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Curvature of the contact distribution

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Abstract. We define and study both the sectional and φ-sectional curvatures of the contact distribution $\mathcal{D}$ on a K-contact manifold M. We prove that 3-dimensional K-contact manifolds of constant curvature are the only ones who carry contact distributions of constant curvature. Also, we prove that $\mathcal{D}$ is of constant φ-sectional curvature if and only if M is a Sasakian space form.

Introduction

A contact metric manifold carries a non-integrable distribution, which is called the contact distribution. Thus such a manifold belongs to the class of non-holonomic manifolds which have been introduced as a need for a geometric interpretation of non-holonomic mechanical systems (cf. Vrănceanu [10], Neimark–Fufaev [7]).

The geometry of manifolds endowed with non-integrable distributions have been intensively studied from many different points of view. We mention here some directions in which such research has been carried out.

First, we refer to the studies on contact manifolds by using associate Riemannian metrics (cf. Blair [2]). As a result of these studies, two important classes of contact manifolds have been investigated: Sasakain and 3-Sasakian manifolds. Then, there are tight relations between CR-manifolds (cf. Greenfield [3]) and

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contact manifolds. A great role in such studies was played by the canonical connection on a CR-manifold introduced by TANAKA [9].

Finally, we point out that from the studies on manifolds endowed with nonintegrable distributions has emerged a new branch of differential geometry which is known as subriemannian geometry (cf. MONTGOMERY [6]). This geometry is mainly based on the study of those geodesics of the manifold which remain tangent to the horizontal distribution. Also, we find here the notion of curvature of a distribution, which is a 2-form that measures its nonintegrability.

Our approach in this paper is different. We introduce and study a curvature tensor field for a contact distribution. This enables us to define and study the sectional curvature and the φ-sectional curvature of a contact distribution on a K-contact manifold. The whole theory is developed by using both the Levi–Civita connection and the Vrănceanu connection. The latter one was specially introduced for studying non-holonomic manifolds.

Now, we outline the content of the paper. First we present the results and formulas on the theory of contact metric manifolds which we need in our study. Then we introduce an adapted frame field on a contact metric manifold and express both the Levi–Civita and Vrănceanu connections with respect to this frame field. In particular, we show that Vrănceanu connection defines two types of covariant derivatives. In Section 4 we show that the restriction of the curvature tensor field of Vrănceanu connection to the contact distribution has the same nice properties as the Levi–Civita connection on a Riemannian manifold (cf. Theorem 4.3), provided the manifold is K-contact. This enables us to define a sectional curvature for the contact distribution on a K-contact manifold. Then we prove a Schur Theorem type for dimensions greater than 3 (cf. Theorem 4.5) and obtain an explicit formula for the curvature tensor field of a contact distribution of constant sectional curvature (see (4.15)). We also prove that contact distributions of constant curvature live on 3-dimensional K-contact manifolds of constant curvature but they do not exist on manifolds of higher dimensions (cf. Theorems 4.7 and 4.9). Finally, in the last section we define the φ-sectional curvature of the contact distribution and prove that Sasakian space forms carry contact distributions of constant φ-sectional curvature (cf. Theorem 5.1). Also, we obtain a formula for the curvature tensor field of a contact distribution of constant φ-sectional curvature (see 5.2a).

1. Preliminaries

Let M be a $(2m+1)$-dimensional manifold and η a differential 1-form on M such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on M. Then M is called a contact manifold and η a contact form on M. It is well known that a contact manifold admits a contact metric structure (φ, ξ, η, g), where φ is a tensor field of type $(1,1)$, ξ is a vector field, and g is a Riemannian metric satisfying (cf. Blair [2], p. 36)

$$\begin{aligned} &(a) \quad \varphi^2 = -I + \eta \otimes \xi, && (b) \quad \eta(\xi) = 1, \\ &(c) \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), && (d) \quad d\eta(X,Y) = g(X, \varphi Y), \end{aligned} \tag{1.1}$$

for any $X, Y \in \Gamma(TM)$. We denote by $M(\varphi, \xi, \eta, g)$ the contact metric manifold M with the contact metric structure (φ, ξ, η, g). The equations in (1.1) imply that

$$\begin{aligned} &(a) \quad \varphi\xi = 0, && (b) \quad g(X, \varphi Y) + g(Y, \varphi X) = 0, \\ &(c) \quad \eta \circ \varphi = 0, && (d) \quad d\eta(\xi, X) = 0, \qquad (e) \quad \eta(X) = g(X, \xi), \end{aligned} \tag{1.2}$$

for any $X, Y \in \Gamma(TM)$.

Here and in the sequel, $\Gamma(TM)$ stands for the module of smooth vector fields on M. The same notation we use for the module of smooth sections of any distribution on M. Also, we should note that we use $d\eta$ given by

$$d\eta(X,Y) = \frac{1}{2}\{X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])\}.$$

A contact metric manifold $M(\varphi, \xi, \eta, g)$ for which the characteristic vector field ξ is a Killing vector field is called a *K-contact manifold.* It is known that M is a K-contact manifold if and only if

$$\widetilde{\nabla}_X \xi = -\varphi X, \quad \forall X \in \Gamma(TM), \tag{1.3}$$

where $\widetilde{\nabla}$ is the Levi–Civita connection on M with respect to g given by (cf. Yano–Kon [11], p. 29)

$$\begin{aligned} 2g(\widetilde{\nabla}_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &+ \ g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y), \end{aligned} \tag{1.4}$$

for any $X, Y \in \Gamma(TM)$.

Finally, we recall that the curvature tensor field $\widetilde{R}$ of a Sasakian space form M of constant φ-sectional curvature c is expressed as follows (cf. Blair [2], p. 113)

$$\widetilde{R}(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\}$$

$$+\frac{c-1}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi$$
$$+\Phi(Z,Y)\varphi X-\Phi(Z,X)\varphi Y+2\Phi(X,Y)\varphi Z\}, \tag{1.5}$$

for any $X,Y,Z\in\Gamma(TM)$, where Φ is the fundamental 2-form of the contact metric structure given by

$$\Phi(X,Y)=g(X,\varphi Y),\quad \forall X,Y\in\Gamma(TM). \tag{1.6}$$

2. The Levi–Civita connection on a contact metric manifold

Let $\mathcal{D}$ be the contact distribution on the contact metric manifold $M(\varphi,\xi,\eta,g)$, that is we have

$$\mathcal{D}_x=\{X_x\in T_xM:\eta(X_x)=0\},\quad \forall x\in M.$$

It follows from (1.1d) that $\mathcal{D}$ is not an integrable distribution. Also, we consider on M the foliation determined by ξ, whose transversal distribution is $\mathcal{D}$. Thus we have the orthogonal decomposition

$$TM=\mathcal{D}\oplus\mathcal{D}^{\perp},\quad \text{where } \mathcal{D}^{\perp}=\mathrm{span}\{\xi\}. \tag{2.1}$$

This enables us to choose an adapted local coordinate system $\{\mathcal{U};x^0,x^i\}$, $i\in\{1,\dots,2m\}$, such that $\xi=\partial/\partial x^0$ on $\mathcal{U}$. Then the contact form η is locally expressed as follows

$$\eta=dx^0+\eta_i dx^i,\quad \text{where } \eta_i=\eta\left(\frac{\partial}{\partial x^i}\right).$$

Now, we consider the vector fields

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-\eta_i\frac{\partial}{\partial x^0},\quad i\in\{1,\dots,2m\}, \tag{2.2}$$

which form a basis in $\Gamma(\mathcal{D})$. Thus $\{\partial/\partial x^0,\delta/\delta x^i\}$ is a frame field on M adapted to the decomposition (2.1).

Next, we examine the Lie brackets of the vector fields from the above frame field. First, from (1.2d) we deduce that

$$\left[\frac{\delta}{\delta x^i},\frac{\partial}{\partial x^0}\right]\in\Gamma(\mathcal{D}),\quad \forall i\in\{1,\dots,2m\}.$$

On the other hand, by direct calculations using (2.2) we obtain

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^0}\right] = \frac{\partial \eta_i}{\partial x^0}\,\frac{\partial}{\partial x^0}.$$

Thus we have

$$\text{(a)}\quad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^0}\right] = 0 \quad\text{and}\quad \text{(b)}\quad \frac{\partial \eta_i}{\partial x^0} = 0, \quad \forall i \in \{1, \ldots, 2m\}. \tag{2.3}$$

Also, by using (2.2) and (2.3) we infer that

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = \left(\frac{\partial \eta_i}{\partial x^j} - \frac{\partial \eta_j}{\partial x^i}\right)\frac{\partial}{\partial x^0}. \tag{2.4}$$

Now we note that the matrix of the Riemannian metric g with respect to the frame field $\{\partial/\partial x^0, \delta/\delta x^i\}$ is

$$[g] = \begin{bmatrix} 1 & 0 \\ 0 & g_{ij} \end{bmatrix}, \tag{2.5}$$

where we put

$$g_{ij} = g\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right). \tag{2.6}$$

The inverse matrix of $[g_{ij}]$ is denoted by $[g^{ij}]$. Also, we express φ and Φ locally, as follows

$$\text{(a)}\ \varphi\left(\frac{\delta}{\delta x^i}\right) = \varphi^j{}_i \frac{\delta}{\delta x^j}, \quad \text{(b)}\ \Phi_{ij} = \Phi\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = g_{ik}\varphi^k{}_j. \tag{2.7}$$

Then, by using (1.1d), (2.4) and (2.7), we obtain

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -2\Phi_{ij}\,\frac{\partial}{\partial x^0}. \tag{2.8}$$

Now we can state the following.

Theorem 2.1. *Let $M(\varphi, \xi, \eta, g)$ be a contact metric manifold. Then the Levi–Civita connection $\widetilde{\nabla}$ on M is completely determined by the following equalities*

$$\begin{aligned}
&\text{(a)}\quad \widetilde{\nabla}_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = F_i{}^k{}_j \frac{\delta}{\delta x^k} - L_{ij}\frac{\partial}{\partial x^0},\\
&\text{(b)}\quad \widetilde{\nabla}_{\frac{\partial}{\partial x^0}} \frac{\delta}{\delta x^j} = \widetilde{\nabla}_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial x^0} = L^k{}_j \frac{\delta}{\delta x^k},\\
&\text{(c)}\quad \widetilde{\nabla}_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^0} = 0,
\end{aligned} \tag{2.9}$$

where we set

$$(a)\quad F_i{}^k{}_j = \frac{1}{2} g^{kh} \left\{ \frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h} \right\},$$

$$(b)\quad L_{ij} = \frac{1}{2}\frac{\delta g_{ij}}{\delta x^0} - \Phi_{ij}, \qquad (c)\quad L^k{}_j = g^{ki} L_{ij}. \tag{2.10}$$

Proof. Take $X = \frac{\delta}{\delta x^i}$, $Y = \frac{\delta}{\delta x^j}$ and $Z = \frac{\delta}{\delta x^h}$ in (1.4) and by using (2.6), (2.8) and (2.9a) we deduce that $F_i{}^k{}_j$ must be given by (2.10a). Similarly, if we take in (1.4) the same X and Y, but $Z = \frac{\partial}{\partial x^0}$ we obtain (2.10b) via (2.3a), (2.8) and (2.9a). Next, the first equality in (2.9b) follows from (2.3a) since $\widetilde{\nabla}$ is torsion-free. The coefficients $L^k{}_j$ from (2.9b) are obtained by similar calculations in (1.4) as we explained for $F_i{}^k{}_j$ and L_{ij}. Finally, (2.9c) represents a well-known property of the characteristic vector field $\xi = \partial/\partial x^0$. □

3. The Vrănceanu connection on a contact metric manifold

Let $M(\varphi, \xi, \eta, g)$ be a contact metric manifold. Then the Vrănceanu connection ∇ defined by the Levi–Civita connection $\widetilde{\nabla}$ is given by

$$\nabla_X Y = P\widetilde{\nabla}_{PX} PY + Q\widetilde{\nabla}_{QX} QY + P[QX, PY] + Q[PX, QY], \tag{3.1}$$

for any $X, Y \in \Gamma(TM)$, where P and Q are the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^\perp$ with respect to the decomposition (2.1). Ianuş [4] has given the formula (3.1) for manifolds endowed with almost product structures. The study of the geometry of foliations via Vrănceanu connection was developed by the authors in a recent book (cf. Bejancu–Farran [1]). The present paper shows that this approach can be very useful and effective in studying the geometry of some classes of contact metric manifolds.

First we state the following.

Theorem 3.1. *The Vrănceanu connection on the contact metric manifold $M(\varphi, \xi, \eta, g)$ is completely determined by the functions $F_i{}^k{}_j$ given by* (2.10a).

Proof. Indeed, by using (3.1), (2.9) and (2.3a) we obtain

$$(a)\quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = F_i{}^k{}_j \frac{\delta}{\delta x^k}, \qquad (b)\quad \nabla_{\frac{\partial}{\partial x^0}} \frac{\delta}{\delta x^i} = 0,$$

$$(c)\quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial x^0} = 0, \qquad (d)\quad \nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^0} = 0, \tag{3.2}$$

which proves the assertion of the theorem. □

Next, let T and R be the torsion and curvature tensor fields of ∇ given by (cf. YANO–KON [11], p. 25)

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{3.3}$$

and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{3.4}$$

respectively. Then, by direct calculations using (3.2), (3.3), (3.4), (2.3a) and (2.8), we deduce that

$$\text{(a)}\quad T\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 2\Phi_{ij}\,\frac{\partial}{\partial x^0}, \qquad \text{(b)}\quad T\left(\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^j}\right) = 0, \tag{3.5}$$

and

$$\begin{aligned} &\text{(a)}\quad R\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^i} = R_i{}^h{}_{jk}\frac{\delta}{\delta x^h}, \\ &\text{(b)}\quad R\left(\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^i} = R_i{}^h{}_{j0}\frac{\delta}{\delta x^h}, \\ &\text{(c)}\quad R\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial x^0} = R\left(\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial x^0} = 0, \end{aligned} \tag{3.6}$$

where we set

$$\begin{aligned} &\text{(a)}\quad R_i{}^h{}_{jk} = \frac{\delta F_i{}^h{}_j}{\delta x^k} - \frac{\delta F_i{}^h{}_k}{\delta x^j} + F_i{}^t{}_j F_t{}^h{}_k - F_i{}^t{}_k F_t{}^h{}_j, \\ &\text{(b)}\quad R_i{}^h{}_{j0} = \frac{\partial F_i{}^h{}_j}{\partial x^0}. \end{aligned} \tag{3.7}$$

Next, we consider an $\mathcal{F}(M) - (r+t)$-multilinear mapping

$$S : \Gamma(\mathcal{D}^*)^r \times \Gamma(\mathcal{D})^t \longrightarrow \mathcal{F}(M),$$

where $\mathcal{F}(M)$ is the algebra of smooth functions on M and $\mathcal{D}^*$ is the dual vector bundle of $\mathcal{D}$. Then we call S a $\mathcal{D}$-tensor field of type (r,t). We should remark that $\mathcal{D}$-tensor fields are particular cases of adapted tensor fields on foliated manifolds (see Section 2.2 of BEJANCU–FARRAN [1]). The local components of S with respect to the frame fields $\{\eta, dx^i\}$ and $\{\partial/\partial x^0, \delta/\delta x^i\}$ are defined as follows:

$$S^{i_1\dots i_r}_{j_1\dots j_t} = S\left(dx^{i_1}, \dots, dx^{i_r}, \frac{\delta}{\delta x^{j_1}}, \dots, \frac{\delta}{\delta x^{j_t}}\right).$$

The Vrănceanu connection induces two covariant derivatives of such tensor fields.

First, it is the *contact covariant derivative* of S with respect to the Vrănceanu connection, which is defined by

$$S^{i_1\dots i_r}_{j_1\dots j_t|k} = \frac{\delta S^{i_1\dots i_r}_{j_1\dots j_t}}{\delta x^k} + \sum_{x=1}^{r} S^{i_1\dots h \dots i_r}_{j_1\dots j_t} F_h{}^{i_x}{}_k - \sum_{y=1}^{t} S^{i_1\dots i_r}_{j_1\dots h\dots j_t} F_{j_y}{}^h{}_k. \tag{3.8}$$

Then the *structural covariant derivative* of S with respect to the Vrănceanu connection is given by

$$S^{i_1\dots i_r}_{j_1\dots j_t|0} = \frac{\partial S^{i_1\dots i_r}_{j_1\dots j_t}}{\partial x^0}. \tag{3.9}$$

It is noteworthy that both covariant derivatives defined above produce adapted tensor fields on M. As an example, by using (3.8) and (2.10a) we deduce that

$$g_{ij|k} = \frac{\delta g_{ij}}{\delta x^k} - g_{hj}\, F_i{}^h{}_k - g_{ih} F_j{}^h{}_k = 0. \tag{3.10}$$

However, by (3.9) we have

$$g_{ij|0} = \frac{\partial g_{ij}}{\partial x^0}, \tag{3.11}$$

which says that the Vrănceanu connection is not a metric connection.

Finally, by (3.2a) and (2.10a), we deduce that on the contact distribution $\mathcal{D}$ the Levi–Civita and Vrănceanu connections are related as follows

$$\widetilde{\widetilde{\nabla}}_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = \nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} - L_{ij}\, \frac{\partial}{\partial x^0}. \tag{3.12}$$

4. The sectional curvature of the contact distribution

In the present section we suppose that $M(\varphi, \xi, \eta, g)$ is a K-contact manifold. Then, by using (1.3), (2.7a) and (2.9b), we deduce that M is K-contact if and only if we have

$$L^i{}_j = -\varphi^i{}_j. \tag{4.1}$$

This enables us to state a new characterization of K-contact manifolds by means of the Vrănceanu connection.

Theorem 4.1. *Let $M(\varphi, \xi, \eta, g)$ be a contact metric manifold. Then M is a K-contact manifold if and only if the Vrănceanu connection is a metric connection.*

Proof. By using (2.10c), (2.10b) and (2.7b) into (4.1) we deduce that M is K-contact if and only if

$$\frac{\partial g_{ij}}{\partial x^0} = 0, \quad \forall i, j \in \{1, \dots, 2m\}. \tag{4.2}$$

Then the assertion follows by using (3.10), (3.11) and (4.2). □

Also, we obtain the following corollary.

Corollary 4.2. *M is a K-contact manifold if and only if*

$$\frac{\partial \varphi^i{}_j}{\partial x^0} = 0, \quad \forall i, j \in \{1, \dots, 2m\}. \tag{4.3}$$

Proof. First, by comparing (3.4) and (2.8), we deduce that

$$\Phi_{ij} = \frac{1}{2}\left(\frac{\partial \eta_j}{\partial x^i} - \frac{\partial \eta_i}{\partial x^j}\right).$$

Then, by (2.3b) we see that Φ_{ij} are functions of $(x^1, \dots, x^{2m})$ alone. Thus we obtain the equivalence of (4.2) and (4.3) via (2.7b). □

Next, by using (3.2), (2.11a) and (4.2), we deduce that the Vrănceanu connection on a K-contact manifold M is completely determined by the local coefficients

$$F_i{}^k{}_j = \frac{1}{2}\, g^{kh} \left\{\frac{\partial g_{hi}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}\right\}, \tag{4.4}$$

which formally look as the local coefficients of a Levi–Civita connection on a $2m$-dimensional manifold. As a consequence of this, from (3.6) and (3.7) we obtain that the curvature tensor field R of Vrănceanu connection is completely determined by the local components

$$R_i{}^h{}_{jk} = \frac{\partial F_i{}^h{}_j}{\partial x^k} - \frac{\partial F_i{}^h{}_k}{\partial x^j} + F_i{}^t{}_j F_t{}^h{}_k - F_i{}^t{}_k\, F_t{}^h{}_j. \tag{4.5}$$

Moreover, we show here that R has some nice properties as the curvature tensor field of a Levi–Civita connection. To state this we put

$$R_{ijkh} = g\left(R\left(\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}\right)\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = g_{jt} R_i{}^t{}_{kh}. \tag{4.6}$$

Now we prove the following.

Theorem 4.3. *Let M be a K-contact manifold. Then the curvature tensor field of the Vrănceanu connection on M satisfies the identities:*

$$\begin{aligned}
&\text{(a)}\quad R_{ijkh}+R_{ijhk}=0, &&\text{(b)}\quad R_{ijkh}+R_{jikh}=0,\\
&\text{(c)}\quad \sum_{(i,j,k)}\{R_{tijk}\}=0, &&\text{(d)}\quad R_{ijkh}=R_{khij},\\
&\text{(e)}\quad \sum_{(i,j,k)}\{R_{tsij|k}\}=0, && \qquad (4.7)
\end{aligned}$$

where $\sum_{(i,j,k)}$ denotes the cyclic sum with respect to (i,j,k) and "$|$" represents the contact covariant derivative with respect to Vrănceanu connection.

Proof. First, (4.7a) is a property of the curvature tensor field of any linear connection on M. Then, taking into account that ∇ is a metric connection and by using (4.6), (3.4), (2.8), (3.2b) and (4.2) we obtain

$$\begin{aligned}
R_{ijkh} &= g\left(\nabla_{\frac{\delta}{\delta x^h}}\nabla_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)-g\left(\nabla_{\frac{\delta}{\delta x^k}}\nabla_{\frac{\delta}{\delta x^h}}\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\\
&=\frac{\delta}{\delta x^h}\left(g\left(\nabla_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\right)-g\left(\nabla_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^i},\nabla_{\frac{\delta}{\delta x^h}}\frac{\delta}{\delta x^j}\right)\\
&\quad-\frac{\delta}{\delta x^k}\left(g\left(\nabla_{\frac{\delta}{\delta x^h}}\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\right)+g\left(\nabla_{\frac{\delta}{\delta x^h}}\frac{\delta}{\delta x^i},\nabla_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j}\right)\\
&=-g\left(\frac{\delta}{\delta x^i},\nabla_{\frac{\delta}{\delta x^h}}\nabla_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j}-\nabla_{\frac{\delta}{\delta x^k}}\nabla_{\frac{\delta}{\delta x^h}}\frac{\delta}{\delta x^j}\right)\\
&\quad+\left[\frac{\delta}{\delta x^h},\frac{\delta}{\delta x^k}\right](g_{ij})=-R_{jikh}-2\Phi_{hk}\frac{\partial g_{ij}}{\partial x^0}=-R_{jikh}.
\end{aligned}$$

Next, from the first Bianchi identity of a linear connection (cf. Kobayashi–Nomizu [5], p. 135), we deduce that

$$\sum_{(i,j,k)}\left\{\left(\nabla_{\frac{\delta}{\delta x^i}}T\right)\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\right)+T\left(T\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right),\frac{\delta}{\delta x^k}\right)\right.$$
$$\left.-R\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k}\right\}=0. \quad (4.8)$$

By using (3.5) and taking into account that both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel with respect to ∇ we deduce that

$$T\left(T\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right),\frac{\delta}{\delta x^k}\right)=0,$$

and

$$\left(\nabla_{\frac{\delta}{\delta x^i}} T\right)\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) \in \Gamma(\mathcal{D}^{\perp}).$$

Thus (4.7c) follows from (4.8) via (4.6), (4.7a) and (4.7b). Now, following O'Neill [8], p. 75, by a combinatorial exercise using (4.7a), (4.7b) and (4.7c), we obtain (4.7d). Finally, the second Bianchi identity for ∇ implies

$$\sum_{(i,j,k)} \left\{ \left(\nabla_{\frac{\delta}{\delta x^i}} R\right)\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) + R\left(T\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right), \frac{\delta}{\delta x^k}\right) \right\} \frac{\delta}{\delta x^t} = 0.$$

Then, by (3.5a) and (3.6c), we infer that

$$R\left(T\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right), \frac{\delta}{\delta x^k}\right) = 0.$$

Thus, the above identity becomes

$$\sum_{(i,j,k)} \{R_t{}^h{}_{jk|i}\} = 0,$$

which implies (4.7e) via (4.6) and (3.10). □

The above theorem has a crucial role in our attempt to define and study a sectional curvature of the contact distribution. First, we consider a 2-dimensional subspace W_x of $\mathcal{D}_x$, which we call a *$\mathcal{D}$-plane* at the point $x \in M$. Next, we take a basis $\{X, Y\}$ of W_x and define

$$\Delta(X,Y) = g(X,X)g(Y,Y) - g(X,Y)^2.$$

Then, by using (2.6), we obtain

$$\Delta(X,Y) = (g_{ik}g_{jh} - g_{ih}g_{jk})X^jX^hY^iY^k, \tag{4.9}$$

where (X^j) and (Y^k) are the local components of X and Y with respect to the basis $\left\{\frac{\delta}{\delta x^i}\right\}$ of $\mathcal{D}_x$. Now, we define the number

$$K(X,Y) = \frac{R_{ijkh}X^jX^hY^iY^k}{\Delta(X,Y)}. \tag{4.10}$$

Next, by using (4.7a) and (4.7b), it is easy to check that $K(X,Y)$ is independent of the basis $\{X, Y\}$ in W_x. Then we define the *sectional curvature* of the contact distribution $\mathcal{D}$ as a real-valued function on the set of all $\mathcal{D}$-planes given by (4.10). By this definition, the curvature tensor field R determines the sectional curvature K of $\mathcal{D}$. To show the converse of this we consider a $\mathcal{D}$-tensor F at $x \in M$ of type $(0,4)$ whose local components F_{ijkh} with respect to the frame field $\{\delta/\delta x^i\}$ satisfy the identities (4.7a), (4.7b) and (4.7c). We call such an F a *curvature-like $\mathcal{D}$-tensor*. Then, following the same reason as in the proof of Proposition 4.1 of O'Neill [8], p. 78, we conclude that F vanishes whenever $F(X,Y,X,Y)=0$, for any $\{X,Y\}$ spanning a $\mathcal{D}$-plane in $\mathcal{D}_x$. As a consequence, we obtain the following lemma showing that K determines R.

Lemma 4.4. *Let F be a curvature-like $\mathcal{D}$-tensor at $x \in M$ such that*

$$K(X,Y) = \frac{F_{ijkh}X^jX^hY^iY^k}{\Delta(X,Y)},$$

whenever $\{X,Y\}$ spans a $\mathcal{D}$-plane. Then we have

$$R_{ijkh} = F_{ijkh}, \quad \text{at } x \in M.$$

If the sectional curvature of $\mathcal{D}$ is a constant for all $\mathcal{D}$-planes W_x and for all points $x \in M$, then $\mathcal{D}$ is called a *contact distribution of constant curvature*. A Schur Theorem type is stated now for $\mathcal{D}$.

Theorem 4.5. *Let $M(\varphi,\xi,\eta,g)$ be a K-contact connected manifold of dimension $2m+1$, with $m>1$. If the sectional curvature of $\mathcal{D}$ does not depend on the $\mathcal{D}$-planes, then $\mathcal{D}$ is of constant curvature.*

Proof. Since K does not depend on $\mathcal{D}$-planes, there exists a function $K(x)$ on M such that $K(X,Y)=K(x)$, for any two linearly independent vector fields $X, Y \in \Gamma(\mathcal{D})$. Then we consider the functions

$$F_{ijkh} = K(x)(g_{ik}g_{jh} - g_{ih}g_{jk}), \tag{4.11}$$

which clearly define a $\mathcal{D}$-tensor field of type $(0,4)$ on M. Moreover, it is easy to verify that F_{ijkh} satisfy (4.7a), (4.7b) and (4.7c) at any $x \in M$. Thus F_{ijkh} define a curvature-like tensor field F on M and whenever $\{X,Y\}$ spans a $\mathcal{D}$-plane, we have

$$K(X,Y) = K(x) = \frac{F_{ijkh}X^jX^hY^iY^k}{\Delta(X,Y)}.$$

Thus, by Lemma 4.4, we obtain

$$R_{ijkh} = K(x)(g_{ik}g_{jh} - g_{ih}g_{jk}). \tag{4.12}$$

Taking into account (4.4)–(4.6), we conclude that R_{ijkh} are functions of $(x^1, \dots, x^{2m})$ alone. Then, by using (4.2) and (4.12), we deduce that

$$\frac{\partial K}{\partial x^0} = 0. \tag{4.13}$$

Now we take the contact covariant derivative in (4.12) with respect to Vrănceanu connection and by using (3.10) we obtain

$$R_{ijkh|t} = \frac{\partial K}{\partial x^t}\,(g_{ik}\,g_{jh} - g_{ih}\,g_{jk}). \tag{4.14}$$

Next, (4.7e) and (4.14) imply

$$\frac{\partial K}{\partial x^t}(g_{ik}\,g_{jh} - g_{ih}\,g_{jk}) + \frac{\partial K}{\partial x^k}(g_{ih}\,g_{jt} - g_{it}\,g_{jh}) + \frac{\partial K}{\partial x^h}(g_{it}\,g_{jk} - g_{ik}\,g_{jt}) = 0.$$

By contracting this equality by $g^{jh}g^{is}$ and taking into account that $m > 1$, we infer that

$$\delta^s_k\frac{\partial K}{\partial x^t} - \delta^s_t\frac{\partial K}{\partial x^k} = 0.$$

Finally, for any $t \in \{1, \dots, 2m\}$ we take $s = k \neq t$ and obtain

$$\frac{\partial K}{\partial x^t} = 0,$$

which together with (4.13) completes the proof of the theorem. □

From the proof of the above theorem we deduce the following corollary.

Corollary 4.6. *The contact distribution on a K-contact manifold is of constant curvature c if and only if the curvature tensor field of Vrănceanu connection is expressed as follows*

$$R_{ijkh} = c(g_{ik}g_{jh} - g_{ih}g_{jk}). \tag{4.15}$$

In order to study the existence of contact distributions of constant curvature we need to relate the curvature tensor fields $\widetilde{R}$ and R of Levi–Civita connection $\widetilde{\nabla}$ and Vrănceanu connection ∇, respectively. First, by using (4.2), (2.10b) and (2.10c), we deduce that

$$L_{ij} = -\Phi_{ij} \quad \text{and} \quad L^k{}_j = -\varphi^k{}_j,$$

provided M is K-contact. Thus (2.9) becomes

$$\text{(a)} \quad \widetilde{\nabla}_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = F_i{}^k{}_j\frac{\delta}{\delta x^k} + \Phi_{ij}\,\frac{\partial}{\partial x^0},$$

$$\text{(b)} \quad \widetilde{\nabla}_{\frac{\partial}{\partial x^0}} \frac{\delta}{\delta x^j} = \widetilde{\nabla}_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial x^0} = -\varphi_j{}^k \frac{\delta}{\delta x^k}, \qquad \text{(c)} \quad \widetilde{\nabla}_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^0} = 0. \tag{4.16}$$

Then, by direct calculations using (4.16), (2.7b) and (2.8), we obtain

$$g\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^h}} \widetilde{\nabla}_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = g_{jt}\left\{\frac{\delta F_i{}^t{}_k}{\delta x^h} + F_i{}^s{}_k\, F_s{}^t{}_h\right\} - \Phi_{ik}\Phi_{jh},$$

and

$$g\left(\widetilde{\nabla}_{[\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}]} \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 2\Phi_{ij}\Phi_{kh}.$$

Finally, by using (3.4) for both $\widetilde{R}$ and R, we infer that

$$\widetilde{R}_{ijkh} = R_{ijkh} - \Phi_{ik}\Phi_{jh} + \Phi_{ih}\Phi_{jk} - 2\Phi_{ij}\Phi_{kh}.$$

In a coordinate-free setting the above formula becomes

$$\begin{aligned} \widetilde{R}(X, Y, Z, U) = {} & R(X, Y, Z, U) - \Phi(X, U)\Phi(Y, Z) \\ & + \Phi(X, Z)\Phi(Y, U) + 2\Phi(X, Y)\Phi(Z, U), \end{aligned} \tag{4.17}$$

for any $X, Y, Z, U \in \Gamma(\mathcal{D})$, where we set

$$\widetilde{R}(X, Y, Z, U) = \widetilde{R}_{ijkh} X^h Y^k U^j Z^i \text{ and } R(X, Y, Z, U) = R_{ijkh} X^h Y^k U^j Z^i.$$

Thus, in particular, from (4.17) we deduce that the sectional curvatures $\widetilde{K}$ and K of M and $\mathcal{D}$ respectively, are related by

$$\widetilde{K}(X, Y) = K(X, Y) - 3\, \frac{(\Phi(X, Y))^2}{\Delta(X, Y)}. \tag{4.18}$$

This formula enables us to state that contact distributions of constant curvature exist only on K-contact manifolds of constant curvature $k = 1$. First, we prove the following.

Theorem 4.7. *Let M be a 3-dimensional K-contact manifold. Then M is of constant curvature $k = 1$ if and only if its contact distribution is of constant curvature $c = 4$.*

Proof. Since M is a K-contact manifold we have

$$\widetilde{K}(X, \xi) = 1,$$

for any unit vector field $X \in \Gamma(\mathcal{D})$. As the fibers of $\mathcal{D}$ are 2-dimensional, we take $\{X, \varphi X\}$ as orthonormal basis in $\Gamma(\mathcal{D})$. Then, taking into account that $\Phi(X, \varphi X) = -1$, from (4.18) we obtain

$$\widetilde{K}(X, \varphi X) = K(X, \varphi X) - 3,$$

which proves the assertion of the theorem. □

As the standard contact structure on any unit odd sphere is K-contact we deduce the following corollary on the existence of contact distributions of constant curvature.

Corollary 4.8. *The contact distribution of the unit sphere S^3 is of constant curvature $c = 4$.*

The next result shows a striking difference between dimensions 3 and higher than 3.

Theorem 4.9. *Let M be a K-contact manifold of dimension $2m+1$, with $m > 1$, and of constant curvature $k = 1$. Then its contact distribution is never of constant curvature.*

Proof. We choose $\{E_1, \ldots, E_m, \varphi E_1, \ldots, \varphi E_m\}$ as orthonormal basis in $\Gamma(\mathcal{D})$. Then we have

$$\text{(a)} \quad \Phi(E_i, E_j) = 0, \ \text{if } i \neq j \quad \text{and} \quad \text{(b)} \quad \Phi(E_i, \varphi E_i) = -1. \tag{4.19}$$

Thus, by using (4.19) into (4.18), we deduce that

$$K(E_i, E_j) = 1, \ \text{for } i \neq j \quad \text{and} \quad K(E_i, \varphi E_i) = 4.$$

Therefore K cannot be a constant on M. □

Corollary 4.10. *The contact distribution of any unit sphere S^{2m+1}, with $m > 1$, is not of constant curvature.*

Finally, we note that $\mathcal{D}$ is an almost complex distribution with the almost complex structure φ. Thus, as in the case of complex manifolds, it is more appropriate to consider the concept of φ-sectional curvature on $\mathcal{D}$ rather than sectional curvature. This is done in the next section.

5. φ-Sectional curvature of the contact distribution

Let $M(\varphi, \xi, \eta, g)$ be a K-contact manifold and $\mathcal{D}$ be its contact distribution. In the previous section we defined the sectional curvature of $\mathcal{D}$ with respect to the $\mathcal{D}$-planes of $\mathcal{D}$. Taking into account that $(\mathcal{D}, \varphi)$ is an almost complex vector bundle over M, it is interesting to consider $\mathcal{D}$-planes that are invariant with respect to φ. Such a plane at $x \in M$ is spanned by an orthonormal basis $\{X, \varphi X\}$, where $X \in \mathcal{D}_x$, and it is called a *φ-section*. The sectional curvature $K(X, \varphi X)$

is denoted by $H(X)$ and it is called the *φ-sectional curvature* of $\mathcal{D}$ at the point $x \in M$ determined by X. If $H(X)$ is independent of both the vector X and the point $x \in M$, we say that $\mathcal{D}$ is of *constant φ-sectional curvature*.

Now, in order to get more information about this special type of sectional curvature, we use the general theory of sectional curvature of $\mathcal{D}$ that we developed in the previous section. First, from (4.18) we deduce that

$$\widetilde{H}(X) = H(X) - 3, \tag{5.1}$$

where $\widetilde{H}(X)$ is the φ-sectional curvature of M with respect to X. Next, we recall that there exists a well developed theory of Sasakian space forms, which are Sasakian manifolds of constant φ-sectional curvature. Then, from (5.1), we obtain the following.

Theorem 5.1. *Let M be a Sasakian manifold and $\mathcal{D}$ be its contact distribution. Then M is a Sasakian space form of constant φ-sectional curvature c if and only if $\mathcal{D}$ is of constant φ-sectional curvature $c + 3$.*

Taking into account the standard examples of Sasakian space forms: $\mathbb{R}^{2m+1}$, S^{2m+1} and $B^m \times \mathbb{R}$ are of constant φ-sectional curvatures $c = -3$, $c = \frac{4}{\alpha} - 3$, $\alpha > 0$ and $\beta - 3$, $\beta < 0$, respectively (cf. Blair [2], p. 114), we can state the following.

Corollary 5.2. *The contact distributions of $\mathbb{R}^{2m+1}$, S^{2m+1} and $B^m \times \mathbb{R}$ are of constant φ-sectional curvatures 0, $\frac{4}{\alpha}$ and β, respectively.*

Finally, we can prove the following interesting theorem.

Theorem 5.3. *If the φ-sectional curvature of the contact distribution $\mathcal{D}$ at any point of the Sasakian manifold M of dimension $2m+1$, $m > 1$, is independent of φ-section at that point, then it is a constant k on the manifold and the curvature tensor field of Vrănceanu connection is given by*

$$\begin{aligned}
&\text{(a)} \quad R(X,Y)Z = \frac{k}{4}\{g(Y,Z)X - g(X,Z)Y + g(Z,\varphi Y)\varphi X \\
&\qquad\qquad - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\}, \\
&\text{(b)} \quad R(X,Y)\xi = R(X,\xi)Z = R(X,\xi)\xi = 0,
\end{aligned} \tag{5.2}$$

for any $X, Y, Z \in \Gamma(\mathcal{D})$.

Proof. First, by (5.1) we deduce that the φ-sectional curvature of M is independent of the choice of φ-section. Then applying Theorem 7.14 from Blair [2],

p. 113, we deduce that M is a Sasakian space form of constant φ-sectional curvature c. Thus, by Theorem 5.1, $\mathcal{D}$ is of constant φ-sectional curvature $k = c + 3$. Moreover, by using (1.6) into (4.17), we deduce that

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + g(Z,\varphi X)\varphi Y - g(Z,\varphi Y)\varphi X - 2g(X,\varphi Y)\varphi Z,$$
$$\forall X,Y,Z \in \Gamma(\mathcal{D}). \quad (5.3)$$

Then, by using (1.5) in (5.3), we obtain (5.2a). Finally, the equalities in (5.2b) are direct consequences of (3.6b), (3.6c) and (3.7b), since ${F_i}^h{}_j$ are functions of $(x^1,\dots,x^{2m})$ alone. □

In the following examples we present explicitly the Vrănceanu connection on $\mathbb{R}^3$ and S^3.

Example 1. Let $M = \mathbb{R}^3$ endowed with the contact form $\eta = \frac{1}{2}(dz - ydx)$ and the Riemannian metric $g = \eta\otimes\eta + \frac{1}{4}\{(dx)^2 + (dy)^2\}$ (cf. Blair [2], p. 48). Then $\xi = 2\frac{\partial}{\partial z}$ and the contact distribution $\mathcal{D}$ is spanned by the orthonormal basis $\left\{X = 2\frac{\partial}{\partial y}, \varphi X = 2\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)\right\}$. By direct calculations using (1.4) we deduce that the Levi–Civita connection $\widetilde{\nabla}$ on $(\mathbb{R}^3, g)$ satisfies:

$$\widetilde{\nabla}_X X = 0,\ \widetilde{\nabla}_X\varphi X = \xi,\ \widetilde{\nabla}_{\varphi X}X = -\xi,\ \widetilde{\nabla}_{\varphi X}\varphi X = 0,\ \widetilde{\nabla}_\xi\xi = 0. \quad (5.4)$$

Also we have

$$[\xi, X] = [\xi,\varphi X] = 0, \quad [X,\varphi X] = 2\xi.$$

Then, by using (3.1), we deduce that the Vrănceanu connection on $(\mathbb{R}^3, g)$ is given by

$$\nabla_X X = 0,\ \nabla_X\varphi X = 0,\ \nabla_{\varphi X}X = 0,\ \nabla_{\varphi X}\varphi X = 0,\ \nabla_\xi\xi = 0,$$
$$\nabla_\xi X = 0,\ \nabla_X\xi = 0,\ \nabla_\xi\varphi X = 0,\ \nabla_{\varphi X}\xi = 0.$$

Thus $R(X,\varphi X)\varphi X = 0$, and therefore the contact distribution of $\mathbb{R}^3$ is of constant φ-sectional curvature $c = 0$.

Example 2. Consider the 3-dimensional unit sphere

$$S^3 = \{(x^1,x^2,x^3,x^4) \in \mathbb{R}^4 : (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\}$$

isometrically immersed in the Euclidean space $(\mathbb{R}^4, <,>)$. Then

$$\xi = -x^3\frac{\partial}{\partial x^1} - x^4\frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3} + x^2\frac{\partial}{\partial x^4}, \quad (5.5)$$

is a unit tangent vector field on S^3. Take the contact distribution $\mathcal{D}$ as the orthogonal complementary distribution to the distribution span$\{\xi\}$ with respect to the induced Riemannian metric g on S^3. Locally, with respect to a coordinate system on $U = \{(x^1, x^2, x^3, x^4) \in S^3 : x^4 > 0\}$ we deduce that $\mathcal{D}$ is spanned by the orthogonal vector fields

$$X_1 = \alpha\frac{\partial}{\partial x^1} + \beta\frac{\partial}{\partial x^2} + \gamma\,\frac{\partial}{\partial x^4} \quad \text{and} \quad X_2 = -\gamma\frac{\partial}{\partial x^2} + \alpha\frac{\partial}{\partial x^3} + \beta\,\frac{\partial}{\partial x^4}, \tag{5.6}$$

where we put

$$\alpha = (x^2)^2 + (x^4)^2, \quad \beta = -x^1x^2 - x^3x^4, \quad \gamma = x^2x^3 - x^1x^4.$$

Then we have:

$$\text{(a)}\ \ [X_1, X_2] = -\,2\{(x^2)^2 + (x^4)^2\}\xi, \quad \text{(b)}\ \ [X_1, \xi] = X_2, \quad \text{(c)}\ \ [X_2, \xi] = -\,X_1. \tag{5.7}$$

The matrix of the Riemannian metric g with respect to the orthogonal frame field $\{X_1, X_2, \xi\}$ is

$$[g] = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.8}$$

Then, by using (1.4) and (5.5)–(5.8), we deduce that the Levi–Civita connection $\widetilde{\nabla}$ on S^3 satisfies the following:

$$\begin{aligned} \widetilde{\nabla}_{X_1}X_1 &= -\widetilde{\nabla}_{X_2}X_2 = -x^1X_1 + x^3X_2, \\ \widetilde{\nabla}_{X_1}X_2 &= -x^3X_1 - x^1X_2 - \alpha\xi, \\ \widetilde{\nabla}_{X_2}X_1 &= -x^3X_1 - x^1X_2 + \alpha\xi. \end{aligned} \tag{5.9}$$

Finally, the Vrănceanu connection on S^3 follows from (3.1) by using (5.9), (5.7b) and (5.7c):

$$\nabla_{X_1}X_1 = -\nabla_{X_2}X_2 = -x^1X_1 + x^3X_2, \quad \nabla_{X_1}X_2 = \nabla_{X_2}X_1 = -x^3X_1 - x^1X_2,$$

$$\nabla_{X_1}\xi = \nabla_{X_2}\xi = 0, \ \nabla_{\xi}X_1 = -X_2, \ \nabla_{\xi}X_2 = X_1.$$

By direct calculations we deduce that the curvature tensor field R of the Vrănceanu connection satisfies the following

$$R(X_1, X_2)X_2 = 4\alpha X_1. \tag{5.10}$$

The tensor field φ is given by

$$\varphi X_1 = X_2, \quad \varphi X_2 = -X_1, \quad \varphi\xi = 0.$$

Also, we have

$$g(X_1, X_1) = g(\varphi X_1, \varphi X_1) = \alpha. \tag{5.11}$$

Thus, by using (5.10) and (5.11), we conclude that $\mathcal{D}$ is of constant φ-sectional curvature $c = 4$.

Conclusions

In the present paper we succeeded to build on the contact distribution of a K-contact manifold, a Riemannian geometry that formally looks like the Riemannian geometry of a manifold. The main tool in this theory is the Vrănceanu connection which enabled us to define, for the first time in literature, the curvature tensor field of the contact distribution. Our results on the contact distribution $\mathcal{D}$ of constant sectional curvature show that, in this case, the restriction of the curvature tensor field of Vrănceanu connection on $\mathcal{D}$ (i.e. $R(X, Y)Z$, for any $X, Y, Z \in \Gamma(\mathcal{D})$) looks like the curvature tensor field of a real space form (cf. (4.15)). Moreover, if $\mathcal{D}$ is of constant φ-sectional curvature then R is given by the same formula as for the curvature tensor field of a complex space form (cf. (5.2a)). Thus, we are entitled to call R as the curvature tensor field of the contact distribution $\mathcal{D}$. As a conclusion we may say that the Vrănceanu connection on $\mathcal{D}$ plays the same role as the Levi–Civita connection on M.

Next, we analyze a possible interplay between the well known subriemannian geometry of the contact distribution and the ideas stemming from our paper.

First, we note that the curvature of $\mathcal{D}$ in subriemannian geometry (see Montgomery [6], p. 49) is just a 2-form on M whose local components are $2\Phi_{ij}$, where Φ_{ij} are given by (2.7b).

Apart from this curvature, our approach introduces a new curvature tensor field R for $\mathcal{D}$, which enables us to study contact distributions of constant sectional curvature and of constant φ-sectional curvature. Also, we recall that the normal subriemannian geodesics are projections on M of the solutions of a Hamiltonian system of differential equations on the cotangent bundle of M. But there are subriemannian geometries which admit minimizing geodesics that do not come from the solutions of a Hamiltonian system. They are called singular geodesics and live in subriemannian geometries whose distributions are not contact distributions everywhere (see Chapters 3 and 5 in Montgomery [6]).

On the other hand, the whole theory we developed in the first three sections can be easily extended to all distributions of codimension one. Thus an interesting question can be raised: is an autoparallel of the Vrănceanu connection a subriemannian geodesic and viceversa? At this moment, we cannot answer this question.

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