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Almost bounded variation of double sequences and some four dimensional summability matrices

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Abstract. In 1937, Mears [Ann. Math., **38** (1937), 594–601] studied absolutely regular matrices for single sequences. In this paper, we define the absolutely almost conservative and absolutely almost regular matrices for double sequences and establish the necessary and sufficient conditions to characterize them.

1. Introduction and preliminaries

A double sequence $x = (x_{jk})$ of real or complex numbers is said to be *bounded* if $\|x\|_\infty = \sup_{j,k} |x_{jk}| < \infty$. The space of all bounded double sequences is denoted by $\mathcal{M}_u$.

A double sequence $x = (x_{jk})$ is said to *converge to the limit L in Pringsheim's sense* (shortly, *p-convergent to L*) [12] if for every $\varepsilon > 0$ there exists an integer N such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case L is called the p-limit of x. If in addition $x \in \mathcal{M}_u$, then x is said to be *boundedly convergent to L in Pringsheim's sense* (shortly, *bp-convergent to L*).

A double sequence $x = (x_{jk})$ is said to *converge regularly to L* (shortly, *r-convergent to L*) if $x \in \mathcal{C}_p$ and the limits $x_j := \lim_k x_{jk}$ ($j \in \mathbb{N}$) and $x^k := \lim_j x_{jk}$

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($k \in \mathbb{N}$) exist. Note that in this case the limits $\lim_j \lim_k x_{jk}$ and $\lim_k \lim_j x_{jk}$ exist and are equal to the p-limit of x.

In general, for any notion of convergence ν, the space of all ν-convergent double sequences will be denoted by $\mathcal{C}_\nu$, the space of all ν-convergent to 0 double sequences by $\mathcal{C}_{\nu 0}$ and the limit of a ν-convergent double sequence x by $\nu\text{-}\lim_{j,k} x_{jk}$, where $\nu \in \{p, bp, r\}$.

Let Ω denote the vector space of all double sequences with the vector space operations defined coordinatewise. Vector subspaces of Ω are called *double sequence spaces*. In addition to above-mentioned double sequence spaces we consider the double sequence space

$$\mathcal{L}_u := \Big\{ x \in \Omega \mid \|x\|_1 := \sum_{j,k} |x_{jk}| < \infty \Big\}$$

of all absolutely summable double sequences.

All considered double sequence spaces are supposed to contain

$$\Phi := \operatorname{span}\big\{ \mathbf{e^{jk}} \mid j, k \in \mathbb{N} \big\},$$

where

$$\mathbf{e^{jk}_{il}} = \begin{cases} 1; & \text{if } (j,k) = (i,\ell), \\ 0; & \text{otherwise.} \end{cases}$$

We denote the pointwise sums $\sum_{j,k} \mathbf{e^{jk}}$, $\sum_j \mathbf{e^{jk}}$ ($k \in \mathbb{N}$), and $\sum_k \mathbf{e^{jk}}$ ($j \in \mathbb{N}$) by $\mathbf{e}$, $\mathbf{e^k}$ and $\mathbf{e_j}$ respectively.

Let E be the space of double sequences converging with respect to a convergence notion ν, F be a double sequence space, and $A = (a_{mnjk})$ be a 4-dimensional matrix of scalars. Define the set

$$F_A^{(\nu)} := \Big\{ x \in \Omega \mid [Ax]_{mn} := \nu\text{-}\sum_{j,k} a_{mnjk} x_{jk} \text{ exists and } Ax := ([Ax]_{mn})_{m,n} \in F \Big\}.$$

Then we say that A maps the space E into the space F if $E \subset F_A^{(\nu)}$ and denote by (E, F) the set of all 4-dimensional matrices A which map E into F.

We say that a 4-dimensional matrix A is $\mathcal{C}_\nu$*-conservative* if $\mathcal{C}_\nu \subset \mathcal{C}_{\nu A}^{(\nu)}$, and $\mathcal{C}_\nu$*-regular* if in addition

$$\nu\text{-}\lim Ax := \nu\text{-}\lim_{m,n} [Ax]_{mn} = \nu\text{-}\lim_{m,n} x_{mn} \ (x \in \mathcal{C}_\nu).$$

For more details on double sequences and 4-dimensional matrices, we refer to [3], [4], [8]–[11], and [13].

The idea of almost convergence for single sequences was introduced by Lorentz [5] and for double sequences by Móricz and Rhoades [7].

A double sequence $x = (x_{jk})$ of real numbers is said to be *almost convergent* to a limit L if

$$p\text{-}\lim_{p,q\to\infty} \sup_{m,n>0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} - L \right| = 0.$$

In this case L is called the f_2-limit of x and we shall denote by f_2 the space of all almost convergent double sequences.

Note that a convergent double sequence need not be almost convergent. However every bounded convergent double sequence is almost convergent and every almost convergent double sequence is bounded.

The space $\mathcal{BV}$ of double sequences $x = (x_{jk})$ of bounded variation was defined by Altay and Başar [1] as follows.

$$\mathcal{BV} := \left\{ x = (x_{jk}) \Big| \sum_{j,k} \left| x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1} \right| < \infty \right\}$$

which is a Banach space normed by

$$\|x\|_{\mathcal{BV}} = \sum_{j,k} |x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1}|.$$

Recently in [14] authors have characterized almost $\mathcal{C}_\nu$-conservative matrices, i.e. those 4-dimensional matrices $A = (a_{mnjk})$ which map the double sequence space $\mathcal{C}_\nu$ into the space f_2 where $\nu \in \{bp, r, p\}$. In this paper we introduce the notion of almost bounded variation for double sequences and use to define the absolutely almost conservative and absolutely almost regular four dimensional matrices and determine conditions to characterize them.

2. Almost bounded variation of double sequences

Motivated by the idea of absolute almost convergence for single sequences [2], we define here the notion of almost bounded variation for double sequences.

Let

$$\phi_{pqst}(x) = \tau_{pqst}(x) - \tau_{p-1,q,s,t}(x) - \tau_{p,q-1,s,t}(x) + \tau_{p-1,q-1,s,t}(x), \qquad \text{and}$$

$$\tau_{pqst}(x) = \frac{1}{(p+1)(q+1)} \sum_{m=0}^{p} \sum_{n=0}^{q} x_{m+s,n+t}.$$

Thus

$$\begin{aligned}
\phi_{pqst}(x) &= \frac{1}{(p+1)(q+1)} \sum_{m=0}^{p} \sum_{n=0}^{q} x_{m+s,n+t} - \frac{1}{p(q+1)} \sum_{m=0}^{p-1} \sum_{n=0}^{q} x_{m+s,n+t} \\
&\quad - \frac{1}{(p+1)q} \sum_{m=0}^{p} \sum_{n=0}^{q-1} x_{m+s,n+t} + \frac{1}{pq} \sum_{m=0}^{p-1} \sum_{n=0}^{q-1} x_{m+s,n+t} \\
&= \frac{1}{(q+1)} \sum_{n=0}^{q} \left[\frac{1}{(p+1)} \sum_{m=0}^{p} x_{m+s,n+t} - \frac{1}{p} \sum_{m=0}^{p-1} x_{m+s,n+t} \right] \\
&\quad - \frac{1}{q} \sum_{n=0}^{q-1} \left[\frac{1}{(p+1)} \sum_{m=0}^{p} x_{m+s,n+t} - \frac{1}{p} \sum_{m=0}^{p-1} x_{m+s,n+t} \right] \\
&= \frac{1}{(q+1)} \sum_{n=0}^{q} \left[\frac{1}{p(p+1)} \sum_{m=1}^{p} m(x_{m+s,n+t} - x_{m-1+s,n+t}) \right] \\
&\quad - \frac{1}{q} \sum_{n=0}^{q-1} \left[\frac{1}{p(p+1)} \sum_{m=1}^{p} m(x_{m+s,n+t} - x_{m-1+s,n+t}) \right] \\
&= \frac{1}{p(p+1)} \sum_{m=1}^{p} m \left[\frac{1}{(q+1)} \sum_{n=0}^{q} y_{m+s,n+t} - \frac{1}{q} \sum_{n=0}^{q-1} y_{m+s,n+t} \right]
\end{aligned}$$

where $y_{m+s,n+t} = (x_{m+s,n+t} - x_{m-1+s,n+t})$. Simplifying further, we get

$$\begin{aligned}
\phi_{pqst}(x) &= \frac{1}{p(p+1)} \sum_{m=1}^{p} m \left[\frac{1}{q(q+1)} \sum_{n=1}^{q} n(y_{m+s,n+t} - y_{m+s,n-1+t}) \right] \\
&= \frac{1}{p(p+1)q(q+1)} \sum_{m=1}^{p} \sum_{n=1}^{q} mn[x_{m+s,n+t} - x_{m-1+s,n+t)} \\
&\quad - x_{m+s,n-1+t} + x_{m-1+s,n-1+t}].
\end{aligned}$$

Now we write

$$\phi_{pqst}(x) = \begin{cases} \dfrac{1}{p(p+1)q(q+1)} \displaystyle\sum_{m=1}^{p} \sum_{n=1}^{q} mn[x_{m+s,n+t} - x_{m-1+s,n+t} \\ \qquad -x_{m+s,n-1+t} + x_{m-1+s,n-1+t}];\ p,q \geq 1 \\ x_{st};\ p = 0 \text{ or } q = 0 \text{ or both } p,q = 0. \end{cases} \tag{2.1}$$

Definition 2.1. A double sequence $x = (x_{jk}) \in \mathcal{M}_u$ is said to be of *almost bounded variation* if

(i) $\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} |\phi_{pqst}(x)|$ converges uniformly in s, t; and

(ii) $\lim_{p,q\to\infty} \tau_{pqst}(x)$, which must exist, should take the same value for all s, t. By $\hat{\mathcal{BV}}$, we denote the space of all double sequences which are of almost bounded variation.

Throughout the paper lim stands for bp-lim and $\sum$ for bp-$\sum$.

Theorem 2.1. *$\hat{\mathcal{BV}}$ is a Banach space normed by*

$$\|x\| = \sup_{s,t} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} |\phi_{pqst}(x)| \tag{2.2}$$

Proof. By uniform convergence, there exist P and Q such that

$$\sum_{p=P+1}^{\infty}\ \sum_{q=Q+1}^{\infty} |\phi_{pqst}(x)| \leq 1$$

for all s, t and for fixed P and Q,

$$\sum_{p=0}^{P}\sum_{q=0}^{Q} |\phi_{pqst}(x)|$$

is bounded because $x \in \hat{\mathcal{BV}}$ and $\hat{\mathcal{BV}} \subset \mathcal{M}_u$. Hence $\|x\|$ is defined.

As in case of $\hat{\mathcal{BV}}$ in [1], it can be easily shown that $\hat{\mathcal{BV}}$ is also a normed linear space.

Now, let (x^b) be a Cauchy sequence in $\hat{\mathcal{BV}}$. Then for each j, k, (x^b_{jk}) is a Cauchy sequence in $\mathbb{C}$. Therefore, $x^b_{jk} \to x_{jk}$ (say). Letting $x = (x_{jk})$, given $\epsilon > 0$ there exists an integer N such that for b, $d > N = N(\epsilon)$ and for each s, t

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} |\phi_{pqst}(x^b - x^d)| < \epsilon,$$

and thus

$$|\tau_{pqst}(x^b - x^d)| < \epsilon.$$

Taking limit $d \to \infty$, we have for $b > N = N(\epsilon)$ and for each s, t

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} |\phi_{pqst}(x^b - x)| \leq \epsilon \tag{2.3}$$

and

$$|\tau_{pqst}(x^b - x)| \leq \epsilon. \tag{2.3'}$$

Now, let $\epsilon > 0$ be given. There exists a b such that (2.3) holds for all s, t. Since $x^b \in \hat{\mathcal{BV}}$, we can choose p_0, q_0 such that

$$\sum_{p=p_0}^{\infty} \sum_{q=q_0}^{\infty} |\phi_{pqst}(x^b)| < \epsilon \quad \text{for all } s,\ t.$$

It follows from (2.3) that

$$\sum_{p=p_0}^{\infty} \sum_{q=q_0}^{\infty} |\phi_{pqst}(x^b) - \phi_{pqst}(x)| \le \epsilon \quad \text{for all } s,\ t.$$

Hence

$$\sum_{p=p_0}^{\infty} \sum_{q=q_0}^{\infty} |\phi_{pqst}(x)| < 2\epsilon \quad \text{for all } s,\ t. \tag{2.4}$$

Thus, starting with any ϵ, we have determined p_0, q_0 such that (2.4) holds. Hence the condition (i) of Definition 2.1 holds.

Now, for given ϵ, let (2.3)' hold for fixed chosen b and for all s, t. Since $x^b \in \hat{\mathcal{BV}}$, we have for all $p \ge p_0, q \ge q_0$

$$|\tau_{pqst}(x^b - Le)| < \epsilon \quad \text{for all } s,\ t.$$

It follows from $(2.3)'$ that

$$|\tau_{pqst}(x^b) - \tau_{pqst}(x)| \le \epsilon \quad \text{for all } s,\ t.$$

Hence

$$|\tau_{pqst}(x) - Le)| < 2\epsilon \quad \text{for all } s,\ t,$$

which is condition(ii) of Definition 2.1. Hence the result. $\square$

3. Absolutely almost conservative matrices

The idea of absolutely regular matrices for single sequences was studied by Mears [6], i.e. those matrices which transform the space v of the sequences of bounded variation into v leaving the limit invariant. Here we define the following:

Definition 3.1. A four dimensional infinite matrix $A = (a_{mnjk})$ is said to be *absolutely almost conservative* if and only if $Ax \in \hat{\mathcal{BV}}$ for all $x \in \mathcal{BV}$.

Definition 3.2. An infinite matrix $A = (a_{mnjk})$ is said to be *absolutely almost regular* if and only if it is absolutely almost conservative and $\lim Ax = \lim x$ for all $x \in \mathcal{BV}$.

We write

$$\triangle_{01} x_{jk} = x_{jk} - x_{j,k-1},$$

$$\triangle_{10} x_{jk} = x_{jk} - x_{j-1,k},$$

$$\triangle_{11} x_{jk} = \triangle_{10}(\triangle_{01} x_{jk}) = \triangle_{01}(\triangle_{10} x_{jk}) = x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1}.$$

Now we find necessary and sufficient conditions for A to be absolutely almost conservative and absolutely almost regular.

We note that, if Ax is defined, then it follows from (2.1) that, for all integers $p, q, s, t \geq 0$

$$\phi_{pqst}(Ax) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha(p,q,j,k,s,t) x_{jk}$$

where

$$\alpha(p,q,j,k,s,t) = \begin{cases} \dfrac{1}{p(p+1)q(q+1)} \displaystyle\sum_{m=1}^{p} \sum_{n=1}^{q} mn[a_{m+s,n+t,j,k} - a_{m-1+s,n+t,j,k} \\ \qquad -a_{m+s,n-1+t,j,k} + x_{m-1+s,n-1+t,j,k}];\ p,\ q \geq 1 \\ a(m,n,j,k);\ p \text{ or } q \text{ or both zero.} \end{cases}$$

The notation $a(m,n,j,k)$ denotes the element a_{mnjk} of the matrix A.

Theorem 3.1. *A matrix $A = (a_{mnjk})$ is absolutely almost conservative if and only if*

(i) *there exists a constant K such that for $i, r = 0, 1, 2, \ldots$; $s, t = 0, 1, 2, \ldots$*

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left| \sum_{j=0}^{i} \sum_{k=0}^{r} \alpha(p,q,j,k,s,t) \right| \leq K;$$

(ii) $u_{jk} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \alpha(p,q,j,k,s,t)$ *uniformly in s, t;*

(iii) $u_{0k} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} \alpha(p,q,j,k,s,t)$ *uniformly in s, t;*

(iv) $u_{j0} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{k=0}^{\infty} \alpha(p,q,j,k,s,t)$ *uniformly in s, t;*

(v) $u = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha(p,q,j,k,s,t)$ *uniformly in s, t;*

$(j, k = 0, 1, 2, \dots)$ *where* (iii), (iv) *and* (v) *satisfy that*

(iii)' *for each* k, $\sum_{j=0}^{\infty} a_{mnjk}$ *converges for all* m, n;

(iv)' *for each* j, $\sum_{k=0}^{\infty} a_{mnjk}$ *converges for all* m, n;

(v)' $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{mnjk}$ *converges for all* m, n;

respectively. In this case, the bp- $\lim Ax$ *is*

$$u\ell - \sum_{k=0}^{\infty} u_{0k} h_k - \sum_{j=0}^{\infty} u_{j0} \ell_j + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_{jk} x_{jk},$$

for every $x = (x_{jk}) \in \mathcal{BV}$, *where*

$$\ell = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \triangle_{11} x_{jk}, \ h_k = \sum_{j=0}^{\infty} \triangle_{10} x_{jk} \quad \text{and} \quad \ell_j = \sum_{k=0}^{\infty} \triangle_{01} x_{jk}.$$

Theorem 3.2. *A matrix* $A = (a_{mnjk})$ *is absolutely almost regular if and only if*

(i) *there exists a constant* K *such that for* $i, r = 0, 1, 2, \dots; s, t = 0, 1, 2, \dots$

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left| \sum_{j=0}^{i} \sum_{k=0}^{r} \alpha(p, q, j, k, s, t) \right| \leq K;$$

(ii) $u_{jk} = 0$ *for each* j, k;

(iii) $u_{0k} = 0$ *for each* k;

(iv) $u_{j0} = 0$ *for each* j;

(v) $u = 1$.

Proof of Theorem 3.1. Let $A = (a_{mnjk})$ be absolutely almost conservative. Put

$$q_{st}(x) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} |\phi_{pqst}(Ax)|$$

It is clear that for fixed m, n and for each j, k

$$\sum_{i=0}^{j} \sum_{r=0}^{k} a_{mnir} x_{ir}$$

is a continuous linear functional on $\mathcal{BV}$. We are given that, for all $x \in \mathcal{BV}$ it tends to a limit as $j, k \to \infty$ (for fixed m, n) and hence by Banach–Steinhaus theorem, this limit, that is to say $(Ax)_{st}$ is also a continuous linear functional on $\mathcal{BV}$. Hence, for fixed s, t and fixed (finite) p, q

$$\sum_{\mu=0}^{p}\sum_{\xi=0}^{q}|\phi_{\mu\xi st}(Ax)| \tag{3.1}$$

is a continuous seminorm on $\mathcal{BV}$. For any given $x \in \mathcal{BV}$, (3.1) is bounded in p, q, s, t. Hence by another application of Banach–Steinhaus theorem, there exists a constant $M > 0$ such that

$$q_{st}(x) \leq M||x||. \tag{3.2}$$

Apply (3.2) with $x = (x_{jk})$ defined by

$$x_{jk} = \begin{cases} 1; & \text{if } j \leq i,\ k \leq r \\ 0; & \text{otherwise.} \end{cases}$$

Note that, in this case, $\|x\| = 2$, and hence (i) must hold.
Since $\mathbf{e^{jk}}$, $\mathbf{e^{k}}$, $\mathbf{e_{j}}$ and $\mathbf{e}$ belong to $\mathcal{BV}$, necessity of (ii), (iii), (iv) and (v) is obvious.

Conversely, let the conditions hold and that $x = (x_{jk}) \in \mathcal{BV}$. We have defined $\hat{\mathcal{BV}}$ as a subspace of $\mathcal{M}_u$. Thus, in order to prove that $Ax \in \hat{\mathcal{BV}}$, it is necessary to prove that Ax exists and is bounded. Since the sum in (i) is bounded, it follows that

$$\sum_{j=0}^{i}\sum_{k=0}^{r} a_{mnjk} \tag{3.3}$$

is bounded for all i, r, m, n. Hence by the convergence of (v)' for fixed m, n the result follows easily.

Now by (v)', the series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(p,q,j,k,s,t)$$

converges for all p, q, s, t. Hence if we write

$$\beta(p,q,j,k,s,t) = \sum_{i=j}^{\infty}\gamma(p,q,i,k,s,t)$$

where

$$\gamma(p,q,i,k,s,t)=\sum_{r=k}^{\infty}\alpha(p,q,i,r,s,t),$$

then $\beta(p,q,j,k,s,t)$ is defined, also, for fixed p, q, s, t we have

$$\left.\begin{aligned}\gamma(p,q,i,k,s,t)\to 0 \text{ as } k\to\infty\\ \beta(p,q,j,k,s,t)\to 0 \text{ as } j\to\infty.\end{aligned}\right\} \tag{3.4}$$

Then (iii) gives that

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|\beta(p,q,0,0,s,t)| \tag{3.5}$$

converges uniformly in s, t. Similarly (iii) and (iii)' yield that, for fixed k

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left|\sum_{j=0}^{\infty}\alpha(p,q,j,k,s,t)\right| \tag{3.6}$$

converges uniformly in s, t; and (iv) and (iv)' yield that, for fixed j

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left|\sum_{k=0}^{\infty}\alpha(p,q,j,k,s,t)\right| \tag{3.7}$$

converges uniformly in s, t. From (ii) for fixed j, k, we have that the series

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|\alpha(p,q,j,k,s,t)| \tag{3.8}$$

converges uniformly in s, t. Since

$$\beta(p,q,j,k,s,t)=\sum_{i=j}^{\infty}\sum_{r=k}^{\infty}\alpha(p,q,i,r,s,t)=\left(\sum_{i=0}^{\infty}-\sum_{i=0}^{j-1}\right)\sum_{r=k}^{\infty}\alpha(p,q,i,r,s,t)$$
$$=\left(\sum_{i=0}^{\infty}\sum_{r=k}^{\infty}-\sum_{i=0}^{j-1}\sum_{r=k}^{\infty}\right)\alpha(p,q,i,r,s,t)$$
$$=\left[\sum_{i=0}^{\infty}\left(\sum_{r=0}^{\infty}-\sum_{r=0}^{k-1}\right)-\sum_{i=0}^{j-1}\left(\sum_{r=0}^{\infty}-\sum_{r=0}^{k-1}\right)\right]\alpha(p,q,i,r,s,t)=\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\alpha(p,q,i,r,s,t)$$
$$-\sum_{i=0}^{\infty}\sum_{r=0}^{k-1}\alpha(p,q,i,r,s,t)-\sum_{i=0}^{j-1}\sum_{r=0}^{\infty}\alpha(p,q,i,r,s,t)+\sum_{i=0}^{j-1}\sum_{r=0}^{k-1}\alpha(p,q,i,r,s,t)$$

$$
\begin{aligned}
&= \beta(p,q,0,0,s,t) - [\beta(p,q,0,0,s,t) - \beta(p,q,0,k,s,t)] \\
&\quad -[\beta(p,q,0,0,s,t) - \beta(p,q,j,0,s,t)] + \sum_{i=0}^{j-1}\sum_{r=0}^{k-1}\alpha(p,q,i,r,s,t) = \beta(p,q,0,k,s,t) \\
&\quad + \beta(p,q,j,0,s,t) - \beta(p,q,0,0,s,t) + \sum_{i=0}^{j-1}\sum_{r=0}^{k-1}\alpha(p,q,i,r,s,t), \qquad (3.9)
\end{aligned}
$$

it follows that, for fixed j, k

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|\beta(p,q,j,k,s,t)| \qquad (3.10)$$

converges uniformly in s, t.

Now

$$
\begin{aligned}
\phi_{pqst}(Ax) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(p,q,j,k,s,t)x_{jk} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left[\sum_{i=j}^{\infty}\sum_{r=k}^{\infty}\alpha(p,q,i,r,s,t)\right](\triangle_{11}x_{jk}) \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\beta(p,q,j,k,s,t)[x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1}] \qquad (3.11)
\end{aligned}
$$

by (3.4) and the boundedness of $x = (x_{jk})$.

Now (i) and the boundedness of the sum (3.5) show that

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|\beta(p,q,j,k,s,t)| \qquad (3.12)$$

is bounded for all j, k, s, t. We can make

$$\sum_{j=j_0+1}^{\infty}\sum_{k=k_0+1}^{\infty}|x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1}|$$

arbitrarily small by choosing j_0 and k_0 sufficiently large. Therefore, it follows that, given $\epsilon > 0$ we can choose j_0, k_0 so that, for all s, t

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left|\sum_{j=j_0+1}^{\infty}\sum_{k=k_0+1}^{\infty}\beta(p,q,j,k,s,t)(x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1})\right| < \epsilon. \qquad (3.13)$$

Now since for each j, k, (3.10) converges uniformly in s, t, it follows that once j_0, k_0 have been chosen we can choose p_0, q_0 so that, for all s, t

$$\sum_{p=p_0+1}^{\infty}\sum_{q=q_0+1}^{\infty}\left|\sum_{j=0}^{j_0}\sum_{k=0}^{k_0}\beta(p,q,j,k,s,t)(x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1})\right| < \epsilon.$$

It follows from (3.13) that the same inequality holds when $\sum_{p=0}^{\infty}$ and $\sum_{q=0}^{\infty}$ are replaced by $\sum_{p=p_0+1}^{\infty}$ and $\sum_{q=q_0+1}^{\infty}$ respectively; hence for all s, t,

$$\sum_{p=p_0+1}^{\infty}\sum_{q=q_0+1}^{\infty}\left|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\beta(p,q,j,k,s,t)(x_{jk}-x_{j-1,k}-x_{j,k-1}+x_{j-1,k-1})\right|<2\epsilon, \quad (3.14)$$

that is

$$\sum_{p=p_0+1}^{\infty}\sum_{q=q_0+1}^{\infty}|\phi_{pqst}(Ax)|<2\epsilon.$$

Thus

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|\phi_{pqst}(Ax)|$$

converges uniformly in s, t. Hence Ax satisfies condition (i) of Definition 2.1; we still have to show that it satisfies condition (ii) of Definition 2.1.

If

$$\phi_{pqst}(Ax)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(p,q,j,k,s,t)x_{jk},$$

then by Abel's partial sum we have

$$\phi_{pqst}(Ax)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left[\sum_{i=0}^{j-1}\sum_{r=0}^{k-1}\alpha(p,q,j,k,s,t)\right]\triangle_{11}x_{jk}.$$

Using (3.9), we get

$$\begin{aligned}\phi_{pqst}(Ax)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\big[&\beta(p,q,j,k,s,t)-\beta(p,q,0,k,s,t)\\&-\beta(p,q,j,0,s,t)+\beta(p,q,0,0,s,t)\big]\triangle_{11}x_{jk}.\end{aligned}$$

Again using Abel's partial sum to first three summations, we get

$$\begin{aligned}\phi_{pqst}(Ax)=&\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(p,q,j,k,s,t)x_{jk}-\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(p,q,j,k,s,t)\triangle_{10}x_{jk}\\&-\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(p,q,j,k,s,t)\triangle_{01}x_{jk}+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(p,q,j,k,s,t)\triangle_{11}x_{jk}.\end{aligned}$$

Hence

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\phi_{pqst}(Ax)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\alpha(p,q,j,k,s,t)x_{jk}$$

$$-\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\alpha(p,q,j,k,s,t)\triangle_{10}x_{jk}-\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\alpha(p,q,j,k,s,t)\triangle_{01}x_{jk}$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\alpha(p,q,j,k,s,t)\triangle_{11}x_{jk}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\alpha(p,q,j,k,s,t)x_{jk}\right]$$

$$-\sum_{k=0}^{\infty}\left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left(\sum_{j=0}^{\infty}\alpha(p,q,j,k,s,t)(x_{jk}-x_{j-1,k})\right)\right]$$

$$-\sum_{j=0}^{\infty}\left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left(\sum_{k=0}^{\infty}\alpha(p,q,j,k,s,t)(x_{jk}-x_{j,k-1})\right)\right]$$

$$+\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(p,q,j,k,s,t)\right](x_{jk}-x_{j-1,k}-x_{j,k-1}+x_{j-1,k-1})$$

$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}u_{jk}x_{jk}-\sum_{k=0}^{\infty}u_{0k}h_k-\sum_{j=0}^{\infty}u_{j0}\ell_j+u\ell,$$

where, for $x\in\mathcal{BV}$

$$\ell_j=\lim_{k\to\infty}x_{jk}=\sum_{k=0}^{\infty}(x_{jk}-x_{j,k-1}),\qquad h_k=\lim_{j\to\infty}x_{jk}=\sum_{j=0}^{\infty}(x_{jk}-x_{j-1,k}),$$

$$\ell=\lim x=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\triangle_{11}x_{jk}.$$

Proof of Theorem 3.2. Suppose that A is absolutely almost regular matrix. Since $\mathbf{e^{jk}},\mathbf{e^{k}},\mathbf{e_{j}}$ and $\mathbf{e}\in\mathcal{BV}$, conditions (ii), (iii), (iv) and (v) hold respectively. Condition (i) follows as in the proof of Theorem 3.1.

Conversely, if a matrix A satisfies the conditions of the theorem, then it is an absolutely almost conservative matrix. For $x\in\mathcal{BV}$, the *bp*-limit of Ax is

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}u_{jk}x_{jk}-\sum_{k=0}^{\infty}u_{0k}h_k-\sum_{j=0}^{\infty}u_{j0}\ell_j+u\ell$$

which reduces to ℓ by using conditions (ii)–(v). Hence A is an absolutely almost regular matrix.

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