $\ell^p(G, \omega) \times \ell^p(G, \omega)$ into $\ell^\infty(G, 1/\widetilde{\omega})$ is separately continuous.*

Proof. Fix a positive function $g \in \ell^p(G, \omega)$, and suppose on the contrary that the map $f \mapsto f * g$ from $\ell^p(G, \omega)$ into $\ell^\infty(G, 1/\widetilde{\omega})$ is not continuous. Then there exists a sequence $(f_n) \subseteq \ell^p(G, \omega)$ of positive functions with $\|f_n\|_{p,\omega} \leq 1$ such that

$$\|f_n * g\|_{\infty, 1/\widetilde{\omega}} \geq n^3$$

for all $n \in \mathbb{N}$. So, if we put

$$f := \sum_{n=1}^{\infty} \frac{f_n}{n^2},$$

then $f \in \ell^p(G, \omega)$; indeed,

$$\|f\|_{p,\omega} \leq \sum_{n=1}^{\infty} \frac{\|f_n\|_{p,\omega}}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

For each $n \in \mathbb{N}$ we have $f \geq n^{-2} f_n$ and thus $f * g \geq n^{-2} f_n * g$. Therefore

$$\|f * g\|_{\infty, 1/\widetilde{\omega}} \geq n^{-2} \|f_n * g\|_{\infty, 1/\widetilde{\omega}} \geq n;$$

in particular, $f*g \notin \ell^\infty(G, 1/\widetilde{\omega})$, a contradiction. It follows that the map $f \mapsto f*g$ from $\ell^p(G, \omega)$ into $\ell^\infty(G, 1/\widetilde{\omega})$ is continuous. Similarly, the map $f \mapsto g * f$ from $\ell^p(G, \omega)$ into $\ell^\infty(G, 1/\widetilde{\omega})$ is continuous. □

Now, we are ready to state and prove the main result of the paper which gives a necessary and sufficient condition for $\ell^p(G, \omega) * \ell^p(G, \omega) \subseteq \ell^\infty(G, 1/\widetilde{\omega})$.

Theorem 2.1. *Let G be a discrete group, ω a weight function on G and $1 \leq p < \infty$. Then the following assertions are equivalent.*

(a) $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\widetilde{\omega})$.

(b) *$f * g$ exists and belongs to $\ell^\infty(G, 1/\widetilde{\omega})$ for all $f, g \in \ell^p(G, \omega)$.*

PROOF. Suppose that (a) holds and let $f, g \in \ell^p(G,\omega)$ and $x \in G$. Define the function $R_x g$ on G by $R_x g(y) = g(yx)$ for all $y \in G$ and note that

$$\|R_x g\|_{p,\omega} = \left(\sum_{y\in G} |g(yx)|^p \omega(y)^p\right)^{1/p} = \left(\sum_{y\in G} |g(y)|^p \omega(yx^{-1})^p\right)^{1/p} \leq \widetilde{\omega}(x)\|g\|_{p,\omega}.$$

We therefore have $R_x g \in \ell^p(G,\omega)$ and thus $\widetilde{R_x g}/\omega \in \ell^q(G)$ by (a). Since $\omega\, f \in \ell^p(G)$, the Holder inequality implies that $f\ \widetilde{R_x g} \in \ell^1(G)$ and thus

$$\|f\ \widetilde{R_x g}\|_1 \leq \|f\|_{p,\omega}\ \|\widetilde{R_x g}\|_{q,1/\omega} = \|f\|_{p,\omega}\ \|R_x g\|_{q,1/\widetilde{\omega}}.$$

Since

$$\frac{|(f * R_x g)(e)|}{\widetilde{\omega}(e)} \leq |(f * R_x g)(e)| = \left|\sum_{y\in G} f(y)\ \widetilde{R_x g}(y)\right|,$$

it follows that

$$\frac{|(f * R_x g)(e)|}{\widetilde{\omega}(e)} \leq \|f\|_{p,\omega}\ \|R_x g\|_{q,1/\widetilde{\omega}}.$$

This together with

$$\|R_x g\|_{q,1/\widetilde{\omega}} = \left(\sum_{y\in G} \frac{|g(yx)|^q}{\widetilde{\omega}(y)^q}\right)^{1/q} = \left(\sum_{y\in G} \frac{|g(y)|^q}{\widetilde{\omega}(yx^{-1})^q}\right)^{1/q} \leq \widetilde{\omega}(x)\ \|g\|_{q,1/\widetilde{\omega}}$$

yield that

$$\begin{aligned}|(f * g)(x)| &= |\sum_{x\in G} f(y)\ g(y^{-1}x)| = |\sum_{x\in G} f(y)\ \widetilde{R_x g}(y)| = |(f * R_x g)(e)| \\ &\leq \omega(e)\ \|f\|_{p,\omega}\ \|R_x g\|_{q,1/\widetilde{\omega}} \leq \omega(e)\ \widetilde{\omega}(x)\ \|f\|_{p,\omega}\ \|g\|_{q,1/\widetilde{\omega}}.\end{aligned}$$

Hence

$$\|f * g\|_{\infty,1/\widetilde{\omega}} \leq \omega(e)\ \|g\|_{p,\omega}\ \|f\|_{q,1/\widetilde{\omega}}.$$

Therefore $f * g$ exists and belongs to $\ell^\infty(G, 1/\widetilde{\omega})$.

Conversely, suppose that (b) holds, and let $f \in \ell^p(G,\omega)$. We show that $f \in \ell^q(G, 1/\widetilde{\omega})$. Lemma 2.1 implies that the map

$$\ell^p(G,\omega) \times \ell^p(G,\omega) \to \ell^\infty(G, 1/\widetilde{\omega})$$

with the formula $(f, g) \mapsto f * g$ is separately continuous. By the uniform boundedness theorem, there exists a constant $M > 0$ such that

$$\|f * g\|_{\infty,1/\widetilde{\omega}} \leq M\ \|f\|_{p,\omega}\ \|g\|_{p,\omega}$$

for all $f, g \in \ell^p(G,\omega)$. For every $g \in \ell^p(G,\widetilde{\omega})$ we have

$$\begin{aligned}\sum_{y\in G} f(y)\, g(y) &= \sum_{y\in G} f(y)\, \widetilde{g}(y^{-1}) = \sum_{y\in G} f(y)\, \widetilde{g}(y^{-1}e) = (f*\widetilde{g})(e)\\ &\le \omega(e)\, \|f*\widetilde{g}\|_{\infty,1/\widetilde{\omega}} \le M\, \omega(e)\, \|f\|_{p,\omega}\, \|\widetilde{g}\|_{p,\omega}\\ &= M\, \omega(e)\, \|f\|_{p,\omega}\, \|g\|_{p,\widetilde{\omega}}.\end{aligned}$$

Thus the functional $\tau(f) : g \mapsto \sum_{y\in G} f(y)\, g(y)$ is bounded on $\ell^p(G,\widetilde{\omega})$. Since $\ell^p(G,\widetilde{\omega})^* = \ell^q(G,1/\widetilde{\omega})$, it follows that $f \in \ell^q(G,1/\widetilde{\omega})$. □

Corollary 2.1. *Let G be a discrete group and ω be a weight function such that $\omega \ge 1$, $\omega = \widetilde{\omega}$ and $1 \in \ell^q(G,1/\omega)$, and $1 \le p < \infty$. Then $f*g$ exists and belongs to $\ell^\infty(G,1/\widetilde{\omega})$ for all $f,g \in \ell^p(G,\omega)$.*

Proof. If $f \in \ell^p(G,\omega)$, then $f\omega \in \ell^p(G)$. Since $1/\omega \in \ell^q(G)$, $f = (f\omega)(1/\omega) \in \ell^1(G)$. Since G is discrete and $\omega \ge 1$,

$$\ell^1(G) \subseteq \ell^q(G) \subseteq \ell^q(G,1/\omega).$$

Thus $f \in \ell^q(G,1/\omega)$ and therefore $\ell^p(G,\omega) \subseteq \ell^q(G,1/\omega)$. Now, we only need to invoke Theorem 2.1. □

Corollary 2.2. *Let G be a discrete group, ω be a weight function on G and $1 \le p \le 2$. Then $f*g$ exists and belongs to $\ell^\infty(G,1/\widetilde{\omega})$ for all $f,g \in \ell^p(G,\omega)$.*

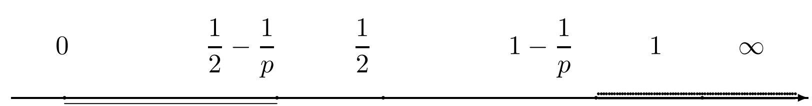
Proof. Since $\omega\widetilde{\omega} \ge 1$, it follows that $\ell^q(G,\omega) \subseteq \ell^q(G,1/\widetilde{\omega})$; also $p \le q$, and so $\ell^p(G,\omega) \subseteq \ell^q(G,\omega)$. Consequently, $\ell^p(G,\omega) \subseteq \ell^q(G,1/\widetilde{\omega})$. Now the result follows from Lemma 2.1 and Theorem 2.1. □

Our next result introduces a large class of weight functions ω on an infinite group G for which $f*g$ exists and belongs to $\ell^\infty(G,1/\widetilde{\omega})$ for all $f,g \in \ell^p(G,\omega)$. In particular, this shows that Corollary 2.2 is not true for $2 < p < \infty$.

Proposition 2.1. *Let $2 < p < \infty$, $\mathbb{Z}$ be the additive group of integer numbers and $\omega_\alpha(n) = (1+|n|)^\alpha$ for all $n \in \mathbb{Z}$, where $0 < \alpha < \infty$. Then*

(1) *$\ell^p(\mathbb{Z},\omega_\alpha) \nsubseteq \ell^q(\mathbb{Z},1/\widetilde{\omega}_\alpha)$ for all $0 < \alpha < \frac{1}{2} - \frac{1}{p}$.*

(2) *$\ell^p(\mathbb{Z},\omega_\alpha) \subseteq \ell^q(\mathbb{Z},1/\widetilde{\omega}_\alpha)$ for all $1 - \frac{1}{p} < \alpha < \infty$.*



Proof. (1). Suppose that $0 < \alpha < \frac{1}{2} - \frac{1}{p}$. Then $\alpha + \frac{1}{p} < \frac{1}{q} - \alpha$ and consequently there exists a real number β such that

$$\alpha + \frac{1}{p} < \beta < \frac{1}{q} - \alpha$$

We therefore have $p\beta - p\alpha > 1$ and $q\beta + q\alpha < 1$. Now, let f be the function on $\mathbb{Z}$ defined by

$$f(n) = (1 + |n|)^{-\beta} \quad (n \in \mathbb{Z}).$$

It follows that

$$\begin{aligned}\|f\|_{p,\omega}^p &= \sum_{n=-\infty}^{\infty} f(n)^p \omega_\alpha(n)^p = 1 + 2\sum_{n=1}^{\infty}(1+|n|)^{-p\beta}(1+|n|)^{p\alpha} \\ &= 1 + 2\sum_{n=1}^{\infty} \frac{1}{(1+|n|)^{p\beta - p\alpha}}\end{aligned}$$

whence $\|f\|_{p,\omega_\alpha}^p < \infty$, and so $f \in \ell^p(\mathbb{Z}, \omega_\alpha)$. But

$$\begin{aligned}\|f\|_{q,1/\omega}^q &= 1 + 2\sum_{n=1}^{\infty} \frac{f(n)^q}{\omega_\alpha(n)^q} = 1 + 2\sum_{n=1}^{\infty}(1+|n|)^{-q\beta}(1+|n|)^{-q\alpha} \\ &= 1 + 2\sum_{n=1}^{\infty}(1+|n|)^{-q\alpha - q\beta}\end{aligned}$$

whence $\|f\|_{q,1/\omega_{alpha}}^q = \infty$ and therefore $\ell^p(\mathbb{Z}, \omega_\alpha) \nsubseteq \ell^q(\mathbb{Z}, 1/\widetilde{\omega}_\alpha)$.

(2). Suppose that $1 - \frac{1}{p} < \alpha < \infty$. Then $\omega_\alpha \geq 1$, $\omega_\alpha = \widetilde{\omega}_\alpha$ and $1 \in \ell^q(G, 1/\omega_\alpha)$. It follows from the proof of Corollary 2.1 that $\ell^p(G, \omega_\alpha) \subseteq \ell^q(G, 1/\omega_\alpha)$. □

A combination of Theorems 1.1 and 2.1 imply that G is finite if $\ell^p(G) \subseteq \ell^q(G)$. As an immediate consequence of Proposition 2.1, we have the following result which shows that this result is not valid in the weighted case.

Corollary 2.3. *For each $1 \leq p < \infty$, there exist an infinite countable discrete group G and a weight function ω on G with $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\widetilde{\omega})$.*

Let us point out that if there is a bounded or multiplicative weight function ω on G with $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\widetilde{\omega})$, then G is finite. Our next result gives an analogue of this fact without the hypothesis that ω is bounded or multiplicative.

Proposition 2.2. *Let G be a discrete group and $2 < p < \infty$. If there is a weight function ω on G with $\ell^p(G, \omega) \subseteq \ell^q(G, 1/\widetilde{\omega})$, then G is countable.*

Proof. It suffices to show that the set $\{x \in G : \omega(x)\,\omega(x^{-1}) \le k\}$ is finite for all $k \ge 2$. To that end, suppose on the contrary that there exist $k_0 \ge 1$ and a sequence (x_n) of distinct elements of G with $\omega(x_n)\,\omega(x_n^{-1}) \le k_0$ for all $n \ge 1$. Define the function $f : G \to \mathbb{R}$ by

$$f(x_n) = \frac{1}{\omega(x_n)\sqrt{n}} \quad \text{and} \quad f(x_n^{-1}) = \frac{1}{\omega(x_n^{-1})\sqrt{n}}$$

for all $n \ge 1$ and $f(x) = 0$ otherwise. Clearly $f \in \ell^p(G,\omega)$ and we have

$$(f * f)(e) = \sum_{n=1}^{\infty} f(x_n)\, f(x_n^{-1}) = \sum_{n=1}^{\infty} \frac{1}{\omega(x_n)\,\omega(x_n^{-1})n} \ge \frac{1}{k_0} \sum_{n=1}^{\infty} \frac{1}{n}.$$

It follows that $f * f$ does not exist. This contradicts Theorem 2.1 together with the assumption that $\ell^p(G,\omega) \subseteq \ell^q(G, 1/\widetilde{\omega})$. $\square$

Let us present some examples illustrating our observations.

Example 1. (a) Let $\mathbb{R}$ be the additive group of real numbers and $2<p<\infty$. Then there is no weight functions ω on $\mathbb{R}$ with $\ell^p(\mathbb{R},\omega) \subseteq \ell^q(\mathbb{R}, 1/\widetilde{\omega})$ by Proposition 2.2. In particular, $\ell^p(\mathbb{R},\omega_\alpha) \nsubseteq \ell^q(\mathbb{R}, 1/\widetilde{\omega}_\alpha)$, where $\omega_\alpha(x) = (1+|x|)^\alpha$ for all $x \in \mathbb{R}$, and $\alpha > 0$. So, there exist $f, g \in \ell^p(\mathbb{R},\omega_\alpha)$ such that $f * g$ does not belong to $\ell^\infty(\mathbb{R}, 1/\widetilde{\omega}_\alpha)$.

(b) Let $\mathbb{Z}$ be the additive group of integer numbers, $\alpha > 0$ and $\omega_\alpha(n) = (1+|n|)^\alpha$ for all $n \in \mathbb{Z}$. Then $\ell^3(\mathbb{Z},\omega_\alpha) \subseteq \ell^{3/2}(\mathbb{Z}, 1/\widetilde{\omega}_\alpha)$ for all $\alpha > 2/3$ and $\ell^3(\mathbb{Z},\omega_\alpha) \nsubseteq \ell^{3/2}(\mathbb{Z}, 1/\widetilde{\omega}_\alpha)$ for all $0 < \alpha < 1/6$.

(c) Let $\mathbb{Z}$ be the additive group of integer numbers, $2 < p < \infty$ and $\omega(n) = \exp(|n|)$ for all $n \in \mathbb{Z}$. Then ω is a weight function on $\mathbb{Z}$ such that $\omega \ge 1$, $\omega = \widetilde{\omega}$ and $1/\omega \in \ell^q(G)$. Then $\ell^p(\mathbb{Z},\omega) \subseteq \ell^q(\mathbb{Z}, 1/\widetilde{\omega})$.

(d) Let $\mathbb{Z}$ be the additive group of integer numbers, $2 < p < \infty$ and $\omega(n) = \exp(n)$ for all $n \in \mathbb{Z}$. Then ω is a multiplicative weight function on $\mathbb{Z}$, and thus $\ell^p(\mathbb{Z},\omega) \nsubseteq \ell^q(\mathbb{Z}, 1/\widetilde{\omega})$.

We end this work by some questions which arises from our results.

Question 1. Does Theorem 2.1 remains valid for all locally compact groups G? In fact, for a measurable weight function ω on G, is the following assertions equivalent?

(a) $L^p(G,\omega) \subseteq L^q(G, 1/\widetilde{\omega})$.

(b) $f * g$ exists and belongs to $L^\infty(G, 1/\widetilde{\omega})$ for all $f, g \in L^p(G,\omega)$.

Question 2. In Proposition 2.1, what happens when $1/2 - 1/p \le \alpha \le 1 - 1/p$? In fact, does the inclusion $\ell^p(\mathbb{Z},\omega_\alpha) \subseteq \ell^q(\mathbb{Z}, 1/\widetilde{\omega}_\alpha)$ hold for all such α?

Question 3. Is the converse of Proposition 2.2 true? In fact, if G is an infinite countable discrete group and $2 < p < \infty$, is there a weight function ω on G with $\ell^p(G,\omega) \subseteq \ell^q(G, 1/\widetilde{\omega})$?

References

[1] F. Abtahi, R. Nasr-Isfahani and A. Rejali, On the L^p−conjecture for locally compact groups, *Arch. Math. (Basel)* **89** (2007), 237–242.

[2] G. Crombez, A characterization of compact groups, *Simon Stevin* **53** (1979), 9–12.

[3] G. Crombez, An elementary proof about the order of the elements in a discrete group, *Proc. Amer. Math. Soc.* **85** (1983), 59–60.

[4] A. El Kinani and A. Benazzouz, Structure m-convex dans l'espace poids $L^p_\Omega(\mathbb{R}^n)$, *Bull. Belg. Math. Soc. Simon Stevin* **10** (2003), 49–57.

[5] R. J. Gaudet and J. L. Gamlen, An elementary proof of part of a classical conjecture, *Bull. Austral. Math. Soc.* **3** (1970), 285–292.

[6] F. P. Greenleaf, Invariant Means on Locally Compact Groups and their Applications, Vol. 16, Math. Studies, *Van Nostrand*, *New York*, 1969.

[7] E. Hewitt and K. Ross, Abstract harmonic analysis I, *Springer-Verlag*, *New York*, 1970.

[8] D. L. Johnson, A new proof of the L^p-conjecture for locally compact groups, *Colloq. Math.* **47** (1982), 101–102.

[9] R. Kunze and E. Stein, Uniformly bounded representations and harmonic analysis of the 2×2 real unimodualr group, *Amer. J. Math.* **82** (1960), 1–62.

[10] Yu. N. Kuznetsova, Weighted L^p-algebras on groups, *Funct. Anal. Appl.* **40** (2006), 234–236.

[11] N. Lohoue, Estimations L^p des coefficients de representation et operateurs de convolution, *Adv. Math.* **38** (1980), 178–221.

[12] P. Milnes, Convolution of L^p functions on non-commutative groups, *Canad. Math. Bull.* **14** (1971), 265–266.

[13] M. Rajagopalan, On the ℓ^p-spaces of a locally compact group, *Colloq. Math.* **10** (1963), 49–52.

[14] M. Rajagopalan, L_p-conjecture for locally compact groups I, *Trans. Amer. Math. Soc.* **125** (1966), 216–222.

[15] M. Rajagopalan, L_p-conjecture for locally compact groups II, *Math. Ann.* **169** (1967), 331–339.

[16] M. Rajagopalan and W. Zelazko, L_p-conjecture for solvable locally compact groups, *J. Indian Math. Soc.* **29** (1965), 87–93.

[17] N. W. Rickert, Convolution of L^p functions, *Proc. Amer. Math. Soc.* **18** (1967), 762–763.

[18] N. W. Rickert, Convolution of L^2 functions, *Colloq. Math.* **19** (1968), 301–303.

[19] S. Saeki, The L^p−conjecture and Young's inequality, *Illinois. J. Math.* **34** (1990), 615–627.

[20] K. Urbanik, A proof of a theorem of Zelazko on L^p-algebras, *Colloq. Math.* **8** (1961), 121–123.

[21] W. Zelazko, On the algebras L^p of a locally comapct group, *Colloq. Math.* **8** (1961), 112–120.

[22] W. Zelazko, A note on L^p algebras, *Colloq. Math.* **10** (1963), 53–56.

[23] W. Zelazko, On the Burnside problem for locally compact groups, *Symp. Math.* **16** (1975), 409–416.

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