Publ. Math. Debrecen
75/3-4 (2009), 375–385

Some characterizations of $L_9(2)$ related to its prime graph

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Abstract. Let G be a finite group. The prime graph of G is denoted by $\Gamma(G)$. In this paper as the main result we determine finite groups G such that $\Gamma(G) = \Gamma(L_9(2))$. Let $\pi_e(G)$ be the set of element orders of G, which is called the spectrum of G. Denote by $h(G)$ the number of isomorphism classes of finite groups H satisfying $\pi_e(H) = \pi_e(G)$. It is proved that some finite groups are uniquely determined by their spectrum, i.e. $h(G) = 1$. As a consequence of our result we prove that the simple group $L_9(2)$ is uniquely determined by its spectrum.

The degree pattern of a finite group is denoted by $D(G)$. At last we prove that if G is a finite group such that $|G| = |L_9(2)|$ and $D(G) = D(L_9(2))$, then $G \cong L_9(2)$.

1. Introduction

We denote by $\pi(n)$ the set of all prime divisors of the integer n. Let G be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of orders of the elements of G is denoted by $\pi_e(G)$ and is called the spectrum of G. It is clear that the set $\pi_e(G)$ is closed and partially ordered by divisibility, hence it is uniquely determined by $\mu(G)$, the subset of its maximal elements. The *prime graph* of G is a graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. The prime graph of G is denoted by $\Gamma(G)$ and sometimes called Kegel–Gruenberg graph. We denote by $t(\Gamma(G))$, the number of connected components of $\Gamma(G)$.

Mathematics Subject Classification: 20D05, 20D60, 20D08.
Key words and phrases: prime graph, order elements, spectrum, degree pattern, projective special linear group.
This research was in part supported by a grant from IPM (No. 85200022).

Hagie in [4] determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. In [6] finite groups with the same prime graph as a CIT simple group are determined. It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group ${}^2G_2(q)$ is uniquely determined by its prime graph [5], [16]. Also the authors in [7] proved that if $p > 11$ is a prime number and $p \not\equiv 1 \pmod{12}$, then $PSL(2,p)$ is uniquely determined by its prime graph. We note that the prime graph of the previous groups are disconnected. In this paper we consider a group with connected prime graph and we prove that

Theorem 1.1. *If G is a finite group and $\Gamma(G) = \Gamma(L_9(2))$, then $G/O_\pi(G) \cong L_9(2)$, where $\pi \subseteq \{2,3,5\}$.*

In [9] it is proved that $L_n(2)$, where $n \geq 3$ is recognizable by spectrum. As a consequence of Theorem 1.1, we can give a new proof for this theorem when $n = 9$.

Theorem 1.2. *The simple group $L_9(2)$ is uniquely determined by its spectrum. In other words, if G is a finite group, then $G \cong L_9(2)$ if and only if $\pi_e(G) = \pi_e(L_9(2))$.*

Let $\pi(G) = \{p_1, p_2, \dots, p_m\}$ and $p_1 < p_2 < \cdots < p_m$. The *degree pattern* of G is denoted by $D(G)$ and defined as follows:

$$D(G) = (\deg(p_1), \deg(p_2), \dots, \deg(p_m)),$$

where $\deg(p_i)$ is the degree of vertex p_i in the prime graph of G. A group G is called *OD-characterizable* if G is uniquely determined by $|G|$ and $D(G)$. It is proved that some finite groups are OD-characterizable, for example sporadic simple groups, $PSL(2,q)$, $PSL(3,q)$, $PSU(3,q)$ [11], [12], [17].

Theorem 1.3. *The simple group $L_9(2)$ is OD-characterizable, in other words if G is a finite group, then $G \cong L_9(2)$ if and only if $|G| = |L_9(2)|$ and $D(G) = D(L_9(2))$.*

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [2], for example.

2. Preliminary results

Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$. Also we denote by $t(2,G)$ the maximal number of vertices in the independent sets of $\Gamma(G)$ containing 2.

Lemma 2.1 (see [13]). *Let G be a finite group satisfying the two conditions:*

(a) *there exists three primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$; i.e. $t(G) \geq 3$;*

(b) *there exists an odd prime in $\pi(G)$ nonadjacent in $\Gamma(G)$ to the prime 2; i.e. $t(2, G) \geq 2$.*

Then there is a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for the maximal normal soluble subgroup K of G. Furthermore $t(S) \geq t(G) - 1$, and one of the following statements holds:

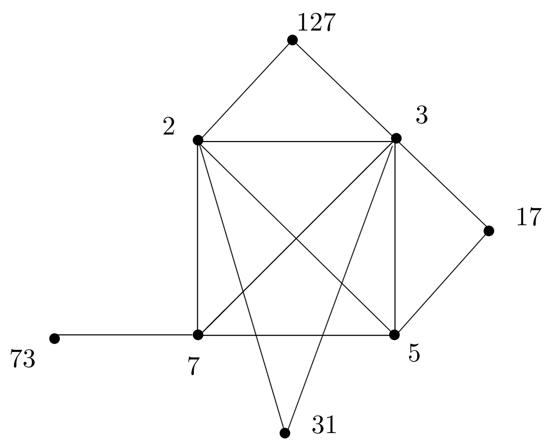
(1) *$S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, S) = 3$.*

(2) *For every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.*

By using [10, Table 2], we have the following result.

Lemma 2.2. $\mu(L_9(2)) = \{16, 56, 120, 124, 186, 210, 217, 252, 254, 255, 381, 465, 511\}$.

Therefore the prime graph of G is as follows:



Lemma 2.3 (see [15]). *The group $L_{n+1}(q)$, $n \geq 4$, contains a Frobenius subgroup with kernel of order q^n and cyclic complement of order $(q^n - 1)/(n+1, q-1)$.*

Lemma 2.4 (see [8]). *Let G be a group, N a normal subgroup of G, and G/N a Frobenius group with Frobenius kernel K and cyclic complement C. If $(|K|, |N|) = 1$ and K is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$ for some prime divisor p of $|N|$.*

Lemma 2.5 ([14, Corollary]). *If G is a solvable group with at least two prime graph components, then G is either Frobenius group or 2-Frobenius group and G has exactly two prime graph components one of which consists of the primes dividing the lower Frobenius complement.*

As a corollary of Lemma 2.5, it follows that if G is a solvable finite group, then $t(\Gamma(G)) \leq 2$.

Lemma 2.6 (Zsigmondy Theorem, (see [18])). *Let p be a prime and n be a positive integer. Then one of the following holds:*

(i) *there is a primitive prime p' for $p^n - 1$, that is, $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$,*

(ii) *$p = 2$, $n = 1$ or 6,*

(iii) *p is a Mersenne prime and $n = 2$.*

Remark 1. Let p be a prime number and $(a, p) = 1$. Let $k \geq 1$ be the smallest positive integer such that $a^k \equiv 1 \pmod{p}$. Then k is called *the order of a with respect to p* and we denote it by $\mathrm{ord}_p(a)$. Obviously by the Fermat's little theorem it follows that $\mathrm{ord}_p(a) \mid (p - 1)$. Also if $a^n \equiv 1 \pmod{p}$, then $\mathrm{ord}_p(a) \mid n$.

3. Groups with the same prime graph as $L_9(2)$

In this section we prove Theorem 1.1 through the following lemmas. So in this section let G be a finite group such that $\Gamma(G) = \Gamma(L_9(2))$.

Lemma 3.1. *Let G be a finite group and $\Gamma(G) = \Gamma(L_9(2))$. Then G is not a solvable group.*

Proof. Let G be a solvable group. We know that $\{17, 31, 73, 127\} \subseteq \pi(G)$. Therefore G has a Hall $\{17, 31, 73, 127\}$-subgroup. If H is a Hall $\{17, 31, 73, 127\}$-subgroup of G, then $t(\Gamma(H)) = 4$, since $\Gamma(H)$ is a subgraph of $\Gamma(G)$ and $\{17, 31, 73, 127\}$ are pairwise nonadjacent in $\Gamma(G)$. Hence G is not a solvable group by Lemma 2.5. □

Lemma 3.2. *Let N be the maximal normal solvable subgroup of G. If $A = \pi(N) \cap \{17, 31, 73, 127\}$, then $|A| \leq 1$.*

Proof. By using Lemma 3.1, it follows that $N \neq G$. If $A \neq \emptyset$, then let H be a Hall A-subgroup of N. If $|A| \geq 3$, then similar to the proof of Lemma 3.1, it follows that $t(\Gamma(H)) \geq 3$, which is a contradiction, by Lemma 2.5.

If $|A| = 2$, then $A = \{p_1, p_2\}$. Now N is a normal subgroup of G and H is a Hall subgroup of N. Hence by the Frattini argument it follows that $G = NN_G(H)$. Let $p_3 \in \{17, 31, 73, 127\} \setminus A$. Obviously $p_3 \notin \pi(N)$ and so $N_G(H)$ has an element of order p_3, say x. Then x is an element of order p_3 which acts fixed point freely on H. Now by using Thompson's Theorem [3, Theorem 10.2.1], it follows that H is nilpotent and so $p_1 \sim p_2$ in $\Gamma(H)$, which is a contradiction. Therefore $|A| \leq 1$. $\square$

Lemma 3.3. *If N is the maximal normal subgroup of G, then $L_9(2) \leq G/N \leq \mathrm{Aut}(L_9(2))$.*

Proof. Easily we can see that $t(G) = 4$ and $t(2, G) = 3$. Therefore by using Lemma 2.1, there exists a nonabelian simple group P such that $P \leq G/N \leq \mathrm{Aut}(P)$. We note that it is proved that $P \cong \mathrm{Socle}(G/N)$. Now we use the classification theorem of finite simple groups. Obviously $\pi(P) \subseteq \pi(G)$ and $\pi(G/N) \subseteq \pi(\mathrm{Aut}(P))$. Also we know that $\{17, 31, 73, 127\} \cap \pi(\mathrm{Aut}(P))$ has at least three elements. If P is a sporadic simple group, then $\{73, 127\} \cap \pi(\mathrm{Aut}(P)) = \emptyset$, which is a contradiction. Also easily we can see that P is not isomorphic to an alternating group. So P is a simple group of Lie type. Now we prove that $\pi(\mathrm{Aut}(P)) \cap \{17, 31, 73, 127\} \subseteq \pi(P)$. In fact we prove that $\{17, 31, 73, 127\} \cap \pi(\mathrm{Out}(P)) = \emptyset$ and so $\{17, 31, 73, 127\} \cap \pi(P)$ has at least three elements. Let $p \in \{17, 31, 73, 127\} \cap \pi(\mathrm{Out}(P))$. By using the notations of [2, Page XVI] we know that $|\mathrm{Out}(P)| = gdf$. By using the tables of [2, PageXVI], we see that for every finite simple group P we have $g \mid 3!$. Also if $P \neq A_n(q)$ and $P \neq {}^2A_n(q)$, then $d \mid 12$. If $P = A_n(q)$ and $p \mid d$, then $p \mid (n+1)$ and hence $n \geq 16$. Now $(q^t - 1) \mid |P|$, for every $2 \leq t \leq n+1$ and hence $|\pi(P)| \geq 10$, by Zsigmondy Theorem, which is a contradiction. Similarly it follows that $P = {}^2A_n(q)$ and $p \mid d$ is impossible. Therefore $p \mid f$ where $q = p_0^f$ and p_0 is a prime number. We know that for finite simple groups in [2, page XVI], always $(q-1)$ is a divisor of $|P|$. Also $(p_0^p - 1) \mid (q-1)$ and $p \in \{17, 31, 73, 127\}$. Easily we can see that if $p_0 \in \{2, 3, 5, 7, 17, 31, 73, 127\}$, then $p_0^p - 1$ has a primitive prime which is not a divisor of $|G|$, and this is a contradiction. Therefore $\pi(\mathrm{Aut}(P)) \cap \{17, 31, 73, 127\} \subseteq \pi(P)$ and so $\{17, 31, 73, 127\} \cap \pi(P)$ has at least three elements. Now we must consider each possibility separately. For convenience we omit the details of the proof and only state a few of them.

Step 1. Let $P \cong A_n(q)$, where $q = p_0^\alpha$. Obviously $p_0 \in \pi(G)$ and we know that $\{17, 31, 73, 127\} \cap \pi(P)$ has at least three elements. If $p_0 = 2$, then $q = 2^\alpha$. We know that $\mathrm{ord}_2 17 = 8$, $\mathrm{ord}_2 31 = 5$, $\mathrm{ord}_2 73 = 9$ and $\mathrm{ord}_2 127 = 7$. Also

$\pi(\prod_{i=2}^{9}(2^i-1)) = \{3,5,7,17,31,73,127\}$. Therefore by using Zsigmondy Theorem it follows that $(2^m-1) \nmid |P|$, for $m \geq 10$. Also

$$\prod_{i=2}^{n+1}(q^i-1) \mid \prod_{i=2}^{9}(2^i-1),$$

and at least three items of 2^5-1, 2^7-1, 2^8-1 and 2^9-1 divide $\prod_{i=2}^{n+1}(2^{i\alpha}-1)$. Therefore $\alpha = 1$ and $q = 2$, which implies that $n \geq 7$. Then $P \cong L_8(2)$ or $P \cong L_9(2)$.

If $P \cong L_8(2)$, then $73 \in \pi(N)$, since 73 does not divide $|\operatorname{Aut}(P)|$. Now by using Lemma 2.3, $P \cong L_8(2)$ contains a Frobenius subgroup KC whose kernel K is an elementary abelian 2-group of order 2^7 and whose complement C is cyclic of order 2^7-1. Therefore G/N contains a Frobenius subgroup T/N which is isomorphic to KC. Now let $\widehat{G} = G/O_{73'}(N)$. It is obvious that $O_{73}(\widehat{G}) \neq 1$. Hence T/N acts on $O_{73}(\widehat{G})$ faithfully and its kernel of order 2^7 acts fixed point freely on $O_{73}(\widehat{G})$. Now by using Lemma 2.4, we conclude that G has an element of order $73(2^7-1)$, which is a contradiction.

Also 13 divides (3^3-1) and (5^4-1). Similarly $19 \mid (7^3-1)$, $307 \mid (17^3-1)$, $331 \mid (31^3-1)$, $37 \mid (73^2-1)$ and $5419 \mid (127^3-1)$, which implies that $p_0 \notin \pi(G) \setminus \{2\}$.

Step 2. Let $P \cong {}^2A_n(q)$, where $q = p_0^\alpha$ and $n \geq 2$. It is obvious that every primitive prime of $x^{2n}-1$ divides x^n+1. If $p_0 = 2$, then for $k \geq 10$, 2^k-1 does not divide $|P|$. Therefore $(2^t+1) \nmid |P|$, for $t \geq 5$. Therefore $\{31,73,127\} \subseteq \pi(N)$, which is a contradiction. If $p_0 = 3$, then $\operatorname{ord}_{17} 3 = 16$ and easily we get a contradiction. Similar arguments show that other cases are impossible.

Step 3. Let $P \cong B_n(q)$ or $C_n(q)$, where $q = p_0^\alpha$ and $n \geq 2$. Similar to the last steps we conclude that $q \neq 2$ is impossible. If $q = 2^\alpha$, then

$$\prod_{i=1}^{n}(q^{2i}-1) \mid \prod_{i=1}^{9}(2^i-1),$$

since $(2^{10}-1) \nmid |P|$. Therefore 2^5-1, 2^7-1 and 2^9-1 do not divide the order of P, which is a contradiction.

Other steps are similar and we omit the details of the proof, for convenience. $\square$

Lemma 3.4. *N is a $\{2,3,5\}$-subgroup of G.*

Proof. Let $N \neq 1$. So there exists a prime p such that $O^p(N) \neq N$, since N is solvable. Then $N/O^p(N)$ is a nontrivial p-group. Let $\widehat{N} = N/O^p(N)$

and $\widehat{G} = G/O^p(N)$, since $O^p(N)$ is a characteristic subgroup of N and $N \lhd G$. If the Frattini subgroup of $\widehat{N}$ is denoted by $\Phi(\widehat{N})$, then $\widehat{N}/\Phi(\widehat{N})$ is an elementary abelian p-group and we have

$$\frac{G}{N} \cong \frac{\widehat{G}}{\widehat{N}} \cong \frac{\widehat{G}/\Phi(\widehat{N})}{\widehat{N}/\Phi(\widehat{N})}.$$

Therefore without loss of generality we can assume that N is an elementary abelian p-group. Let $C = C_G(N)$ and note that C is a normal subgroup of G. Therefore CN is a normal subgroup of G. If $C \nleq N$, then CN/N contains a subgroup which is isomorphic to $L_9(2)$, since $\mathrm{Socle}(G/N) \cong L_9(2)$. Then 73 divides $|L_9(2)|$ and hence 73 is a divisor of $|CN/N| = |C/(C \cap N)|$ and so $C_G(N)$ contains an element of order 73. By assumption p divides $|N|$ and hence G contains an element of order $73p$. Now the prime graph of G shows that $p = 7$. Similar argument shows that G contains an element of order 127×7, which is a contradiction. Therefore we may assume that $C \leq N$ and $L_9(2)$ acts faithfully on N.

By assumption, $L_9(2) \leq G/N$ and $L_9(2)$ contains a Frobenius subgroup KF whose kernel K is an elementary abelian 2-group of order 2^8 and whose complement F is cyclic of order 2^8-1. Hence G/N contains a Frobenius subgroup T/N of the form $2^8 : 2^8 - 1$. Now by using Lemma 2.4, we conclude that $3 \times 5 \times 17 \times p = (2^8 - 1)p \in \pi_e(G)$. Therefore $p \in \{2, 3, 5, 17\}$. Now we prove that $p = 17$ is impossible. If $\{2, 17\} \subseteq \pi(N)$, then let H be a Hall $\{2, 17\}$-subgroup of N. Then by Frattini argument it follows that $G = NN_G(H)$. Therefore 73 divides $|N_G(H)|$, and so an element of order 73, say x acts fixed point freely on H. Hence H is nilpotent by Thompson's Theorem [3, Theorem 10.2.1] and G has an element of order 34, which is not the case. If 17 but not 2, divides the order of N then, since $L_2(9)$ contains a non-cyclic abelian 2-subgroup, G again contains an element of order 34, which is a contradiction. Hence N is a $\{2, 3, 5\}$-subgroup of G. □

Lemma 3.5. *The finite group G/N is isomorphic to $L_9(2)$.*

Proof. Up to now it follows that $L_9(2) \leq G/N \leq \mathrm{Aut}(L_9(2))$. Also $|\mathrm{Out}(L_9(2))| = 2$, which implies that $G/N \cong L_9(2)$ or $G/N \cong \mathrm{Aut}(L_9(2))$.

Let $G/N \cong \mathrm{Aut}(L_9(2))$. If σ is a graph automorphism of order 2 of $L_9(2)$, then by using Theorem 19.9 in [1] we have

$$C_{L_9(2)}(\sigma) \cong PSO^+(9, 2),$$

and so $|C_{L_9(2)}(\sigma)| = 2^{16}(2^8-1)(2^6-1)(2^4-1)(2^2-1)$. Therefore $2 \sim 17$ in $\Gamma(G)$, which is a contradiction. Hence $G/N \cong L_9(2)$. □

Now the proof of Theorem 1.1 is completed. As a consequence of this result we can prove that $L_9(2)$ is recognizable by its order elements.

Proof of Theorem 1.2. If $\pi_e(G) = \pi_e(L_9(2))$, then obviously $\Gamma(G) = \Gamma(L_9(2))$ and so by using Theorem 1.1 it follows that $G/N \cong L_9(2)$. We note that in the proof of Lemma 3.4, we prove that if $p \mid |N|$, then $3 \times 5 \times 17 \times p = (2^8 - 1)p \in \pi_e(G)$. Now by using Lemma 2.2, we get a contradiction, if $|N|$ is not a 2-group. If N is a 2-group, then by using [9] we get a contradiction. Hence $N = 1$ and so $G \cong L_9(2)$. Therefore $L_9(2)$ is characterizable by its spectrum. $\square$

4. OD-characterization of $L_9(2)$

In this section we prove Theorem 1.3 and it follows that the simple group $L_9(2)$ is uniquely determined by its order and its degree pattern. Therefore throughout this section let G be a finite group such that $|G| = |L_9(2)| = 2^{36} \times 3^5 \times 5^2 \times 7^3 \times 17 \times 31 \times 73 \times 127$ and $D(G) = D(L_9(2)) = (5, 6, 4, 4, 2, 2, 1, 2)$. We prove Theorem 1.3 through the following lemmas. We note that the proof of this theorem has similar steps as the proof of Theorem 1.1. We note that in graph theory a complete graph on n vertices is denoted by K_n. It is obvious that the degree of each vertex of K_n is $n - 1$.

Lemma 4.1. *If $|G| = |L_9(2)|$ and $D(G) = D(L_9(2))$, then the prime graph of G is connected.*

Proof. If p is a vertex of the prime graph, then $N(p)$ is defined as follows:

$$N(p) = \{p' \mid p' \in \pi(G),\ p \sim p' \text{ in } \Gamma(G)\}.$$

By assumption we know that $\deg(3) = 6$ and $\deg(2) = 5$. First let $2 \nsim 3$ in $\Gamma(G)$. Then each vertex of $\Gamma(G)$, except 2 and 3, is joined to vertex 3. Also 2 is joined to 5 vertices. Obviously $N(2) \cap N(3)$ is nonempty and so there is a path between 2 and 3, too. Therefore $\Gamma(G)$ is connected. If $2 \sim 3$, then $N(2) \cup N(3)$ has at least 7 elements, since $\deg(3) = 6$. But we know that there is no element of order 0, and so $\Gamma(G)$ is connected. $\square$

Lemma 4.2. *Let N be the maximal normal solvable subgroup of G. Then*

$$\pi(N) \cap \{5, 17, 31, 73, 127\} = \emptyset.$$

Especially G is nonsolvable.

Proof. We know that if $p \in \{17, 31, 73, 127\}$, then $p^2 \nmid |G|$. So if $p_1, p_2 \in \{17, 31, 73, 127\} \cap \pi(N)$, then let H be a $\{p_1, p_2\}$-Hall subgroup of N. Then H is an abelian subgroup of N, since $p_1 \nmid (p_2 - 1)$ and $p_2 \nmid (p_1 - 1)$. Hence $p_1 \sim p_2$ in $\Gamma(G)$. Let $A = \{17, 31, 73, 127\} \cap \pi(N)$.

If $|A| = 4$, then there exists a complete subgraph of order 4 in $\Gamma(G)$, which is a contradiction since the degree of these elements are at most 2. If $|A| = 3$, then similarly the induced subgraph on A is K_3. This implies that $\Gamma(G)$ is disconnected since for all $p \in A$, $\deg(p) \leq 2$, which is a contradiction.

If $A = \{p_1, p_2\}$, then $p_1 \sim p_2$ in $\Gamma(G)$. But we know that at least one of them is joined to 3, since $\deg(3) = 6$. Without loss of generality assume that $p_1 \sim 3$. Let $r \in \{17, 31, 73, 127\} \setminus A$. Let P be a Sylow p_1-subgroup of N. By Frattini argument it follows that $G = NN_G(P)$. Since $r \nmid |N|$, we conclude that $r \in \pi(N_G(P))$. Therefore if $g \in N_G(P)$ is an element of order r, then $P\langle g\rangle$ is a subgroup of order $p_1 r$. As we mentioned above this subgroup is abelian and so $p_1 \sim r$. Hence $\deg(p_1) \geq 3$, which is a contradiction since $\deg(p_1) \leq 2$.

If $A = \{p\}$, then let $\{p_1, p_2, p_3\} = \{17, 31, 73, 127\} \setminus A$. Now similar to the last case we conclude that $p_i \sim p$, for $1 \leq i \leq 3$, which is a contradiction.

Now we prove that $5 \notin \pi(N)$. Easily we can see that if $p \in \{17, 73, 127\}$, then any group of order $5p$ or $5p^2$ is abelian. As we proved $\{17, 73, 127\} \cap \pi(N) = \emptyset$. Again similar to the previous case let Q be a Sylow 5-subgroup of N. By using Frattini argument it follows that 5 is joined to every element in $\{17, 73, 127\}$. But we know that $\deg(3) = 6$ and so 3 is joined to at least two elements in $\{17, 73, 127\}$. We know that $\deg(73) = 1$ and hence $17, 127 \in N(3) \cap N(5)$. But this implies that $3, 5 \in N(17) \cap N(127)$ and since $\deg(73) = 1$ and $5 \sim 73$ we conclude that $2 \notin N(17) \cup N(73) \cup N(127)$, which is a contradiction since $\deg(2) = 5$. Therefore N is a $\{2, 3, 7\}$-subgroup. Especially it follows that G is nonsolvable. $\square$

Lemma 4.3. *Let N be the maximal normal solvable subgroup of G. Then*

$$L_9(2) \leq G/N \leq \mathrm{Aut}(L_9(2)).$$

Proof. We proved that $\{5, 17, 31, 73, 127\} \subseteq \pi(\overline{G}) = \pi(G/N)$. Let $S = \mathrm{Socle}(G/N)$ and easily we can conclude that $S \leq G/N \leq \mathrm{Aut}(S)$. Therefore $\{5, 17, 31, 73, 127\} \subseteq \pi(\mathrm{Aut}(S))$. We claim that 2 is not joined to every element of $\{5, 17, 31, 73, 127\}$. Otherwise since $\deg(2) = 5$, it follows that $2 \nsim 3$. Also $73 \nsim 3$, since $\deg(73) = 1$. Hence we get a contradiction since $\deg(3) = 6$. Therefore there exists at least one element of $\{5, 17, 31, 73, 127\}$, say p, such that $2 \nsim p$. We know that $S \cong P_1 \times \cdots \times P_k$, where each P_i is a nonabelian simple group. We claim that $k = 1$. Otherwise let $k \geq 2$. In this case $p \notin \pi(S)$ and $p \in \pi(\mathrm{Aut}(S))$, which implies that $p \in \pi(\mathrm{Out}(S))$. Let $\varphi \in \overline{G}$ be an automorphism of S of order p. Note that $2 \nsim p$. Now exactly similar to the proof of Lemma 3.3, we conclude that there exists i, where $1 \leq i \leq k$, and φ is an outer automorphism of P_i of order p. Note that $p \geq 5$. The rest of the proof of this lemma is similar to the proofs of Lemma 3.3. So for convenience we omit the rest of the proof of this lemma. Similar to the proof of Lemma 3.3 and by using the classification of finite simple groups, it follows that $k = 1$ and S is a nonabelian simple group such that $\{5, 17, 31, 73, 127\} \subseteq \pi(S)$. Hence $|S| = 2^a \times 3^b \times 5^2 \times 7^c \times 17 \times 31 \times 73 \times 127$. Now similar to the proof of Lemma 3.3, we conclude that $S = \mathrm{Socle}(G/N) \cong L_9(2)$. Therefore $L_9(2) \leq G/N \leq \mathrm{Aut}(L_9(2))$. □

Proof of Theorem 1.3. Using the above lemmas we conclude that $L_9(2) \leq G/N \leq \mathrm{Aut}(L_9(2))$. On the other hand we know that $|G| = |L_9(2)|$. Therefore $|N| = 1$ and $G \cong L_9(2)$, as required. □

Acknowledgements. The author express his gratitude to the referee for several valuable pointers and for very useful comments which improved the results in the manuscript, specially in Theorem 1.1 and Lemma 3.4. The author would like to thank the Institute for Studies in Theoretical Physics and Mathematics (IPM) for the financial support. This paper is dedicated to my family for their unending love and support.

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(Received August 22, 2008; revised June 5, 2009)