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Generalizations of Mitrinović, Adamović and Lazarević's inequalities and their applications

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Abstract. In this paper, by introducing a parameter, we give new generalizations of Mitrinović–Adamović's inequality and Lazarević's inequality, and we apply these results to present new refinements of Cusa–Huygens' inequality and Wilker's inequality. These results improve many known results in the literature.

1. Introduction

Mitrinović and Adamović [11] proved that the inequality

$$\cos x < \left(\frac{\sin x}{x}\right)^3 \tag{1}$$

holds for all $x \in (0, \pi/2)$, and showed that the exponent 3 is the largest possible. A hyperbolic analogue of inequality (1) was presented by Lazarević, which is stated as follows:

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3, \tag{2}$$

where $x \neq 0$, and the exponent 3 is the least possible (see [8, p. 131]).

Inequalities (1) and (2) were recorded by Mitrinović and Vasić in their famous monograph [12], and from then they have evoked the interest of many mathematicians. Surveys on various generalizations and developments of these inequalities can be found in the monograph of Kuang [10]. Recently, several new

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proofs, variations, extensions and applications of this pair of inequalities have been discussed in the literature. The interested reader is referred to the papers [1], [3], [5], [6], [15], [19], [20], [21], [23] and to the references therein.

In this paper, we shall generalize the inequalities (1) and (2) by introducing a parameter. It is important to note that the results presented here contain the refinements of Mitrinović–Adamović's inequality and Lazarević's inequality as special cases. In Section 4 of this paper, we show that our results can be applied to the refinements of the Cusa–Huygens's inequality and the Wilker's inequality.

2. Main results

Our main results are the following theorems.

Theorem 1. *If $x \in (0, \pi/2)$, then the inequality*

$$1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cos x < \left(\frac{\sin x}{x}\right)^{\lambda} \tag{3}$$

holds if and only if $\lambda < 0$ or $\lambda \geq \lambda_0$, where $\lambda_0 \simeq 1.420330769$ is the root of the equation $\lambda/3 + (2/\pi)^{\lambda} - 1 = 0$. Moreover, the inequality (3) is reversed if and only if $0 < \lambda \leq 7/5$.

Theorem 2. *If $x \neq 0$, then the inequality*

$$1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cosh x < \left(\frac{\sinh x}{x}\right)^{\lambda} \tag{4}$$

holds if and only if $\lambda < 0$ or $\lambda \geq 7/5$. Moreover, the inequality (4) is reversed if and only if $0 < \lambda \leq 1$.

3. Proof of the main results

Proof of Theorem 1. Define a function $f : (0, \pi/2) \to \mathbb{R}$ by

$$f(x) = \ln\left(1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cos x\right) - \ln\left(\frac{\sin x}{x}\right)^{\lambda},$$

where λ is a parameter and $\lambda < 3$. Differentiating $f(x)$ with respect to x gives

$$\begin{aligned} f'(x) &= \frac{-\lambda \sin x}{3-\lambda+\lambda\cos x} - \frac{\lambda\cos x}{\sin x} + \frac{\lambda}{x} \\ &= \frac{\lambda\left(-x\sin^2 x + (\lambda-3)(x\cos x - \sin x) - \lambda x\cos^2 x + \lambda \sin x\cos x\right)}{(3-\lambda+\lambda\cos x)x\sin x} \\ &\triangleq \frac{\lambda g_1(x)}{(3-\lambda+\lambda\cos x)x\sin x}. \end{aligned}$$

Computing the derivative of $g_1(x)$ gives

$$\begin{aligned} g_1'(x) &= (\sin x)(-2x\cos x - \sin x + 3x - \lambda x + 2\lambda x\cos x - \lambda\sin x) \\ &\triangleq (\sin x)g_2(x). \end{aligned}$$

Similarly, we have

$$\begin{aligned} g_2'(x) &= 2x\sin x - 3\cos x + 3 - \lambda - 2\lambda x\sin x + \lambda\cos x \\ &= (\sin x)\left((3-\lambda)\tan\frac{x}{2} + (2-2\lambda)x\right) \triangleq (\sin x)g_3(x), \\ g_3'(x) &= \frac{3-\lambda}{2}\left(\sec^2\frac{x}{2} - \frac{4\lambda-4}{3-\lambda}\right). \end{aligned}$$

To prove the required results, we consider the following four cases.

Case I. If $0 < \lambda \le 7/5$, then clearly $(4\lambda-4)/(3-\lambda) \le 1$. We have

$$g_3'(x) = \frac{3-\lambda}{2}\left(\sec^2\frac{x}{2} - \frac{4\lambda-4}{3-\lambda}\right) > 0$$

for all $x \in (0,\pi/2)$. On the other hand $g_3(0) = 0$, and thus we conclude that the function g_3 is increasing and positive on $(0,\pi/2)$. Similarly, by using the functional relationships stated above, we deduce that the functions g_2 and g_1 are also increasing and positive on $(0,\pi/2)$. Hence, we have $f'(x) > 0$ for all $x \in (0,\pi/2)$, we thus infer that f is increasing on $(0,\pi/2)$.

Now, from

$$\lim_{x\to 0^+} f(x) = 0,$$

we deduce that

$$f(x) = \ln\left(1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cos x\right) - \ln\left(\frac{\sin x}{x}\right)^{\lambda} > 0$$

for all $x \in (0,\pi/2)$, which leads to the reverse inequality of (3).

Case II. When $\lambda < 0$. It is easy to see that in this case we also have that $g_3'(x) > 0$ for all $x \in (0, \pi/2)$. In the same way as in previous case, we can deduce that the functions g_3, g_2, g_1 are increasing and positive on $(0, \pi/2)$. Note that $\lambda < 0$, we thus have $f'(x) < 0$ for all $x \in (0, \pi/2)$, which implies that f is decreasing on $(0, \pi/2)$. Consequently, we have $f(x) < f(0^+) = 0$ for all $x \in (0, \pi/2)$, i.e.

$$\ln\left(1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cos x\right) - \ln\left(\frac{\sin x}{x}\right)^{\lambda} < 0,$$

which evidently implies the desired inequality (3).

Case III. When $\lambda \geq \lambda_0 \simeq 1.420330769$.

We first consider the case of $\lambda = \lambda_0$. Since the function

$$x \mapsto g_3'(x) = \frac{3 - \lambda_0}{2}\left(\sec^2\frac{x}{2} - \frac{4\lambda_0 - 4}{3 - \lambda_0}\right)$$

is increasing on $(0, \pi/2)$, and $g_3'(0) < 0$, $g_3'(\pi/2-) = +\infty$, we conclude that there exists $x_1 \in (0, \pi/2)$ such that $g_3'(x) < 0$ for all $x \in (0, x_1)$, $g_3'(x_1) = 0$ and $g_3'(x) > 0$ for all $x \in (x_1, \pi/2)$. Thus, the function g_3 is decreasing from $g_3(0) = 0$ to $g_3(x_1) < 0$ and then increasing to $g_3(\pi/2) > 0$. Hence, there is $x_2 \in (x_1, \pi/2)$ such that $g_3(x) < 0$ for all $x \in (0, x_2)$, $g_3(x_2) = 0$ and $g_3(x) > 0$ for all $x \in (x_2, \pi/2)$.

Now, from $g_2'(x) = (\sin x)g_3(x)$, we conclude that the function g_2 is decreasing from $g_2(0) = 0$ to $g_3(x_2) < 0$ and then increasing to $g_2(\pi/2) > 0$. Hence, there is $x_3 \in (x_2, \pi/2)$ such that $g_2(x) < 0$ for all $x \in (0, x_3)$, $g_2(x_3) = 0$ and $g_2(x) > 0$ for $x \in (x_3, \pi/2)$. Further, from $g_1'(x) = (\sin x)g_2(x)$, we deduce that the function g_1 is decreasing from $g_1(0) = 0$ to $g_1(x_3) < 0$ and then increasing to $g_1(\pi/2) > 0$. Hence, there is $x_4 \in (x_3, \pi/2)$ such that $g_1(x) < 0$ for all $x \in (0, x_4)$, $g_1(x_4) = 0$ and $g_1(x) > 0$ for all $x \in (x_4, \pi/2)$.

Finally, from the relation

$$f'(x) = \frac{\lambda_0 g_1(x)}{(3 - \lambda_0 + \lambda_0\cos x)x\sin x},$$

we conclude that the function f is decreasing from $f(0^+) = 0$ to $f(x_4) < 0$ and then increasing to $f(\pi/2) = 0$. Therefore, for all $x \in (0, \pi/2)$ we have

$$f(x) = \ln\left(1 - \frac{\lambda_0}{3} + \frac{\lambda_0}{3}\cos x\right) - \ln\left(\frac{\sin x}{x}\right)^{\lambda_0} < 0,$$

that is,

$$1-\frac{\lambda_0}{3}+\frac{\lambda_0}{3}\cos x<\left(\frac{\sin x}{x}\right)^{\lambda_0}. \tag{5}$$

Next, we prove the validity of inequality (3) for $\lambda>\lambda_0$. We shall use the well-known Bernoulli inequality [12, p. 34], which states that if $-1<u\neq 0$ and $0<\alpha<1$, then

$$(1+u)^\alpha>1+\alpha u \quad \text{for } \alpha>1 \text{ or } \alpha<0; \tag{6}$$

$$(1+u)^\alpha<1+\alpha u \quad \text{for } 0<\alpha<1. \tag{7}$$

It is easy to verify that $(\lambda_0/3)\cos x-\lambda_0/3>-1$ and $\lambda/\lambda_0>1$. By using the Bernoulli inequality (6) and the inequality (5), we have

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^\lambda &= \left(\left(\frac{\sin x}{x}\right)^{\lambda_0}\right)^{\lambda/\lambda_0} > \left(1+\left(\frac{\lambda_0}{3}\cos x-\frac{\lambda_0}{3}\right)\right)^{\lambda/\lambda_0}\\ &\geq 1+\frac{\lambda}{\lambda_0}\left(\frac{\lambda_0}{3}\cos x-\frac{\lambda_0}{3}\right)=1-\frac{\lambda}{3}+\frac{\lambda}{3}\cos x.\end{aligned}$$

This proves the desired inequality (3).

Case IV. When $7/5<\lambda<\lambda_0\simeq 1.420330769$. Via the same discussions as in the previous case, we can find that the function

$$f(x)=\ln\left(1-\frac{\lambda}{3}+\frac{\lambda}{3}\cos x\right)-\ln\left(\frac{\sin x}{x}\right)^\lambda$$

is decreasing from $f(0^+)=0$ to $f(\eta)<0$ $(0<\eta<\pi/2)$ and then increasing to $f(\pi/2)>0$, where $f(\pi/2)=\ln(1-\lambda/3)-\lambda\ln(2/\pi)>\ln(1-\lambda_0/3)-\lambda_0\ln(2/\pi)=0$.

It means that the inequality (3) as well as its reverse version are not true in general under the assumption that $7/5<\lambda<\lambda_0$. This proves the validity of the assertion that the constant λ_0 is the minimum positive value of λ for which the inequality (3) holds, and the constant $7/5$ is the maximum value of λ for which the reverse inequality of (3) holds.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Define a function $\phi:(0,\infty)\to\mathbb{R}$ by

$$\phi(x)=\ln\left(1-\frac{\lambda}{3}+\frac{\lambda}{3}\cosh x\right)-\ln\left(\frac{\sinh x}{x}\right)^\lambda,$$

where λ is a parameter and $\lambda > 0$. Differentiating $\phi(x)$ with respect to x gives

$$\begin{aligned}\phi'(x) &= \frac{\lambda\left(x\sinh^2 x + (\lambda-3)(x\cosh x - \sinh x) - \lambda x\cosh^2 x + \lambda\sinh x\cosh x\right)}{(3-\lambda+\lambda\cosh x)x\sinh x}\\ &\triangleq \frac{\lambda\varphi_1(x)}{(3-\lambda+\lambda\cosh x)x\sinh x}.\end{aligned}$$

Further, we have

$$\begin{aligned}\varphi_1'(x) &= (\sinh x)(2x\cosh x + \sinh x - 3x + \lambda x - 2\lambda x\cosh x + \lambda\sinh x)\\ &\triangleq (\sinh x)\varphi_2(x),\\ \varphi_2'(x) &= (\sinh x)\left((3-\lambda)\tanh\frac{x}{2} + (2-2\lambda)x\right)\\ &\triangleq (\sinh x)\varphi_3(x),\\ \varphi_3'(x) &= \frac{1}{2\cosh^2\frac{x}{2}}\left(3-\lambda-(4\lambda-4)\cosh^2\frac{x}{2}\right).\end{aligned}$$

Observe that for all $x \neq 0$ we have

$$1-\frac{\lambda}{3}+\frac{\lambda}{3}\cosh(-x)-\left[\frac{\sinh(-x)}{(-x)}\right]^{\lambda} = 1-\frac{\lambda}{3}+\frac{\lambda}{3}\cosh x-\left(\frac{\sinh x}{x}\right)^{\lambda},$$

thus, in order to prove Theorem 2, it is enough to prove that the inequalities asserted by Theorem 2 hold for all $x > 0$.

As in the proof of Theorem 1 we consider four cases.

Case I. If $\lambda \geq 7/5$, then $(3-\lambda)/(4\lambda-4) \leq 1$. In this case for all $x > 0$ we have

$$\varphi_3'(x) = \frac{2\lambda-2}{\cosh^2\frac{x}{2}}\left(\frac{3-\lambda}{4\lambda-4}-\cosh^2\frac{x}{2}\right) < 0.$$

On the other hand $\varphi_3(0) = 0$, and thus we conclude that the function φ_3 is decreasing and negative on $(0,+\infty)$. Similarly, by using the functional relationships stated above, we deduce that the functions φ_2 and φ_1 are also decreasing and negative on $(0,+\infty)$. Hence, we infer that ϕ is decreasing on $(0,+\infty)$. Taking into account that $\phi(0^+) = 0$ we obtain that for all $x > 0$

$$\phi(x) = \ln\left(1-\frac{\lambda}{3}+\frac{\lambda}{3}\cosh x\right) - \ln\left(\frac{\sinh x}{x}\right)^{\lambda} < 0,$$

which leads to the required inequality (4).

Case II. When $0 < \lambda \leq 1$. It is easy to see that for all $x > 0$

$$\varphi_3'(x) = \frac{1}{2\cosh^2\frac{x}{2}}\left(3 - \lambda + (4 - 4\lambda)\cosh^2\frac{x}{2}\right) > 0.$$

Consequently, as in the previous case we have that the functions $\varphi_1, \varphi_2, \varphi_3$ are increasing and positive on $(0, +\infty)$. We thus have $\phi'(x) > 0$ for all $x > 0$, which implies that ϕ is increasing on $(0, +\infty)$. Consequently, we have $\phi(x) > \phi(0^+) = 0$ for all $x \in (0, +\infty)$, which implies the reverse inequality of (4), that is,

$$1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cosh x > \left(\frac{\sinh x}{x}\right)^{\lambda}.$$

Case III. When $\lambda < 0$. In this case $-\lambda > 0$, and by appealing to the result proved in the previous case, we obtain that for all $x > 0$

$$\left(\frac{\sinh x}{x}\right)^{-\lambda} < 1 - \frac{(-\lambda)}{3} + \frac{(-\lambda)}{3}\cosh x,$$

that is,

$$\left(\frac{\sinh x}{x}\right)^{\lambda} > \frac{1}{1 + \lambda/3 - (\lambda/3)\cosh x}.$$

This in turn implies that

$$\left(\frac{\sinh x}{x}\right)^{\lambda} > \frac{1}{1 + \lambda/3 - (\lambda/3)\cosh x}$$
$$> \frac{1 - (\lambda/3 - (\lambda/3)\cosh x)^2}{1 + \lambda/3 - (\lambda/3)\cosh x} = 1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cosh x,$$

which is exactly the required inequality (4).

Case IV. When $1 < \lambda < 7/5$. Note that the function

$$x \mapsto \varphi_3'(x) = \frac{3 - \lambda}{2}\left(\frac{1}{\cosh^2\frac{x}{2}} - \frac{4\lambda - 4}{3 - \lambda}\right)$$

is decreasing on $(0, +\infty)$, and $\varphi_3'(0) > 0$, $\varphi'(+\infty) < 0$. Consequently, there exists $x_1 \in (0, +\infty)$ such that $\varphi_3'(x) > 0$ for all $x \in (0, x_1)$, $\varphi_3'(x_1) = 0$ and $\varphi_3'(x) < 0$ for all $x > x_1$. Thus, the function φ_3 is increasing from $\varphi_3(0) = 0$ to $\varphi_3(x_1) > 0$ and then decreasing to $\varphi_3(+\infty) = -\infty$.

Similarly, by investigating the monotonicity of functions φ_2, φ_1, ϕ, we deduce that there exist x_2, x_3, x_4 such that the function φ_2 is increasing from $\varphi_2(0) = 0$ to $\varphi_3(x_2) > 0$ and then decreasing to $\varphi_2(+\infty) = -\infty$; φ_1 is increasing from $\varphi_1(0) = 0$ to $\varphi_1(x_3) > 0$ and then decreasing to $\varphi_1(+\infty) = -\infty$; ϕ is increasing from $\phi(0) = 0$ to $\phi(x_4) > 0$ and then decreasing to $\phi(+\infty) = -\infty$. These in turn imply that the inequality (4) as well as its reverse are not true in general under the assumption that $1 < \lambda < 7/5$. This proves the validity of the assertion that the constant $7/5$ is the minimum positive value of λ for which the inequality (4) holds, and the constant 1 is the maximum value of λ for which the reverse inequality of (4) holds. $\square$

4. Some applications

In this section, we show some applications of Theorems 1 and 2. By using Theorems 1 and 2, and the Bernoulli inequality (6), we get immediately the following refinements of Mitrinović–Adamović's inequality and Lazarević's inequality.

Corollary 1. *Let $x \in (0, \pi/2)$ and $\lambda \in [\lambda_0, 3]$, where $\lambda_0 \simeq 1.420330769$ is the root of the equation $\lambda/3 + \big(2/\pi\big)^{\lambda} - 1 = 0$. Then the following inequality holds*

$$\cos x \leq \left(1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cos x\right)^{3/\lambda} < \left(\frac{\sin x}{x}\right)^{3}.$$

Corollary 2. *Let $x \neq 0$ and $\lambda \in [7/5, 3]$. Then the following inequality holds*

$$\cosh x \leq \left(1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cosh x\right)^{3/\lambda} < \left(\frac{\sinh x}{x}\right)^{3}.$$

The inequality

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}, \tag{8}$$

which holds for all $x \in (0, \pi/2)$, is known in the literature as Cusa–Huygens' inequality (see [17]). As a consequence of Theorem 1, by using the Bernoulli inequality (7), we obtain that the following refinement of Cusa–Huygens' inequality.

Corollary 3. *If $x \in (0, \pi/2)$ and $\lambda \in [1, 7/5]$, then we have*

$$\frac{\sin x}{x} < \left(1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cos x\right)^{1/\lambda} \leq \frac{2 + \cos x}{3}.$$

In [14], Qi et al. presented a reversed version of Cusa–Huygens' inequality, as follows:

$$\frac{\sin x}{x} > \frac{1+\cos x}{2}, \tag{9}$$

where $x \in (0, \pi/2)$. The following result, which follows easily from Theorem 1 and Bernoulli's inequality (6), is a refinement of (9).

Corollary 4. *Let $x \in (0, \pi/2)$ and $\lambda \in [\lambda_0, 2]$, where $\lambda_0 \simeq 1.420330769$ is the root of the equation $\lambda/3 + \left(2/\pi\right)^{\lambda} - 1 = 0$. Then the following inequality holds*

$$\frac{\sin x}{x} > \left(1 - \frac{\lambda}{3} + \frac{\lambda}{3}\cos x\right)^{1/\lambda} > \frac{1+\cos x}{2}.$$

The inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2,$$

which holds for all $x \in (0, \pi/2)$, is known in the literature as Wilker's inequality (see [16], [18]). Recently, Zhu [23] gave a hyperbolic analogue of the Wilker inequality, as follows:

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2,$$

where $x \neq 0$. The following result is a generalized and refined version of the above Wilker-type inequality.

Corollary 5. *Let $x \neq 0$ and $p \geq 1$. Then the following inequality holds*

$$\left(\frac{\sinh x}{x}\right)^{2p} + \left(\frac{\tanh x}{x}\right)^{p} > \left(\frac{x}{\sinh x}\right)^{2p} + \left(\frac{x}{\tanh x}\right)^{p} > 2.$$

Proof. From the Lazarević inequality (2), we observe that

$$\left(\frac{x}{\sinh x}\right)^2 \frac{x}{\tanh x} = \cosh x \left(\frac{x}{\sinh x}\right)^3 < 1,$$

so that

$$\begin{aligned}\left(\frac{\sinh x}{x}\right)^{2p} + \left(\frac{\tanh x}{x}\right)^{p} &> \left[\left(\frac{\sinh x}{x}\right)^{2p} + \left(\frac{\tanh x}{x}\right)^{p}\right]\left[\left(\frac{x}{\sinh x}\right)^2 \frac{x}{\tanh x}\right]^p \\ &= \left(\frac{x}{\sinh x}\right)^{2p} + \left(\frac{x}{\tanh x}\right)^{p}.\end{aligned}$$

On the other hand, putting $\lambda = 1$ in Theorem 2 gives

$$\cosh x > 3\left(\frac{\sinh x}{x}\right) - 2.$$

By using the power means inequality and the above inequalities, we have

$$\begin{aligned}\left(\frac{x}{\sinh x}\right)^{2p} + \left(\frac{x}{\tanh x}\right)^{p} &\geq 2^{1-p}\left[\left(\frac{x}{\sinh x}\right)^{2} + \frac{x}{\tanh x}\right]^{p} \\ &= 2^{1-p}\left[\left(\frac{x}{\sinh x}\right)^{2} + \left(\frac{x}{\sinh x}\right)\cosh x\right]^{p} \\ &> 2^{1-p}\left[\left(\frac{x}{\sinh x}\right)^{2} + \left(\frac{x}{\sinh x}\right)\left(3\left(\frac{\sinh x}{x}\right) - 2\right)\right]^{p} \\ &= 2^{1-p}\left[2 + \left(1 - \frac{x}{\sinh x}\right)^{2}\right]^{p} > 2,\end{aligned}$$

which completes the proof. □

In the same way, as above, one can prove the following generalized and refined version of the Wilker inequality.

Corollary 6. *If $x \in (0, \pi/2)$ and $p \geq 1$, then*

$$\left(\frac{\sin x}{x}\right)^{2p} + \left(\frac{\tan x}{x}\right)^{p} > \left(\frac{x}{\sin x}\right)^{2p} + \left(\frac{x}{\tan x}\right)^{p} > 2.$$

We note that new researches, which are concerned with the inequalities (8), (9), Wilker's inequality and its hyperbolic analogue, are in active progress. Readers can refer to the papers [2]–[4], [7], [22] and to the references therein. In [2], the second author extended the inequalities (8) and (9) to generalized Bessel functions of the first kind, while in [4], Baricz and Sándor extended the results from [20] to Bessel functions of the first kind. The paper [22] contains the full solution of Oppenheim, and generalizes the inequalities (8) and (9), while the paper [7] contains the extensions of the results from [22] to Bessel and modified Bessel functions of the first kind. Moreover, during the course of writing this paper we have found the paper of Klén et al. [9], where another improvements of the inequalities (8) and (9) were presented.

Finally, it is worth to mention that the right-hand side of the above inequality in Corollary 6 is actually a general form of [20, Lemma 3]. Moreover, we note that the inequality

$$\left(\frac{\sin x}{x}\right)^{2p} + \left(\frac{\tan x}{x}\right)^{p} > 2,$$

which holds for all $x \in (0, \pi/2)$ and $p \geq 1$, has been discussed in a more general setting by Wu and Srivastava [20], and actually the above generalized Wilker inequality holds true for all $p > 0$ or $p \leq -1$, conform [20, Theorem 1].

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