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On almost conservative matrix methods for double sequence spaces

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Abstract. In this paper, we use the idea of almost convergence of double sequences to define and characterize the almost $\mathcal{C}_\nu$-conservative matrices, that is, those 4-dimensional matrices which transform ν-convergent double sequences into the almost convergent double sequences; where ν stands for p-, bp-, and r-convergence.

1. Introduction and preliminaries

Here we give notions and notation for double sequence spaces. For other notations we refer the reader to [1].

A double sequence $x = (x_{jk})$ of real or complex numbers is said to be *bounded* if

$$\|x\|_\infty = \sup_{j,k} |x_{jk}| < \infty.$$

The space of all bounded double sequences is denoted by $\mathcal{M}_u$.

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A double sequence $x = (x_{jk})$ is said to *converge to the limit* L *in Pringsheim's sense* (shortly, *p-convergent to* L) [9] if for every $\varepsilon > 0$ there exists an integer N such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case L is called the p-limit of x. If in addition $x \in \mathcal{M}_u$, then x is said to be *boundedly convergent to* L *in Pringsheim's sense* (shortly, *bp-convergent to* L).

A double sequence $x = (x_{jk})$ is said to *converge regularly to* L (shortly, *r-convergent to* L) if $x \in \mathcal{C}_p$ and the limits $x_j := \lim_k x_{jk}$ $(j \in \mathbb{N})$ and $x^k := \lim_j x_{jk}$ $(k \in \mathbb{N})$ exist. Note that in this case the limits $\lim_j \lim_k x_{jk}$ and $\lim_k \lim_j x_{jk}$ exist and are equal to the p-limit of x.

In general, for any notion of convergence ν, the space of all ν-convergent double sequences will be denoted by $\mathcal{C}_\nu$, the space of all ν-convergent to 0 double sequences by $\mathcal{C}_{\nu 0}$ and the limit of a ν-convergent double sequence x by ν-$\lim_{j,k} x_{jk}$, where $\nu \in \{p, bp, r\}$.

Let Ω denote the vector space of all double sequences with the vector space operations defined coordinatewise. Vector subspaces of Ω are called *double sequence spaces*. In addition to above-mentioned double sequence spaces we consider the double sequence space

$$\mathcal{L}_u := \Big\{x \in \Omega \mid \|x\|_1 := \sum_{j,k} |x_{jk}| < \infty\Big\}$$

of all absolutely summable double sequences.

All considered double sequence spaces are supposed to contain

$$\Phi := \operatorname{span}\{\mathbf{e^{jk}} \mid j, k \in \mathbb{N}\},$$

where

$$\mathbf{e^{jk}_{il}} = \begin{cases} 1, & \text{if } (j,k) = (i,\ell), \\ 0, & \text{otherwise.} \end{cases}$$

We denote the pointwise sums $\sum_{j,k} \mathbf{e^{jk}}$, $\sum_j \mathbf{e^{jk}}$ $(k \in \mathbb{N})$, and $\sum_k \mathbf{e^{jk}}$ $(j \in \mathbb{N})$ by $\mathbf{e}$, $\mathbf{e^k}$ and $\mathbf{e_j}$ respectively.

Let E be the space of double sequences converging with respect to a convergence notion ν, F be a double sequence space, and $A = (a_{mnjk})$ be a 4-dimensional matrix of scalars. Define the set

$$F_A^{(\nu)} := \Big\{x \in \Omega \mid [Ax]_{mn} := \nu - \sum_{j,k} a_{mnjk} x_{jk} \text{ exists and } Ax := ([Ax]_{mn})_{m,n} \in F\Big\}.$$

Then we say that A maps the space E into the space F if $E \subset F_A^{(\nu)}$ and denote by (E, F) the set of all 4-dimensional matrices A which map E into F.

For more details on double sequences and 4-dimensional matrices, we refer to [5], [11], [12].

The idea of almost convergence for single sequences was introduced by Lorentz [4] and for double sequences by Móricz and Rhoades [6].

A double sequence $x = (x_{jk})$ of real numbers is said to be *almost convergent* to a limit L if

$$p\text{-}\lim_{p,q\to\infty} \sup_{m,n>0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} - L \right| = 0.$$

In this case L is called the f_2-limit of x and we shall denote by f_2 the space of all almost convergent double sequences.

Note that a convergent double sequence need not be almost convergent. However every bounded convergent double sequence is almost convergent and every almost convergent double sequence is bounded.

4-dimensional matrices mapping every almost convergent double sequence into a *bp*-convergent double sequence with the same limit were considered by Móricz and Rhoades in [6]. 4-dimensional matrices mapping every almost convergent double sequence into an almost convergent double sequence with the same limit were characterized by Mursaleen [8]).

In this paper, we characterize almost $\mathcal{C}_\nu$-conservative matrices, i.e. those 4-dimensional matrices $A = (a_{mnjk})$ which map the double sequence space $\mathcal{C}_\nu$ into the space f_2 where $\nu \in \{bp, r, p\}$.

To derive them we apply characterizations of 4-dimensional matrices from the class $(\mathcal{C}_\nu, \mathcal{C}_{bp})$ for $\nu \in \{bp, r, p\}$.

2. The class of matrices $(\mathcal{C}_\nu, \mathcal{C}_{bp})$

The conditions for a 4-dimensional matrix to map the spaces $\mathcal{C}_{bp}$, $\mathcal{C}_r$, $\mathcal{C}_p$ into the space $\mathcal{C}_{bp}$ are well known (see for example [2]).

Theorem 2.1. (a) *The matrix $A = (a_{mnjk})$ is in $(\mathcal{C}_r, \mathcal{C}_{bp})$ if and only if the following conditions hold:*

(i) $\sup_{m,n} \sum_{j,k} |a_{mnjk}| < \infty$,

(ii) *the limit* $bp\text{-}\lim_{m,n} a_{mnjk} = a_{jk}$ *exists* $(j, k \in \mathbb{N})$,

(iii) *the limit* $bp\text{-}\lim_{m,n} \sum_{j,k} a_{mnjk} = v$ *exists,*

(iv) *the limit* $bp\text{-}\lim_{m,n} \sum_j a_{mnjk_0} = u^{k_0}$ *exists* $(k_0 \in \mathbb{N})$,

(v) *the limit* $bp\text{-}\lim_{m,n}\sum_k a_{mnj_0k} = v_{j_0}$ *exists* $(j_0 \in \mathbb{N})$.

In this case, $a = (a_{jk}) \in \mathcal{L}_u$, $(u^k), (v_j) \in \ell$ *and*

$$bp\text{-}\lim_{m,n}[Ax]_{m,n} = \sum_{j,k} a_{jk}x_{jk} + \sum_j \left(v_j - \sum_k a_{jk}\right)x_j + \sum_k \left(u^k - \sum_j a_{jk}\right)x^k + \left(v + \sum_{j,k} a_{jk} - \sum_j v_j - \sum_k u^k\right) r\text{-}\lim_{m,n} x_{mn} \quad (x \in \mathcal{C}_r).$$

(b) *The matrix* $A = (a_{mnjk})$ *is in* $(\mathcal{C}_r, \mathcal{C}_{bp})$ *and* $bp\text{-}\lim Ax = r\text{-}\lim_{m,n} x_{mn}$ $(x \in \mathcal{C}_r)$ *if and only if the conditions* (i)–(v) *hold with* $a_{jk} = u^k = v_j = 0$ $(j,k \in \mathbb{N})$ *and* $v = 1$.

Theorem 2.2. (a) *The matrix* $A = (a_{mnjk})$ *is in* $(\mathcal{C}_{bp}, \mathcal{C}_{bp})$ *if and only if it satisfies the conditions* (i), (ii), *and* (iii) *of Theorem 2.1 and*

(vi) $bp\text{-}\lim_{m,n}\sum_j |a_{mnjk_0} - a_{jk_0}| = 0$ $(k_0 \in \mathbb{N})$,

(vii) $bp\text{-}\lim_{m,n}\sum_k |a_{mnj_0k} - a_{j_0k}| = 0$ $(j_0 \in \mathbb{N})$.

In this case, $a = (a_{jk}) \in \mathcal{L}_u$ *and*

$$bp\text{-}\lim_{m,n}[Ax]_{m,n} = \sum_{j,k} a_{jk}x_{jk} + \left(v - \sum_{j,k} a_{jk}\right) bp\text{-}\lim_{m,n} x_{mn} \qquad (x \in \mathcal{C}_{bp}).$$

(b) *The matrix* $A = (a_{mnjk})$ *is in* $(\mathcal{C}_{bp}, \mathcal{C}_{bp})$ *and* $bp\text{-}\lim Ax = bp\text{-}\lim_{m,n} x_{mn}$ $(x \in \mathcal{C}_{bp})$ *if and only if the conditions* (i), (ii), (iii) *of Theorem 2.1 and* (vi) *and* (vii) *hold with* $a_{jk} = 0$ $(j,k \in \mathbb{N})$ *and* $v = 1$.

Theorem 2.3. (a) *The matrix* $A = (a_{mnjk})$ *is in* $(\mathcal{C}_p, \mathcal{C}_{bp})$ *if and only if the conditions* (i)–(iii) *of Theorem 2.1 hold and*

(viii) *for every* $j \in \mathbb{N}$, *there exists* $K \in \mathbb{N}$ *such that* $a_{mnjk} = 0$ *for* $k > K$ $(m,n \in \mathbb{N})$,

(ix) *for every* $k \in \mathbb{N}$, *there exists* $J \in \mathbb{N}$ *such that* $a_{mnjk} = 0$ *for* $j > J$ $(m,n \in \mathbb{N})$,

In this case, $a = (a_{jk}) \in \mathcal{L}_u$, $(a_{jk_0})_j, (a_{j_0k})_k \in \varphi$ $(j_0, k_0 \in \mathbb{N})$ *and*

$$bp\text{-}\lim_{m,n}[Ax]_{m,n} = \sum_{j,k} a_{jk}x_{jk} + \sum_j \left(v - \sum_{j,k} a_{jk}\right) p\text{-}\lim_{m,n} x_{mn} \quad (x \in \mathcal{C}_p).$$

(b) *The matrix* $A = (a_{mnjk})$ *is in* $(\mathcal{C}_p, \mathcal{C}_{bp})$ *and* $bp\text{-}\lim Ax = p\text{-}\lim_{m,n} x_{mn}$ $(x \in \mathcal{C}_p)$ *if and only if the conditions* (ii), (iii) *of Theorem 2.1 and* (viii)–(xiii) *hold with* $a_{jk} = 0$ $(j,k \in \mathbb{N})$ *and* $v = 1$.

3. Almost $\mathcal{C}_\nu$-conservative matrices

In this section, we define and characterize almost $\mathcal{C}_\nu$-conservative and almost $\mathcal{C}_\nu$-regular matrices.

Definition 3.1. A four dimensional matrix $A = (a_{mnjk})$ is said to be *almost $\mathcal{C}_\nu$-conservative* if it transforms every ν-convergent double sequence $x = (x_{jk})$ into the almost convergent double sequence, where $\nu \in \{p, bp, r\}$; that is, $A \in (\mathcal{C}_\nu, f_2)$.

Definition 3.2. A four dimensional matrix $A = (a_{mnjk})$ is said to be *almost $\mathcal{C}_\nu$-regular* if it is almost $\mathcal{C}_\nu$-conservative and $f_2\text{-}\lim Ax = \nu\text{-}\lim x$ for every $x \in \mathcal{C}_\nu$.

Comparing this definition with the definition of almost regular 4-dimensional matrix by Mursaleen and Savaş ([7]) we see that the authors in fact considered almost $\mathcal{C}_{bp}$-regular matrices.

Almost conservative and almost regular matrices for single sequences were characterized by King [3].

Theorem 3.1. (a) *A matrix $A = (a_{mnjk})$ is almost $\mathcal{C}_{bp}$-conservative if and only if the following conditions hold:*

(i) $\sup_{m,n} \sum_{j,k} |a_{mnjk}| =: M < \infty$,

(ii) *the limit $bp\text{-}\lim_{p,q} \alpha(j,k,p,q,s,t) = a_{jk}$ exists $(j,k \in \mathbb{N})$ uniformly in $s,t \in \mathbb{N}$,*

(iii) *the limit $bp\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t) = u$ exists uniformly in $s,t \in \mathbb{N}$,*

(iv) *the limit $bp\text{-}\lim_{p,q} \sum_{k} |\alpha(j,k,p,q,s,t) - a_{jk}| = 0$ exists $(j \in \mathbb{N})$ uniformly in $s,t \in \mathbb{N}$,*

(v) *the limit $bp\text{-}\lim_{p,q} \sum_{j} |\alpha(j,k,p,q,s,t) - a_{jk}| = 0$ exists $(k \in \mathbb{N})$ uniformly in $s,t \in \mathbb{N}$,*

where

$$\alpha(j,k,p,q,s,t) = \frac{1}{pq} \sum_{m=s}^{s+p-1} \sum_{n=t}^{t+q-1} a_{mnjk}.$$

In this case, $a = (a_{jk}) \in \mathcal{L}_u$, and

$$f_2\text{-}\lim Ax = \sum_{j,k} a_{jk} x_{jk} + \Big(u - \sum_{j,k} a_{jk}\Big) bp\text{-}\lim_{i,l} x_{il}, \tag{1}$$

that is,

$$bp\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t) x_{jk} = \sum_{j,k} a_{jk} x_{jk} + \Big(u - \sum_{j,k} a_{jk}\Big) bp\text{-}\lim_{i,l} x_{il}$$

uniformly in $s, t \in \mathbb{N}$.

(b) $A = (a_{mnjk})$ *is almost* $\mathcal{C}_{bp}$*-regular if and only if the conditions* (i)–(v) *hold with* $a_{jk} = 0$ $(j, k \in \mathbb{N})$ *and* $u = 1$.

Proof. (a) *Necessity.* Let $A \in (\mathcal{C}_{bp}, f_2)$. The condition (i) follows, since $(\mathcal{C}_{bp}, f_2) \subset (\mathcal{C}_{bp}, \mathcal{M}_u)$ (see [2], §5, 5). Since $\mathbf{e^{jk}}$ and $\mathbf{e}$ are in $\mathcal{C}_{bp}$, the conditions (ii) and (iii) follow respectively.

It is obvious that if $A \in (\mathcal{C}_{bp}, f_2)$, then the matrix $B^{st} := (b^{st}_{pqjk})_{p,q,j,k} := (\alpha(j, k, p, q, s, t))_{p,q,j,k}$ is in $(\mathcal{C}_{bp}, \mathcal{C}_{bp})$ for every $s, t \in \mathbb{N}$. In particular, the double sequence $b^{st} = (b^{st}_{jk})$ with $b^{st}_{jk} := bp\text{-}\lim_{p,q} b^{st}_{pqjk} = a_{jk}$ is in $\mathcal{L}_u$ and

$$bp\text{-}\lim_{p,q} \sum_k |b^{st}_{pqjk} - b^{st}_{jk}| = bp\text{-}\lim_{p,q} \sum_k |\alpha(j, k, p, q, s, t) - a_{jk}| = 0$$

for every $s, t \in \mathbb{N}$.

To verify the conditions (iv) and (v), we need to prove that these limits are uniform in $s, t \in \mathbb{N}$. Suppose on contrary that for given $j_0 \in \mathbb{N}$

$$bp\text{-}\lim_{p,q} \sup_{s,t} \sum_k |\alpha(j_0, k, p, q, s, t) - a_{j_0 k}| \neq 0.$$

Then there exists $\varepsilon > 0$ and index sequences $(p_i), (q_i)$ such that

$$\sup_{s,t} \sum_k |\alpha(j_0, k, p_i, q_i, s, t) - a_{j_0 k}| \geq \varepsilon \quad (i \in \mathbb{N}).$$

So for every $i \in \mathbb{N}$, we can choose $s_i, t_i \in \mathbb{N}$ such that

$$\sum_k |\alpha(j_0, k, p_i, q_i, s_i, t_i) - a_{j_0 k}| \geq \varepsilon \quad (i \in \mathbb{N}).$$

Since

$$\sum_k |\alpha(j_0, k, p_i, q_i, s_i, t_i)| \leq \sup_{m,n} \sum_{j,k} |a_{mnjk}| < \infty,$$

$(a_{jk}) \in \mathcal{L}_u$ and by (ii) going to a subsequence of $(p_i,\ q_i,\ s_i,\ t_i)$ on need we may find an index sequence (k_i) such that

$$\sum_{k=1}^{k_i} |\alpha(j_0, k, p_i, q_i, s_i, t_i) - a_{j_0 k}| \leq \frac{\varepsilon}{8} \quad \text{and}$$

$$\sum_{k=k_{i+1}+1}^{\infty} |\alpha(j_0, k, p_i, q_i, s_i, t_i)| + \sum_{k=k_{i+1}+1}^{\infty} |a_{j_0 k}| \leq \frac{\varepsilon}{8} \quad (i \in \mathbb{N}).$$

So

$$\sum_{k=k_i+1}^{k_{i+1}} |\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k}| \geq \frac{3\varepsilon}{4} \quad (i \in \mathbb{N}).$$

We define the double sequence $x = (x_{jk})$ by

$$x_{jk} = \begin{cases} (-1)^i \mathrm{sgn}(\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k}) & \text{for } k_i < k \leq k_{i+1} \ (i \in \mathbb{N}),\ j = j_0 \\ 0 & \text{for } j \neq j_0. \end{cases}$$

Then $x \in \mathcal{C}_{bp0}$ with $\|x\|_\infty \leq 1$, but for i even we have

$$\begin{aligned}
&\frac{1}{p_iq_i}\sum_{m=s_i}^{s_i+p_i-1}\sum_{n=t_i}^{t_i+q_i-1}(Ax)_{mn} - \sum_{j,k} a_{jk}x_{jk} = \sum_k \alpha(j_0,k,p_i,q_i,s_i,t_i)x_{j_0k} \\
&\quad - \sum_k a_{j_0k}x_{j_0k} \geq \sum_{k=k_i+1}^{k_{i+1}} (\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k})x_{j_0k} \\
&\quad - \sum_{k=1}^{k_i} |\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k}| - \sum_{k=k_{i+1}+1}^{\infty} |\alpha(j_0,k,p_i,q_i,s_i,t_i)| \\
&\quad - \sum_{k=k_{i+1}+1}^{\infty} |a_{j_0k}| \geq \sum_{k=k_i+1}^{k_{i+1}} |\alpha(j_0,k,p_i,q_i,s_i,t_i) - a_{j_0k}| - \frac{\varepsilon}{8} - \frac{\varepsilon}{8} \geq \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned}$$

Analogously for i odd we get

$$\frac{1}{p_iq_i}\sum_{m=s_i}^{s_i+p_i-1}\sum_{n=t_i}^{t_i+q_i-1}(Ax)_{mn} - \sum_{j,k} a_{jk}x_{jk} \leq -\frac{\varepsilon}{2}.$$

Hence $\frac{1}{pq}\sum_{m=s}^{s+p-1}\sum_{n=t}^{t+q-1}(Ax)_{mn}$ does not converge as $p,q \to \infty$ uniformly in $s,t \in \mathbb{N}$, that is, $Ax \notin f_2$, giving the contradiction. Hence (iv) holds. In the same way we get that (v) holds.

Sufficiency. Let the conditions (i)–(v) hold. Then for any s, t the matrix $B^{st} := (\alpha(j,k,p,q,s,t))_{p,q,j,k}$ is in $(\mathcal{C}_{bp}, \mathcal{C}_{bp})$. In particular

$$bp\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t)x_{jk} = \sum_{j,k} a_{jk}x_{jk} + \left(u - \sum_{j,k} a_{jk}\right) bp\text{-}\lim_{i,l} x_{il} \quad (s,t \in \mathbb{N}).$$

To prove that the limit is uniform in $s,t \in \mathbb{N}$, we consider

$$\sum_{j,k}(\alpha(j,k,p,q,s,t) - a_{jk})(x_{jk} - bp\text{-}\lim_{i,l} x_{il}).$$

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$|x_{jk} - bp\text{-}\lim_{i,l} x_{il}| \leq \frac{\varepsilon}{8M} \quad \text{for } j,k \geq N.$$

By (ii), (iv) and (v) we can choose $P \in \mathbb{N}$ such that for $p,q \geq P$ and every $s,t \in \mathbb{N}$ we have

$$\sum_{j=1}^{N-1} \sum_{k=1}^{N-1} |\alpha(j,k,p,q,s,t) - a_{jk}| \leq \frac{\varepsilon}{8\|x\|_\infty}$$

$$\sum_{j=1}^{N-1} \sum_{k} |\alpha(j,k,p,q,s,t) - a_{jk}| \leq \frac{\varepsilon}{8\|x\|_\infty}$$

$$\sum_{k=1}^{N-1} \sum_{j=N}^{\infty} |\alpha(j,k,p,q,s,t) - a_{jk}| \leq \frac{\varepsilon}{8\|x\|_\infty}.$$

Then

$$\begin{aligned}
&\Big|\sum_{j,k} (\alpha(j,k,p,q,s,t) - a_{jk})(x_{jk} - bp\text{-}\lim_{i,l} x_{il})\Big| \\
&\qquad \leq 2 \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} |\alpha(j,k,p,q,s,t) - a_{jk}| \, \|x\|_\infty \\
&\qquad\quad + 2 \sum_{j=1}^{N-1} \sum_{k} |\alpha(j,k,p,q,s,t) - a_{jk}| \, \|x\|_\infty \\
&\qquad\quad + 2 \sum_{k=1}^{N-1} \sum_{j=N}^{\infty} |\alpha(j,k,p,q,s,t) - a_{jk}| \, \|x\|_\infty \\
&\qquad\quad + \sum_{j=N}^{\infty} \sum_{k=N}^{\infty} (|\alpha(j,k,p,q,s,t)| + |a_{jk}|) \big|x_{jk} - bp\text{-}\lim_{i,l} x_{il}\big| \\
&\qquad \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2M \frac{\varepsilon}{8M} = \varepsilon \qquad (s,t \in \mathbb{N}).
\end{aligned}$$

Hence

$$p\text{-}\lim_{p,q} \sum_{j,k} (\alpha(j,k,p,q,s,t) - a_{jk})(x_{jk} - bp\text{-}\lim_{i,l} x_{il}) = 0$$

uniformly in s,t, that is

$$f_2\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t) x_{jk} = \sum_{j,k} a_{jk} x_{jk} + \Big(u - \sum_{j,k} a_{jk}\Big) bp\text{-}\lim_{i,l} x_{il}.$$

(b) The sufficiency follows from (1) and the necessity follows from the inclusion $\{\mathbf{e^{jk}},\ \mathbf{e} |\ j,k \in \mathbb{N}\} \subset \mathcal{C}_{bp}$. □

Theorem 3.2. (a) *A matrix* $A = (a_{mnjk})$ *is almost* $\mathcal{C}_r$*-conservative if and only if the conditions* (i)–(iii) *of Theorem 3.1 hold and*

(iv) *the limit* $bp\text{-}\lim_{p,q} \sum_j \alpha(j, k_0, p, q, s, t) = u^{k_0}$ *exists uniformly in* s, t $(k_0 \in \mathbb{N})$,

(v) *the limit* $bp\text{-}\lim_{p,q} \sum_k \alpha(j_0, k, p, q, s, t) = v_{j_0}$ *exists uniformly in* s, t $(j_0 \in \mathbb{N})$.

In this case, $a = (a_{jk}) \in \mathcal{L}_u$, $(u^k), (v_j) \in \ell$ *and*

$$f_2\text{-}\lim Ax = \sum_{j,k} a_{jk} x_{jk} + \sum_j \Big(v_j - \sum_k a_{jk}\Big) x_j + \sum_k \Big(u^k - \sum_j a_{jk}\Big) x^k + \Big(u + \sum_{j,k} a_{jk} - \sum_j v_j - \sum_k u^k\Big) r\text{-}\lim x. \tag{2}$$

(b) $A = (a_{mnjk})$ *is almost* $\mathcal{C}_\nu$*-regular if and only if the conditions* (i)–(iii) *of Theorem 3.1 and* (iv), (v) *hold with* $u_{jk} = u^k = v_j = 0$ $(j, k \in \mathbb{N})$ *and* $u = 1$.

Proof. (a) *Necessity.* The condition (i) holds, since $(\mathcal{C}_r, f_2) \subset (\mathcal{C}_r, \mathcal{M}_u)$. The conditions (ii), (iii), (iv) and (v) follow, since $\mathbf{e^{jk}}$, $\mathbf{e}$, $\mathbf{e^k}$, $\mathbf{e_j} \in \mathcal{C}_r$ $(j, k \in \mathbb{N})$.

Sufficiency. Let the conditions (i)–(v) hold and suppose first that $x = (x_{jk}) \in \mathcal{C}_r$ satisfies $x_j = x^k = 0$ $(j, k \in \mathbb{N})$. Then also $r\text{-}\lim x = 0$.

By Theorem 2.1 the matrix $B^{st} := (\alpha(j, k, p, q, s, t))_{p,q,j,k}$ is in $(\mathcal{C}_r, \mathcal{C}_{bp})$ for any $s, t \in \mathbb{N}$. In particular

$$bp\text{-}\lim_{p,q} \sum_{j,k} \alpha(j, k, p, q, s, t) x_{jk} = \sum_{j,k} a_{jk} x_{jk} \quad (s, t \in \mathbb{N}).$$

To prove that the limit is uniform in $s, t \in \mathbb{N}$, we consider

$$\sum_{j,k} (\alpha(j, k, p, q, s, t) - a_{jk}) x_{jk}.$$

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$|x_{jk}| \le \frac{\varepsilon}{4M} \quad \text{for } j \ge N \text{ or } k \ge N \quad (j, k \in \mathbb{N}).$$

By (ii) we can choose $P \in \mathbb{N}$ such that for $p, q \ge P$ and any $s, t \in \mathbb{N}$ we have

$$\sum_{j=1}^{N-1} \sum_{k=1}^{N-1} |\alpha(j, k, p, q, s, t) - a_{jk}| \le \frac{\varepsilon}{2\|x\|_\infty}.$$

Then

$$\left|\sum_{j,k}(\alpha(j,k,p,q,s,t)-a_{jk})x_{jk}\right| \leq \sum_{j=1}^{N-1}\sum_{k=1}^{N-1}|\alpha(j,k,p,q,s,t)-a_{jk}|\,\|x\|_\infty$$
$$+\sum_{(j,k)\in\mathbb{N}^2\setminus[1,N-1]^2}(|\alpha(j,k,p,q,s,t)|+|a_{jk}|)|x_{jk}| \leq \frac{\varepsilon}{2}+2M\frac{\varepsilon}{4M}=\varepsilon \quad (s,t\in\mathbb{N}).$$

Hence

$$p\text{-}\lim_{p,q}\sum_{j,k}(\alpha(j,k,p,q,s,t)-a_{jk})x_{jk}=0$$

uniformly in s, t, that is

$$f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)x_{jk}=\sum_{j,k}a_{jk}x_{jk}.$$

Now let $x=(x_{jk})$ be any element of $\mathcal{C}_r$ with $\xi := r\text{-}\lim x$, then for the double sequence $z := (z_{jk})$ with $z_{jk} := x_{jk}-x_j-x^k+\xi$ we have $\lim_k z_{jk}=0$ $(j\in\mathbb{N})$ and $\lim_j z_{jk}=0$ $(k\in\mathbb{N})$. Hence

$$f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)(x_{jk}-x_j-x^k+\xi)=f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)z_{jk}$$
$$=\sum_{j,k}a_{jk}z_{jk}=\sum_{j,k}a_{jk}(x_{jk}-x_j-x^k+\xi).$$

The existence of the limit

$$f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)x_{jk}=\sum_{j,k}a_{jk}z_{jk}+f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)(x_j-\xi)$$
$$+f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)(x^k-\xi)+f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)\xi$$

then would follow if the limits on the right side exist.

The third limit

$$f_2\text{-}\lim_{p,q}\sum_{j,k}\alpha(j,k,p,q,s,t)\xi=\xi v$$

exists by (iii).

We will show that the first limit equals to $\sum_j v_j(x_j-\xi)$. For that end let $\varepsilon>0$ and $N\in\mathbb{N}$ such that

$$|x_j-\xi|\leq\frac{\varepsilon}{4M}\quad\text{for } j\geq N.$$

By (v) we can choose $P \in \mathbb{N}$ such that for $p, q \geq P$ and any $s, t \in \mathbb{N}$ we have

$$\sum_{j=1}^{N-1} \left| \sum_k \alpha(j,k,p,q,s,t) - v_j \right| \leq \frac{\varepsilon}{4\|x\|_\infty}.$$

Then

$$\left| \sum_j \Big(\sum_k \alpha(j,k,p,q,s,t) - v_j \Big)(x_j - \xi) \right| \leq 2 \sum_{j=1}^{N-1} \left| \sum_k \alpha(j,k,p,q,s,t) - v_j \right| \|x\|_\infty$$
$$+ \sum_{j=N}^{\infty} \left(\sum_k |\alpha(j,k,p,q,s,t)| + |v_j| \right) |x_{jk}| \leq \frac{\varepsilon}{2} + 2M \frac{\varepsilon}{4M} = \varepsilon \quad (s, t \in \mathbb{N}).$$

Hence

$$f_2\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t)(x_j - \xi) = \sum_j v_j (x_j - \xi).$$

Analogously

$$f_2\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t)(x^k - \xi) = \sum_k u^k (x^k - \xi).$$

Hence the limit

$$f_2\text{-}\lim_{p,q} \sum_{j,k} \alpha(j,k,p,q,s,t) x_{jk}$$

exists and the formula (2) holds.

(b) The sufficiency follows from (2) and the necessity follows from the inclusion $\{\mathbf{e^{jk}}, \mathbf{e}, \mathbf{e^k}, \mathbf{e_j} \mid j, k \in \mathbb{N}\} \subset \mathcal{C}_r$. $\square$

Theorem 3.3. (a) *A matrix $A = (a_{mnjk})$ is almost $\mathcal{C}_p$-conservative if and only if the conditions* (i)–(iii) *of Theorem 3.1 and* (viii), (ix) *of Theorm 2.3 hold.*

In this case, $a = (a_{jk}) \in \mathcal{L}_u$, $(a_{jk_0})_j, (a_{j_0k})_k \in \varphi$ $(j_0, k_0 \in \mathbb{N})$ and

$$f_2\text{-}\lim Ax = \sum_{j,k} a_{jk} x_{jk} + \Big(u - \sum_{j,k} a_{jk} \Big) p\text{-}\lim_{i,l} x_{il}, \tag{3}$$

(b) *$A = (a_{mnjk})$ is almost $\mathcal{C}_p$-regular if and only if the conditions* (i)–(iii) *of Theorem 3.1 and* (viii), (ix) *of Theorem 2.3 hold with $a_{jk} = 0$ $(j, k \in \mathbb{N})$ and $u = 1$.*

Proof. (a) *Necessity* of conditions (i)–(iii) follows in the same way as in Theorem 3.1. The conditions (viii), (ix) of Theorem 2.3 follow since $(\mathcal{C}_p, f_2) \subset (\mathcal{C}_p, \mathcal{M}_u)$ (see [2], §5, 6).

Sufficiency. First note that the condition (viii) of Theorem 2.3 implies that $\alpha(j_0, k, p, q, s, t) = 0$ for given $j_0 \in \mathbb{N}$, $k > K$ and any $p, q, s, t \in \mathbb{N}$. Hence also $a_{j_0 k} = 0$ for $k > K$. Now in view of (ii) the condition (iv) of Theorem 3.1 follows. Analogously the condition (v) of Theorem 3.1 as well as $(a_{jk_0})_j \in \varphi$ $(k_0 \in \mathbb{N})$ follows from the condition (ix) of Theorem 2.3. So in view of Theorem 3.1 A is almost $\mathcal{C}_{bp}$-conservative.

Now let $x \in \mathcal{C}_p$. Then there exists $N \in \mathbb{N}$ such that

$$\sup_{k,l>N} |x_{kl}| < \infty.$$

We consider x as a decomposition $x = y + z$ where y is an element of $\mathcal{C}_{bp}$ defined by $y_{kl} := x_{kl}$ for $k, l > N$ and $y_{kl} := 0$ for $k \le N$ or $l \le N$ and $z := x - y$. So $Ay \in f_2$ and

$$f_2\text{-}\lim Ay = \sum_{j,k>N} a_{jk} x_{jk} + \Big(u - \sum_{j,k} a_{jk}\Big) p\text{-}\lim_{i,l} x_{il}.$$

To prove that $Ax \in f_2$ we need to verify that $Az \in f_2$. For that end let $K \in \mathbb{N}$ be such that $a_{mnjk} = 0$ for $k > K$, $j = 1, \dots, N$ and any $m, n \in \mathbb{N}$. Let also $J \in \mathbb{N}$ be such that $a_{mnjk} = 0$ for $j > J$, $k = 1, \dots, N$ and any $m, n \in \mathbb{N}$. Then

$$Az = \sum_{j=1}^{N} \sum_{k=1}^{K} z_{jk} A\mathbf{e}^{\mathbf{jk}} + \sum_{k=1}^{N} \sum_{j=N+1}^{J} z_{jk} A\mathbf{e}^{\mathbf{jk}} \in f_2$$

and

$$\begin{aligned} f_2\text{-}\lim Az &= \sum_{j=1}^{N} \sum_{k=1}^{K} a_{jk} z_{jk} + \sum_{k=1}^{N} \sum_{j=N+1}^{J} a_{jk} z_{jk} \\ &= \sum_{j=1}^{N} \sum_{k} a_{jk} z_{jk} + \sum_{k=1}^{N} \sum_{j=N+1}^{\infty} a_{jk} z_{jk}. \end{aligned}$$

Hence $Ax = Ay + Az \in f_2$ and the formula (3) holds.

(b) can be proved in the same way as in Theorem 3.1. □

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