

## A note on extension theory and direct limits

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**Abstract.** The direct limit of a direct sequence consisting of normal spaces and closed inclusion mappings is a normal space. This result does not generalize to direct systems because there exist direct systems of compact Hausdorff spaces and closed inclusion mappings whose limit spaces fail to be normal. Introducing additional assumptions, K. Morita obtained normality of the limit space also for systems of normal spaces. Replacing normality of the spaces in the system by the property that a given  $K$  (in particular, a CW-complex  $K$ ) is an absolute extensor for these spaces, the analogue of Morita's theorem remains valid, i.e.,  $K$  is an absolute extensor for the limit space. This holds even under a weaker version of Morita's additional assumptions. In the case of direct sequences, these weaker conditions are always satisfied and therefore, the improved Morita theorem implies the result for direct sequences of normal spaces.

### 1. Introduction

Extension theory (see [2] or [8]) is based on the following idea. Let  $K$  be a CW-complex and  $X$  a space. Then we write  $X\tau K$  (or  $K \in AE(X)$ ) if  $K$  is an *absolute extensor* for  $X$ , i.e., for each closed subset  $A$  of  $X$  and map  $f : A \rightarrow K$ , there exists a map  $F : X \rightarrow K$  that extends  $f$ . Note that  $X\tau K$  makes sense for an arbitrary space  $K$ . It is this expanded notion that we use below.

In case  $K = [0, 1]$  (or  $K = \mathbb{R}$ ), by Tietze's theorem,  $X\tau K$  is equivalent to the statement that  $X$  is a normal space. In case  $K = S^n$ , for normal spaces  $X$ ,  $X\tau K$  is equivalent to the statement that the covering dimension  $\dim X \leq n$  [5],

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Theorem 3.2.10. In case  $K$  is an Eilenberg–MacLane complex  $K(G, n)$  for an Abelian group  $G$  and  $X$  is paracompact, then  $X\tau K$  is equivalent to the statement that the cohomological dimension  $\dim_G X \leq n$  [4], [7], [14], [9]. By definition, the extension dimension  $\text{extdim } X \leq K$ , provided  $X\tau K$  (see [1] and [2]).

It is known (see e.g., the proof of Proposition A.5.1. (iv) in [6]) that the direct limit of a direct sequence consisting of normal spaces and closed inclusion mappings is a normal space (see Proposition 1). It is natural to ask whether this result generalizes from direct sequences to direct systems. We show that the answer is negative because there exist direct systems of compact Hausdorff spaces and closed inclusion mappings whose limit spaces fail to be normal (Theorem 1).

In [12] K. MORITA considered direct systems  $\mathbf{X} = (X_a, \psi_a^b, A)$  of normal spaces and closed inclusion mappings, satisfying some additional conditions concerning the unions  $\bigcup_{d \in D} X_d \subseteq X$ , for subsets  $D \subseteq A$ . He proved that for such systems the limit space  $X$  is a normal space (see Proposition 3). In Theorem 3 we prove the analogue of Morita’s theorem for systems consisting of spaces, for which  $K$  is an absolute extensor and of closed inclusions. Moreover, we impose Morita’s additional conditions only on subsets  $D \subseteq A$ , whose cardinality  $\text{card}(D) < \text{card}(A)$ . In the case of sequences, i.e., when  $A = \mathbb{N}$ ,  $D$  is finite and the additional conditions are automatically satisfied. Therefore, Theorem 3 yields Proposition 1 as a special case. Theorem 4 is a slight improvement of Theorem 3 in which closed inclusions are replaced by closed embeddings.

## 2. Direct limits of normal spaces

By a direct system  $\mathbf{X} = (X_a, \psi_a^b, A)$  is meant a system of spaces  $X_a$ , indexed by a directed index set  $(A, \preceq)$ , and mappings  $\psi_a^b : X_a \rightarrow X_b$ , for  $a \preceq b$ , such that  $\psi_a^a = \text{id}$  and  $\psi_b^c \psi_a^b = \psi_a^c$ , for  $a \preceq b \preceq c$ . With  $\mathbf{X}$  is associated its direct limit  $X$ . It is the quotient space  $X = \tilde{X} / \sim$ , where  $\tilde{X}$  is the coproduct (direct sum)  $\tilde{X} = \bigsqcup_{a \in A} X_a$  and  $\sim$  is the equivalence relation, where  $x \in X_a$  and  $y \in X_b$  are considered  $\sim$ -equivalent provided there exists an element  $c \in A$ ,  $a, b \preceq c$ , such that  $\psi_a^c(x) = \psi_b^c(y)$ . If  $\psi : \tilde{X} \rightarrow X$  is the corresponding quotient mapping, then there are canonical mappings  $\psi_a : X_a \rightarrow X$ ,  $a \in A$ , defined by putting  $\psi_a = \psi \mid X_a$ . It is readily seen that  $\psi_b \psi_a^b = \psi_a$  for  $a \preceq b$ ,  $X = \bigcup_{a \in A} \psi_a(X_a)$  and a subset  $H$  of  $X$  is closed (open) in  $X$  if and only if  $(\psi_a)^{-1}(H) \cap X_a$  is closed (open) in  $X_a$  for every  $a \in A$ . For more details see e.g., Appendix II of [3].

Direct systems whose connecting mappings  $\psi_a^b : X_a \hookrightarrow X_b$  are inclusion mappings are special. In this case  $x, y \in \tilde{X}$  and  $x \sim y$  imply  $x = y$ , because

for  $x \in X_a$  and  $y \in X_b$ , there is a  $c \succeq a, b$  such that  $x = \psi_a^c(x) = \psi_b^c(y) = y$ . Consequently,  $X = \bigcup_{a \in A} X_a$  and the canonical mappings  $\psi_a : X_a \rightarrow X$  are inclusion mappings. Moreover, a subset  $H$  of  $X$  is closed (open) in  $X$  if and only if  $H \cap X_a$  is closed (open) in  $X_a$ , for every  $a \in A$ .

Direct sequences are direct systems where the indexing set  $A = \mathbb{N}$ . The usual notation is  $\mathbf{X} = (X_n, \psi_n^{n+1})$  because the connecting mappings  $\psi_n^m$ , where  $n < m$ , are the compositions  $\psi_{m-1}^m \circ \dots \circ \psi_n^{n+1}$ .

**Proposition 1.** *If  $\mathbf{X} = (X_n, \psi_n^{n+1})$  is a direct sequence of normal spaces  $X_n$ , each  $X_n$  is a closed subset of  $X_{n+1}$  and  $\psi_n^{n+1} : X_n \rightarrow X_{n+1}$  is the inclusion mapping, then the direct limit  $X = \text{dir lim } \mathbf{X}$  is also a normal space and each  $X_n$  is closed in  $X$ .*

Proposition 1 is obtained from Proposition 2 by putting  $K = I$ .

**Proposition 2.** *Let  $\mathbf{X} = (X_n, \psi_n^{n+1})$  be a direct sequence of spaces  $X_n$  such that each  $X_n$  is a closed subset of  $X_{n+1}$  and  $\psi_n^{n+1} : X_n \rightarrow X_{n+1}$  is the inclusion mapping. If  $K$  is a space and  $X_n \tau K$  for all  $n \in \mathbb{N}$ , then the direct limit  $X = \text{dir lim } \mathbf{X}$  has the property that  $X \tau K$  and each  $X_n$  is closed in  $X$ .*

By Remark 1 below, Proposition 2 follows from Theorem 3.

**Theorem 1.** *There exists a direct system of compact Hausdorff spaces (hence normal spaces) and inclusion mappings such that the limit space  $X$  is not normal.*

Indeed, with every topological space  $X$  one can associate the direct system  $\mathbf{C}$  which consists of all compact subspaces  $C_\lambda \subseteq X$ ,  $\lambda \in \Lambda$ , and the connecting mappings  $C_\lambda \rightarrow C_{\lambda'}$ ,  $\lambda \preceq \lambda'$ , are inclusions. We refer to  $\mathbf{C}$  as to the *inclusion system of compact subspaces of  $X$* . If  $X$  is Hausdorff, then each  $C_\lambda$  is closed in  $X$ .

**Lemma 1.** *Every Hausdorff locally compact space  $X$  is the direct limit of its inclusion system of compact subspaces  $\mathbf{C}$ .*

PROOF. Every Hausdorff locally compact space is a compactly generated space (see the definition of a  $k$ -space in 3.1.17 of [5]). To conclude that  $X = \text{dir lim } \mathbf{C}$ , one needs to verify that, whenever a subset  $B \subseteq X$  intersects every compact subset  $C_\lambda$  of  $X$  in a closed subset of  $C_\lambda$ , then  $B$  is closed in  $X$ . For this fact see Theorem 3.3.18 of [5]. □

**Theorem 2.** *If  $X$  is a Hausdorff locally compact space which fails to be normal, then its inclusion system of compact subspaces  $\mathbf{C}$  is a direct system of normal spaces and closed inclusion mappings such that  $X = \text{dir lim } \mathbf{C}$  is not normal.*

PROOF OF THEOREM 1. It suffices to notice that there exist Hausdorff locally compact spaces, which are not normal. Such a space is the Tychonoff plank, i.e., the space  $X = [0, \omega_1] \times [0, \omega] \setminus \{(\omega_1, \omega)\}$ . One may find a proof that  $X$  is not normal in 3.12.19.(a) of [5] or [15], p. 106.

### 3. General direct limit theorem

In view of Theorem 1, it is natural to ask for additional conditions under which the direct limit  $X$  of a direct system of normal spaces and closed inclusion mappings  $\mathbf{X}$  is a normal space. Such conditions were given by K. MORITA, who proved the following theorem (see Theorem 2 of [12]).

**Proposition 3** (K. Morita). *Let  $X$  be a space and let  $\mathcal{F}$  be a closed covering of  $X$  such that the topology of  $X$  is the weak topology induced by the collection  $\mathcal{F}$ , i.e.,  $U \subseteq X$  is open (closed) in  $X$  if and only if  $U \cap F$  is open (closed) in  $F$  for every space  $F \in \mathcal{F}$ . If each  $F \in \mathcal{F}$  is a normal space, then so is  $X$ , provided the following additional conditions are fulfilled.*

- (i) *For every subcollection  $\mathcal{G}$  of  $\mathcal{F}$ ,  $\cup \mathcal{G}$  is a closed subset of  $X$ .*
- (ii) *For every subcollection  $\mathcal{G}$  of  $\mathcal{F}$ , the relative topology of  $\cup \mathcal{G}$  is the weak topology induced by  $\mathcal{G}$ .*

Since the union of two closed normal subspaces is a normal space, there is no loss of generality in assuming that  $\mathcal{F}$  is closed with respect to finite unions. In that case  $\mathcal{F}$ , ordered by the inclusion  $\subseteq$ , becomes a directed set and one obtains a direct system  $\mathbf{X} = (F, \psi_F^{F'}, \mathcal{F})$ , where  $\psi_F^{F'}$  are inclusions  $F \hookrightarrow F'$ , for  $F \subseteq F'$ . Its direct limit is the space  $X$ .

The analogue of Proposition 3, where normality is replaced by paracompactness, was proved by E. MICHAEL in [11], Theorem 8.2 and by K. MORITA in [13], Theorem 1. It is useful in cohomological dimension theory.

**Proposition 4** (E. Michael). *Let  $X$  be a space and let  $\mathcal{F}$  be a closed covering of  $X$  such that the topology of  $X$  is the weak topology induced by the collection  $\mathcal{F}$ . If each  $F \in \mathcal{F}$  is a paracompact space, then so is  $X$ , provided conditions (i) and (ii) from Proposition 3 are fulfilled.*

We now state an improved version of Morita's theorem.

**Theorem 3.** *Let  $K$  be a space and let  $X$  be the direct limit of a direct system of spaces  $\mathbf{X} = (X_a, \psi_a^b, A)$ , where the index set  $(A, \preceq)$  is directed, each*

$X_a, a \in A$ , is a subset of  $X$  satisfying the condition  $X_a \tau K$  and the connecting mappings  $\psi_a^b : X_a \rightarrow X_b, a \preceq b$ , are inclusions. Moreover, for every subset  $D \subseteq A$  of cardinality  $\text{card}(D) < \text{card}(A)$ , let the following conditions hold.

- (i)  $X_D = \bigcup_{d \in D} X_d$  is a closed subset of  $X$ ,
- (ii) the topology of  $X_D$ , inherited from  $X$ , coincides with the weak topology induced by the family of sets  $\{X_d \mid d \in D\}$ .

Then  $X \tau K$ .

PROOF. If  $A$  is finite, it has an element  $d$  such that  $a \preceq d$ , for every  $a \in A$ . Therefore,  $X = X_d$  has the desired property that  $X \tau K$ . Now assume that  $A$  is infinite. For an arbitrary  $a \in A$ , put  $D = \{a\}$  and note that  $\text{card}(D) = 1 < \text{card}(A)$ . Therefore, by (i),  $X_a = X_D$  is closed in  $X$ .

By the well-ordering theorem, we may assume that  $A = [0, \gamma)$ , where  $\gamma$  is an initial ordinal and thus,  $\text{card}(\alpha) < \text{card}(\gamma)$ , for every ordinal  $\alpha < \gamma$ . We shall denote by  $\leq$  the ordering in  $[0, \gamma)$  as distinguished from  $\preceq$ , used for the ordering of  $A$ . Let  $H \subseteq X$  be closed and  $f : H \rightarrow K$  a map. For each  $a \in A$ , let  $H_a = H \cap X_a$  and  $\theta_a = f \mid H_a : H_a \rightarrow K$ . If  $a \preceq b$ , then  $H_a \subseteq H_b$  and  $\theta_a = \theta_b \mid H_a$ . We will construct, by transfinite induction, maps  $\Theta_a : X_a \rightarrow K, a \in A$ , such that  $\Theta_a \mid H_a = \theta_a$  and  $\Theta_a = \Theta_b \mid X_a$ , for  $a \preceq b$ . Clearly, these maps induce a unique mapping  $\Theta : X \rightarrow K$  such that  $\Theta \mid X_a = \Theta_a$ , for  $a \in A$ , and  $\Theta \mid H = f$ .

Since  $H_0$  is closed in  $X_0, \theta_0 : H_0 \rightarrow K$  is a map and  $X_0 \tau K$ , we may choose a map  $\Theta_0 : X_0 \rightarrow K$  such that  $\Theta_0 \mid H_0 = \theta_0$ . Now let us make the inductive assumption. Let  $0 < \delta < \gamma$  and suppose that for all  $0 \leq a < \delta$ , we have determined a map  $\Theta_a : X_a \rightarrow K$  such that:

- (iii)  $\Theta_a \mid H_a = \theta_a$ , and
- (iv) if  $0 \leq a < b < \delta$ , then  $\Theta_a \mid (X_a \cap X_b) = \Theta_b \mid (X_a \cap X_b)$ .

Put  $D = [0, \delta)$  and note that there is a unique function  $\Theta_D : X_D \rightarrow K$  such that  $\Theta_D \mid X_d = \Theta_d$ , for each  $d \in D$ . Clearly,  $\Theta_D \mid H_d = \Theta_d \mid H_d = \theta_d$ . Since each  $\Theta_d$  is continuous, the assumption (ii) implies that also  $\Theta_D$  is continuous.

We now define an extension  $\Theta_D^\delta : X_D \cup H_\delta \rightarrow K$  by putting  $\Theta_D^\delta \mid X_D = \Theta_D$  and  $\Theta_D^\delta \mid H_\delta = \theta_\delta$ . To see that this function is well defined and continuous, we need to verify that both summands  $X_D$  and  $H_\delta$  are closed in  $X$  and

$$\Theta_D \mid (X_D \cap H_\delta) = \theta_\delta \mid (X_D \cap H_\delta). \tag{1}$$

Indeed,  $X_D$  is closed in  $X$  because of (i) and  $H_\delta = X_\delta \cap H$  is the intersection of two sets closed in  $X$ . To verify (1), it suffices to show that

$$\Theta_d \mid (X_d \cap H_\delta) = \theta_\delta \mid (X_d \cap H_\delta), \quad (\forall d \in D). \tag{2}$$

First note that by the directedness of  $A$ , there is  $b \in A$  such that  $\delta, d \preceq b$ . By assumption,  $\theta_\delta = \theta_b \mid X_\delta$  and thus,  $\theta_\delta \mid (X_\delta \cap H_d) = \theta_b \mid (X_\delta \cap H_d)$ . Similarly,  $\theta_d = \theta_b \mid X_d$  and thus,  $\theta_d \mid (X_d \cap H_\delta) = \theta_b \mid (X_d \cap H_\delta)$ . Since  $X_\delta \cap H_d = X_\delta \cap X_d \cap H = X_d \cap H_\delta$ , we see that

$$\theta_\delta \mid (X_d \cap H_\delta) = \theta_d \mid (X_d \cap H_\delta). \tag{3}$$

Since  $d \in D$ ,

$$\Theta_d \mid (X_d \cap H_\delta) = \theta_d \mid (X_d \cap H_\delta) = \theta_\delta \mid (X_d \cap H_\delta), \tag{4}$$

as desired.

Since  $X_D \cup H_\delta$  is closed in  $X$  and  $X_\delta \tau K$ ,  $\Theta_D^\delta$  admits a continuous extension  $\Theta_\delta : X_D \cup X_\delta \rightarrow K$ . Clearly,  $\Theta_\delta \mid H_\delta = \Theta_D^\delta \mid H_\delta = \theta_\delta$ , which is condition (iii), for  $a = \delta$ . To verify also condition (iv), we need to show that  $\Theta_a \mid (X_a \cap X_\delta) = \Theta_\delta \mid (X_a \cap X_\delta)$ , for  $0 \leq a < \delta$ , i.e., for  $a \in D$ . Indeed,  $\Theta_\delta \mid (X_a \cap X_\delta) = \Theta_D^\delta \mid (X_a \cap X_\delta) = \Theta_D \mid (X_a \cap X_\delta)$ , because  $X_a \cap X_\delta \subseteq X_a \subseteq X_D$ . Since  $X_a \cap X_\delta \subseteq X_a$ , we see that  $\Theta_D \mid (X_a \cap X_\delta) = \Theta_a \mid (X_a \cap X_\delta)$ , as required.

By transfinite induction, we obtain mappings  $\Theta_a : X_a \rightarrow K$ , defined for all  $a \in [0, \gamma) = A$ , satisfying (iii) and (iv). To complete the proof of the theorem we must prove that, for elements  $a \preceq b$  from  $A$  we have  $\Theta_a = \Theta_b \mid X_a$ . First note that  $a \preceq b$  implies  $X_a \subseteq X_b$  and thus,  $X_a \cap X_b = X_a$ . By (iv), if  $0 \leq a < b < \gamma$ , then  $\Theta_a \mid (X_a \cap X_b) = \Theta_b \mid (X_a \cap X_b)$ . The same formula holds if  $0 \leq b < a < \gamma$ . Consequently, in all cases  $\Theta_a = \Theta_b \mid X_a$ .  $\square$

*Remark 1.* Proposition 2 is an immediate consequence of Theorem 3. Indeed, if  $\mathbf{X}$  is a direct sequence of normal spaces  $X_n$  and closed inclusion mappings, then every subset  $D \subseteq \mathbb{N}$  with  $\text{card}(D) < \text{card}(\mathbb{N}) = \aleph_0$  is a finite set. Therefore, it has a maximal element  $n_D$  and thus,  $X_D = X_{n_D}$ . It is now clear that  $X_D$  satisfies conditions (i) and (ii) from Theorem 3 and thus,  $X_n \tau K$  implies  $X \tau K$ .

The next theorem is a slight strengthening of Theorem 3 because we replace closed inclusions by closed embeddings.

**Theorem 4.** *Let  $K$  be a space and let  $X$  be the direct limit of a direct system of spaces  $\mathbf{X} = (X_a, \psi_a^b, A)$ , where the index set  $(A, \preceq)$  is directed, each  $X_a, a \in A$ , is a space satisfying the condition  $X_a \tau K$  and the connecting mappings  $\psi_a^b : X_a \rightarrow X_b, a \preceq b$ , are closed embeddings. Moreover, for every subset  $D \subseteq A$  of cardinality  $\text{card}(D) < \text{card}(A)$ , let the following conditions hold.*

- (i)  $\bigcup_{d \in D} \psi_d(X_d)$  is a closed subset of  $X$ ,
- (ii) the topology of  $\bigcup_{d \in D} \psi_d(X_d)$ , inherited from  $X$ , coincides with the weak topology induced by the family of sets  $\{\psi_d(X_d) \mid d \in D\}$ .

Then  $X \tau K$ .

In the proof we need the following lemma.

**Lemma 2.** *If in a direct system  $\mathbf{X}$  the connecting mappings  $\psi_a^b : X_a \rightarrow X_b$  are closed embeddings, then so are the mappings  $\psi_a : X_a \rightarrow X = \text{dir lim } \mathbf{X}$ . Consequently, the mappings  $\psi_a : X_a \rightarrow \psi_a(X_a)$  are homeomorphisms.*

PROOF. Let us first see that  $\psi_a : X_a \rightarrow X$  is an injection. If  $x, y \in X_a$  and  $\psi_a(x) = \psi_a(y)$ , then  $x \sim y$  and thus, there exists an index  $b$ ,  $a \preceq b$ , such that  $\psi_a^b(x) = \psi_a^b(y)$ . Since  $\psi_a^b$  is an injection, it follows that  $x = y$ . We still need to prove that  $\psi_a : X_a \rightarrow X$  is a closed mapping, i.e., if  $H \subseteq X_a$  is a closed set in  $X_a$ , then  $\psi_a(H)$  is a closed set in  $X$ . This is equivalent to showing that  $\psi_b^{-1}(\psi_a(H))$  is a closed set in  $X_b$ , for every  $b \in A$ . Choose  $c \in A$  so that  $a, b \preceq c$  and note that  $\psi_c \psi_b^c = \psi_b$  implies  $\psi_b^{-1}(\psi_a(H)) = (\psi_b^c)^{-1}(\psi_c^{-1}(\psi_a(H)))$ . Since  $\psi_a = \psi_c \psi_a^c$ , we see that  $\psi_b^{-1}(\psi_a(H)) = (\psi_b^c)^{-1} \psi_c^{-1}(\psi_c(\psi_a^c(H))) = (\psi_b^c)^{-1}(\psi_a^c(H))$ , because  $\psi_c : X_c \rightarrow X$  is an injection and thus,  $\psi_c^{-1}(\psi_c(Y)) = Y$ , for every subset  $Y \subseteq X_c$ . Since  $\psi_a^c : X_a \rightarrow X_c$  is a closed mapping and  $H$  is a closed subset of  $X_a$ , it follows that  $\psi_a^c(H)$  is a closed subset of  $X_c$  and thus,  $(\psi_b^c)^{-1}(\psi_a^c(H)) = \psi_b^{-1}(\psi_a(H))$  is a closed subset of  $X_b$ , as desired. The second assertion of the lemma is an obvious consequence of the first one.  $\square$

PROOF OF THEOREM 4. We will derive Theorem 4 from Theorem 3. For every  $a \in A$ , put  $X'_a = \psi_a(X_a)$ . By the second assertion of Lemma 2,  $\psi_a : X_a \rightarrow X'_a$  is a homeomorphism and therefore,  $X_a \tau K$  implies  $X'_a \tau K$ . Note that  $a \preceq b$  implies  $X'_a \subseteq X'_b$ . Indeed, if  $x' \in X'_a$ , there is an  $x \in X_a$  such that  $x' = \psi_a(x)$ . Since  $\psi_a = \psi_b \psi_a^b$ , it follows that  $x' = \psi_b \psi_a^b(x) \in \psi_b(X_b) = X'_b$ . Clearly, taking for  $\psi'^b_a : X'_a \rightarrow X'_b$  the inclusion mappings  $X'_a \hookrightarrow X'_b$ , we obtain a direct system  $\mathbf{X}' = (X'_a, \psi'^b_a, A)$ , whose connecting mappings are inclusions. To see that they are closed mappings, note that for  $D = \{a\}$ , one has  $\bigcup_{d \in D} \psi_d(X_d) = \psi_a(X_a) = X'_a$  and thus, by assumption (i) of Theorem 4,  $X'_a$  is a closed subset of  $X$  for each  $a \in A$ . It follows that the inclusion mapping  $\psi'^b_a : X'_a \rightarrow X'_b$  is closed. Indeed, if  $H$  is a closed subset of  $X'_a$ , then  $\psi'^b_a(H) = H \subseteq X'_b$  is also a closed subset of  $X'_b$ , because  $X'_a \subseteq X'_b$  is closed in  $X$ , hence it is also closed in  $X'_b$ .

By the construction of direct limits, the limit  $X'$  of the new system  $\mathbf{X}'$  equals  $X$  and the corresponding canonical mappings  $\psi'_a : X'_a \rightarrow X'$  are just inclusions of  $X'_a = \psi_a(X_a)$  to  $X' = X$ . Let us show that the direct limit topologies of  $X$  and  $X'$

coincide so that the systems  $\mathbf{X}$  and  $\mathbf{X}'$  have the same limit space. First assume that a set  $H \subseteq X = X'$  is closed in  $X$  and thus, the sets  $\psi_a^{-1}(H)$  are closed in  $X_a$ . It is readily seen that  $\psi_a^{-1}(H) = \psi_a^{-1}(X'_a \cap H)$ , and thus the sets  $\psi_a^{-1}(X'_a \cap H)$  are closed in  $X_a$ . By Lemma 2, the mapping  $\psi_a : X_a \rightarrow X$  is a closed mapping and thus,  $\psi_a(\psi_a^{-1}(X'_a \cap H))$  is closed in  $X$ . Note that  $\psi_a(\psi_a^{-1}(X'_a \cap H)) = X'_a \cap H$ , because for a mapping  $f : Y \rightarrow Z$  and a set  $S \subseteq Z$ , one has  $ff^{-1}(S) = S \cap f(Y)$ . It follows that  $X'_a \cap H$  is a closed subset of  $X$ . Since  $X'_a \cap H \subseteq X'_a$ , we conclude that the set  $X'_a \cap H$  is closed also in  $X'_a$ , for every  $a \in A$ . However, this is equivalent to saying that the set  $H$  is closed in  $X'$ . Conversely, if  $H$  is closed in  $X'$ , then the sets  $X'_a \cap H$  are closed in  $X'_a$ , hence also closed in  $X$ . By the continuity of  $\psi_a : X_a \rightarrow X$ , it follows that the sets  $\psi_a^{-1}(X'_a \cap H) = \psi_a^{-1}(H)$  are closed in  $X_a$ , which implies that  $H$  is closed in  $X$ .

By the assumptions (i) and (ii),  $\bigcup_{d \in D} X'_d$  is a closed subset of  $X = X'$  and the topology of  $\bigcup_{d \in D} X'_d$ , inherited from  $X = X'$ , coincides with the weak topology induced by the family of sets  $\{X'_d \mid d \in D\}$ . All this proves that  $\mathbf{X}'$  is a direct system, which consists of spaces  $X'_a \tau K$  and closed inclusion mappings,  $X$  is its direct limit and conditions (i) and (ii) from Theorem 3 are fulfilled. Consequently, that theorem yields the desired conclusion that  $X \tau K$ .  $\square$

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