Publ. Math. Debrecen
76/1-2 (2010), 21–30

On some operator algebras generated by unilateral weighted shifts

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Abstract. In this study, we present the necessary and sufficient conditions for the algebra generated by unilateral weighted shifts to be isometrically isomorphic to the polydisc algebra.

1. Introduction and preliminaries

It is well-known that one of the most studied bounded operator on a separable Hilbert space is the unilateral weighted shift operator. A wide description of this operator and some basic results concerning it can be found in [4]. For an excellent introduction to the theory of unilateral weighted shift operators and an extensive bibliography, one can refer to well-known survey article [10]. Here, it is shown that each unilateral weighted shift is unitarily equivalent to the multiplication by the function z on a weighted L^2 or H^2 space. This identification presents interaction between operator theory and analytic function theory. It is a rich source for examples and counter-examples in both operator theory and analytic function theory. Then, this identification is extended to the N-variable unilateral weighted shift operator in [5]. Recently, the unilateral weighted shifts and the algebras generated by these operators are studied by [6], [7], [2]. In [2], it is shown that the necessary and sufficient conditions for the algebra generated by this system is to be isometrically isomorphic to the ball algebra.

Mathematics Subject Classification: 46H35, 47A13, 47B37, 47L75.
Key words and phrases: unilateral weighted shifts, operator algebras, functional Hilbert spaces, polydisc algebra.

In this paper, a functional model of the unilateral weighted shifts system is obtained. By using this functional model, the necessary and sufficient conditions for the algebra generated by this system which is isometrically isomorphic to the polydisc algebra are found.

We proceed with the definitions and notations.

Let $I = (i_1, i_2, \dots, i_N)$ be a multi-index. Following the notation in [5], we write $I \geq 0$ whenever $i_j \geq 0$, $j = 1, 2, \dots, N$. We use the notation $|I| = |i_1 + i_2 + \cdots + i_N|$. For the multi-index I, $I \pm \varepsilon_j$ denotes the multi-index $(i_1, i_2, \dots, i_j \pm 1, \dots, i_N)$, where ε_j is another multi-index $(\delta_{1j}, \delta_{2j}, \dots, \delta_{ij}, \dots, \delta_{Nj})$. Let z^I denote $z_1^{i_1} z_2^{i_2} \dots z_N^{i_N}$ where $z = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N$.

Let $\{e_I\}_{I \geq 0}$ be an orthonormal basis on a separable complex Hilbert space H, and let $\{w_{I,j}\}_{I \geq 0,\ 1 \leq j \leq N}$ be a bounded net of complex numbers. Denote by A_j the bounded linear operators whose actions on the elements of the basis $\{e_I\}_{I \geq 0}$ of H are given as $A_j e_I = w_{I,j} e_{I+\varepsilon_j}$, $1 \leq j \leq N$. A family of N operators, denoted by $A = (A_1, A_2, \dots, A_N)$, is called a system of unilateral weighted shifts, and the numbers of $\{w_{I,j}\}_{I \geq 0,\ 1 \leq j \leq k}$ are called the weights of the system. Hereafter, we will refer to the system of N-variable unilateral weighted shift A as the system A with weights $w_{I,j}$. The necessary condition for establishing the functional model of a system is that the operators in the system should commute. As is shown in [5], the necessary and sufficient condition for $A = (A_1, A_2, \dots, A_N)$ to be commutative is

$$w_{I,j} w_{I+\varepsilon_j,i} = w_{I,i} w_{I+\varepsilon_i,j} \quad \forall I, \quad 1 \leq i, j \leq N.$$

In addition, unless stated otherwise, we assume that A is a commutative system with weights $w_{I,j}$. Furthermore, since the system A generated by $w_{I,j} \neq 0$, $j = 1, 2, \dots, N$ is unitarily equivalent to the system B generated by $|w_{I,j}|$ for arbitrary multi-index $I \geq 0$, without loss of generality, we may consider the case, when the weights of system A positive. With the aid of positive $\{w_{I,j}\}_{I \geq 0}$, we define $\beta_{I+\varepsilon_j} = w_{I,j} \beta_I$ with $\beta_0 = 1$. The fact that $\{\beta_I\}_{I \geq 0}$ is well-defined is shown in [5]. We define $H^2(\beta)$ with the help of $\{\beta_I\}_{I \geq 0}$ in the following manner:

$$H^2(\beta) = \left\{ f(z) : f(z) = \sum_{I \geq 0} f_I z^I,\ f_I \in \mathbb{C},\ \sum_{I \geq 0} |f_I|^2 \beta_I^2 < \infty \right\}.$$

It is clear that $H^2(\beta)$ is a Hilbert space and $\left\{ \frac{z^I}{\beta_I} \right\}_{I \geq 0}$ is an orthonormal basis for $H^2(\beta)$. The multiplication operators, with the independent variables z_j, in $H^2(\beta)$ and A_z are respectively defined as $A_{z_j} \frac{z^I}{\beta_I} = z_j \frac{z^I}{\beta_I}$ $(j = 1, 2, \dots, N)$, $A_z = (A_{z_1}, A_{z_2}, \dots, A_{z_N})$.

The system A_z and the system A generated by $w_{I,j}$ are unitarily equivalent.

The open unit disc in $\mathbb{C}$, whose boundary is the circle $\mathbb{T}$, is denoted by Δ and its closure by $\bar{\Delta}$. The polydisc Δ^N is a subset of $\mathbb{C}^N$, which are cartesian products of N copies of Δ. The closure of Δ^N is denoted by $\bar{\Delta}^N$. It is possible to represent $\bar{\Delta}$ as the cartesian product of $[0,1]$ and $\mathbb{T}$, i.e. $\bar{\Delta} = [0,1] \times \mathbb{T}$. As a consequence of this representation, $\bar{\Delta}^N = ([0,1] \times \mathbb{T})^N$. And, any element of $\bar{\Delta}^N$ is represented by $z = (z_1, z_2, \ldots, z_N) \in \bar{\Delta}^N$, $z_j = r_j e^{i\theta_j}$, $j = 1, 2, \ldots, N$ where $r_j \in [0,1]$ and $\theta_j \in [0, 2\pi)$.

Let $\mathcal{A}(\Delta^N)$ denote the polydisc algebra that is the class of all continuous complex functions on the closure $\bar{\Delta}^N$ of Δ^N whose restriction to Δ^N is analytic.

2. Functional Hilbert spaces

It is clear that $H^2(\beta)$ is determined by the weights of the system A. Now, we choose the weights of the system A in such a way that $H^2(\beta)$ is a functional space. For convenience, we restate here the definition of a functional Hilbert space as given in [1, p. 19]: a functional Hilbert space H of complex-valued functions on a set X; the Hilbert space structure of H is related to X in two ways. It is required that (1) if f and g are in H and if α and β are scalars, then $(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$ for each x in X, that is the evaluation functionals on H are linear and (2) to each x in X there corresponds a positive constant γ_x such that $|f(x)| \leq \gamma_x \|f\|$ for all f in H, that is the evaluation functionals on H are bounded.

Let ν be a positive, regular, probability Borel measure on $[0,1]^N = [0,1] \times \cdots \times [0,1] (N$ times). Then the measure μ is given on $\bar{\Delta}^N$ by

$$d\mu = \frac{1}{(2\pi)^N} d\nu(r_1, r_2, \ldots, r_N) d\theta_1 d\theta_2 \ldots d\theta_N. \tag{2.1}$$

Let Ω denote the family of the systems A such that μ be a measure defined as in (2.1) in $\bar{\Delta}^N$ so that $\left\{\frac{z^I}{\beta_I}\right\}_{I \geq 0} \in L^2(\bar{\Delta}^N, \mu)$ and $\left\{\frac{1}{\sqrt{(2\pi)^N}} \frac{z^I}{\beta_I}\right\}_{I \geq 0}$ is an orthonormal system in $L^2(\bar{\Delta}^N, \mu)$, where $L^2(\bar{\Delta}^N, \mu)$ is the space of complex-valued functions on $\bar{\Delta}^N$, which are Lebesgue measurable and square integrable with respect to measure μ. It is possible to show that the multi-variable moment problem

$$\beta_I^2 = \int_{[0,1]^N} r_1^{2i_1} r_2^{2i_2} \ldots t_N^{2i_N} d\nu(r_1, r_2, \ldots, r_N)$$

has a solution for β_I^2's corresponding to the system A included in the class Ω. That is, a measure ν defined on $[0,1]^N$ exists for β_I^2's. Let ν_A denote the measure corresponding to the system A. The so-called multi-variable moment problem is studied in [9].

Now, let $H^2(\bar{\Delta}^N,\mu)$ be a subspace generated by the orthonormal system $\{\frac{z^I}{\beta_I}\}_{I\geq 0} \in L^2(\bar{\Delta}^N,\mu)$. Then we have

$$H^2(\bar{\Delta}^N,\mu) = H^2(\mu) = \left\{ f : f = \sum_{I\geq 0} f_I z^I, \quad z \in \Delta^N, \quad \sum_{I\geq 0} |f_I|^2 \beta_I^2 < \infty \right\}.$$

Moreover, let Ω' be a subset of Ω defined by

$$\Omega' = \{A \in \Omega : \nu_A(U(1,1,\dots,1)) > 0 \text{ for arbitrary neighborhood } U(1,1,\dots,1) \text{ of the point } (1,1,\dots,1) \in [0,1]^N\}.$$

Theorem 1. *If $A \in \Omega'$, then $H^2(\mu)$ is a functional space.*

Proof. Consider the region $G = \{(z_1,z_2,\dots,z_N) \in \mathbb{C}^N : |z_i| \leq g_i < 1 \ \ i = 1,2,\dots,N\}$. Since there exist u_i's such that $g_i < u_i < 1$ for arbitrary g_i, we can write

$$\begin{aligned} \beta_I^2 &= \int_{[0,1]^N} r_1^{2i_1} r_2^{2i_2} \dots r_N^{2i_N}\, d\nu(r_1,r_2,\dots,r_N) \\ &\geq u_1^{2i_1} u_2^{2i_2} \dots u_N^{2i_N}\, d\nu(r_1',r_2',\dots,r_N'), \quad r_i' \in [u_i,1]. \end{aligned}$$

Since $\|f\| = \sum_{I\geq 0} |f_I|^2 \beta_I^2$ converges, there exists an M such that $|f_I|^2\beta_I^2 < M^2$ for all but finitely many multi-index $I \geq 0$.

The series

$$\begin{aligned} |f(z)| &= \left| \sum_{I\geq 0} f_I z^I \right| \leq \sum_{I\geq 0} |f_I| \frac{z^I}{\beta_I} \beta_I \\ &\leq \sum_{I\geq 0} |f_I| \beta_I \left(\frac{g_1}{u_1}\right)^{i_1} \dots \left(\frac{g_N}{u_N}\right)^{i_N} \frac{1}{\sqrt{d\nu(r_1',r_2',\dots,r_N')}}. \end{aligned} \tag{2.2}$$

converges in the compact region G. Since the series (2.2) converges on arbitrary compact subset of Δ^N, $f(z)$ is an analytic function according to Weierstrass Theorem [8, p. 38]. The next thing to be done is to verify the second condition in the definition of the functional Hilbert space. Let us define the linear functional τ_{z_0} as $\tau_{z_0}(f) = f(z_0)$ for arbitrary $z_0 = (z_{10},\dots,z_{N0}) \in \Delta^N$. In the Hilbert space

$$\ell^2(\beta) = \left\{ \{a_I\}_{I\geq 0} : a_I \in \mathbb{C}, \quad \sum_{I\geq 0} |a_I|^2 \beta_I^2 < \infty \right\},$$

using Schwartz's inequality $|(a_I, b_I)| \leq \|a_I\| \, \|b_I\|$, we obtain the inequality

$$|\tau_{z_0}(f)| = |f(z_0)| = \left|\sum_{I\geq 0} f_I z_0^I\right| = \left|\sum_{I\geq 0} f_I \frac{z_0^I}{\beta_I^2}\beta_I^2\right|$$
$$\leq \sum_{I\geq 0} |f_I|^2 \beta_I^2 \sum_{I\geq 0} \left(\frac{|\overline{z_0}|^I}{\beta_I^2}\right)^2 \beta_I^2 = \gamma_{z_0}\|f\|.$$

where $\gamma_{z_0} = \sum_{I\geq 0}\big(\frac{|\overline{z_0}|^I}{\beta_I}\big)^2$. The proof of the convergence of $\sum_{I\geq 0}\big(\frac{|\overline{z_0}|^I}{\beta_I}\big)^2$ is similar to that of previous one. □

On the other hand, the multiplication operators, with the independent variables z_j's, are defined in $H^2(\mu)$ as $A_{z_j}\varphi = z_j\varphi$, $\forall\varphi \in H^2(\mu)$. We note that, the multiplication operators, with the independent variables z_j's, in $H^2(\beta)$ and the multiplication operators, with the independent variables z_j's, in $H^2(\mu)$ are unitarily equivalent.

3. Main theorem

Theorem 2. *Let $A \in \Omega$. A necessary and sufficient condition for the operator algebra generated by the system A to be isometrically isomorphic to the polydisc algebra is that A belongs to Ω'.*

Proof. Without loss of generality, we can take $N = 2$. Let $A \in \Omega'$. Since the polynomials are dense in the polydisc algebra [9, p. 22], it is enough to show that

$$\|P(A_1, A_2)\|_{B(H^2(\mu))} = \max_{(z_1,z_2)\in\bar{\Delta}^2} |P(z_1, z_2)|$$

where P is a two-variable polynomial.

Since the system $A = (A_1, A_2)$ and the system $A_z = (A_{z_1}, A_{z_2})$ are unitary equivalent,

$$\|P(A_1, A_2)\| \leq \max_{(z_1,z_2)\in\bar{\Delta}^2} |P(z_1, z_2)|. \tag{3.1}$$

Now, consider the polynomial $P_{mn}(z_1, z_2) = \big(\frac{1+z_1}{2}\big)^m \big(\frac{1+z_2}{2}\big)^n$ and let D_1, D_2 be defined by

$$D_1 = \left\{(z_1, z_2) : |z_i| \leq 1, \ \left|\frac{1+z_i}{2}\right| \leq 1 - \eta_i, \ i = 1, 2\right\}$$

$$D_2 = \bar{\Delta}^2 \backslash D_1$$

where $0 < \eta_i < 1$, $i = 1, 2$. Since $\bar{\Delta}^2 = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$, it follows that

$$\begin{aligned}
\|P_{mn}(z_1, z_2)\|_{H^2}^2 &= \int_{\bar{\Delta}^2} \left|\frac{1+z_1}{2}\right|^{2m} \left|\frac{1+z_2}{2}\right|^{2n} d\nu(r_1, r_2) d\theta_1 d\theta_2 \\
&= \int_{D_1} \left|\frac{1+z_1}{2}\right|^{2m} \left|\frac{1+z_2}{2}\right|^{2n} d\nu(r_1, r_2) d\theta_1 d\theta_2 \\
&\quad + \int_{D_2} \left|\frac{1+z_1}{2}\right|^{2m} \left|\frac{1+z_2}{2}\right|^{2n} d\nu(r_1, r_2) d\theta_1 d\theta_2,
\end{aligned}$$

$$\mathfrak{a} \overset{\text{def}}{=} \int_{D_1} \left|\frac{1+z_1}{2}\right|^{2m} \left|\frac{1+z_2}{2}\right|^{2n} d\nu(r_1, r_2) d\theta_1 d\theta_2 \le (1-\eta_1)^{2m} (1-\eta_2)^{2n} (2\pi)^2. \tag{3.2}$$

It is clear that

$$\begin{aligned}
D_2 = \bar{\Delta}^2 \backslash D_1 &\supset \left\{(z_1, z_2) : |z_i| \le 1,\ \left|\frac{1+z_i}{2}\right| > 1 - \eta_i,\ i = 1, 2\right\} \\
&\supset \left\{(z_1, z_2) : |z_i| \le 1,\ 1 - \frac{\eta_i}{2} < |z_i| \le 1, -\gamma_i < arg z_i < \gamma_i,\ i = 1, 2\right\}
\end{aligned}$$

where

$$\gamma_i = \arccos \left[\frac{3}{2}\Big(1 - \frac{\eta_i}{2}\Big) - \frac{1}{2\big(1 - \frac{\eta_i}{2}\big)}\right], \quad (i = 1, 2).$$

Furthermore, we have that $\gamma_i \simeq \sqrt{2\eta_i}$ $(i = 1, 2)$ for sufficiently small values of η_i. On the other hand, we obtain

$$\begin{aligned}
\mathfrak{b} &\overset{\text{def}}{=} \int_{D_2} \left|\frac{1+z_1}{2}\right|^{2m} \left|\frac{1+z_2}{2}\right|^{2n} d\nu(r_1, r_2) d\theta_1 d\theta_2 \\
&\ge \Big(1 - \frac{\eta_1}{2}\Big)^{2m} \Big(1 - \frac{\eta_2}{2}\Big)^{2n} 8\sqrt{\eta_1 \eta_2}\ \nu(R'')
\end{aligned} \tag{3.3}$$

where $R'' = \{(r_1'', r_2'') : r_i'' \in \big(1 - \frac{\eta_i}{2}, 1\big]\ i = 1, 2\}$.

Comparing (3.2) and (3.3), we see that $\mathfrak{a} = o(\mathfrak{b}) \binom{m \to \infty}{n \to \infty}$ and consequently

$$\|P_{mn}(z_1, z_2)\| \approx \mathfrak{b}.$$

In $\bar{\Delta}^2$, $|P(z_1, z_2)|$ takes its maximum value on T^2 [9, p. 21]. Suppose that $|P(z_1, z_2)|$ takes its maximum value at the point $(1, 1)$. (Otherwise, if $|P(z_1, z_2)|$ takes its maximum value at the point (z_{10}, z_{20}), then it is possible to define a new polynomial as $\tilde{P}(z_1, z_2) = P(z_1 z_{10}, z_2 z_{20})$). Then, for $\forall \delta > 0$, $\exists \eta > 0$ such that $\forall (z_1, z_2) \in D$,

$$\Big| |P(z_1, z_2)| - \max_{(z_1, z_2) \in \bar{\Delta}^2} |P(z_1, z_2)| \Big| < \delta$$

where

$$D = \Big\{(z_1, z_2) : |z_i| \le 1,\ \Big|\frac{1+z_i}{2}\Big| > 1-\eta,\ i = 1,2\Big\}.$$

From the definition of the norm, the equation follows.

$$\begin{aligned}\|P(A_1, A_2)P_{mn}(z_1, z_2)\|^2_{H^2(\mu)} = &\int_{D_1} |P(z_1, z_2)|^2 |P_{mn}(z_1, z_2)|^2 d\nu(r_1, r_2) d\theta_1 d\theta_2 \\ &+ \int_{D_2} |P(z_1, z_2)|^2 |P_{mn}(z_1, z_2)|^2 d\nu(r_1, r_2) d\theta_1 d\theta_2.\end{aligned}$$

The results obtained from the following inequalities can be proved in a way similar to that given above.

$$\begin{aligned}\mathfrak{a}' &\stackrel{\text{def}}{=} \int_{D_1} |P(z_1, z_2)|^2 |P_{mn}(z_1, z_2)|^2 d\nu(r_1, r_2) d\theta_1 d\theta_2 \\ &\le \Big(\max_{(z_1,z_2)\in\bar{\Delta}^2} |P(z_1, z_2)| + \delta\Big)^2 (1-\eta_1)^{2m}(1-\eta_2)^{2n}(2\pi)^2 \end{aligned} \tag{3.4}$$

$$\begin{aligned}\mathfrak{b}' &\stackrel{\text{def}}{=} \int_{D_2} |P(z_1, z_2)|^2 |P_{mn}(z_1, z_2)|^2 d\nu(r_1, r_2) d\theta_1 d\theta_2 \\ &\ge \Big(\max_{(z_1,z_2)\in\bar{\Delta}^2} |P(z_1, z_2)| - \delta\Big)^2 \Big(1-\frac{\eta_1}{2}\Big)^{2m}\Big(1-\frac{\eta_2}{2}\Big)^{2n} 8\sqrt{\eta_1\eta_2}\ \nu(R''). \end{aligned} \tag{3.5}$$

Comparing (3.4) and (3.5), we see that $\mathfrak{a}' = o(\mathfrak{b}')\binom{m\to\infty}{n\to\infty}$ and consequently

$$\|P(A_1, A_2)P_{mn}(z_1, z_2)\| \approx \mathfrak{b}'.$$

For arbitrary $\delta > 0$

$$\|P(A_1, A_2)P_{mn}(z_1, z_2)\|^2 \ge \Big(\max_{(z_1,z_2)\in\bar{\Delta}^2} |P(z_1, z_2)| - \delta\Big)^2 \|P_{mn}(z_1, z_2)\|^2$$

from which it follows that

$$\|P(A_1, A_2)\| \ge \max_{(z_1,z_2)\in\bar{\Delta}^2} |P(z_1, z_2)|. \tag{3.6}$$

(3.1) and (3.6) imply that

$$\|P(A_1, A_2)\| = \max_{(z_1,z_2)\in\bar{\Delta}^2} |P(z_1, z_2)|.$$

This equality shows that the operator algebra generated by the system A is a polydisc algebra.

Conversely, assume that $A \in \Omega$ and that the operator algebra generated by the system A is isometrically isomorphic to polydisc algebra. In this case

$$\|P(A_1, A_2)\| = \max_{(z_1,z_2)\in\bar{\Delta}^2} |P(z_1, z_2)|. \tag{3.7}$$

Now, suppose that the point $(1,1)$ has a neighborhood $U_\delta(1,1)$ such that the measure ν of the neighborhood is zero i.e. $\nu(U_\delta(1,1)) = 0$. Then, there exist $g_1, g_2 < 1$ such that

$$\operatorname{supp}\nu \subset [0,1]^2 \backslash \{(r_1, r_2) : 1 - g_i < r_i \le 1,\ i = 1,2\}. \tag{3.8}$$

According to (3.8), for $\forall f, g \in H^2(\mu)$

$$(f, g) = \int_{D'} f(z)\overline{g(z)}\, d\nu(r_1, r_2) d\theta_1 d\theta_2$$

where

$$D' = \bar{\Delta}^2 \backslash \{(z_1, z_2) : 1 - g_i < |z_i| \le 1,\ i = 1,2\}$$

from which it follows that

$$\|P(A_1, A_2)\| \le \max_{(z_1,z_2)\in D'} |P(z_1, z_2)|$$

which contradicts with (3.7). Therefore $A \in \Omega'$. □

Corollary 1. *Let $A \in \Omega$. If $\lim_{|I|\to\infty} \sqrt[|I|]{\beta_I} = 1$, then the operator algebra generated by system A is isometrically isomorphic to polydisc algebra.*

Proof. It suffices to show that for every neighborhood U of the point $(1, 1, \ldots, 1)$, $\nu(U(1, 1, \ldots, 1)) > 0$. If $\nu(U(1, 1, \ldots, 1)) = 0$, there exists real numbers g_i such that $\nu([g_1, 1] \times [g_2, 1] \cdots \times [g_N, 1]) = 0$, where $0 < g_i < 1$, $i = 1, 2, \ldots, N$. Therefore, we have

$$\beta_I^2 = \int_{[0,1]^N} r_1^{2i_1} r_2^{2i_2} \ldots r_N^{2i_N}\, d\nu(r_1, r_2, \ldots, r_N)$$

$$\beta_I^2 \le g_1^{2i_1} g_2^{2i_2} \ldots g_N^{2i_N}\, d\nu([0,1]^N).$$

Set $g = \max\{g_1, g_2, \ldots, g_N\}$.

$$\beta_I^2 \le g^{2(i_1+i_2+\cdots+i_N)} \Longrightarrow \beta_I \le g^{i_1+i_2+\cdots+i_N}.$$

Taking the upper limit of β_I, we have

$$\limsup_{|I|\to\infty} \sqrt[|I|]{\beta_I} \le g < 1$$

which contradicts with the assumption. Thus, A must belong to Ω'. □

It is noted that the theorems in our paper can be stated by using the Reinhardt measure. In Theorem 1.9 [1], Curto and Yan prove that, in the case of pairs of operators, the interior of the polynomially convex hull of the support of the measure is always in the set of bounded point evaluations. By using this result, our theorems can be given as follows;

Theorem 1'. *If the Reinhardt measure contains the point* $(1, 1, \dots, 1)$ *in its support, then the evaluation functionals are bounded.*

Theorem 2'. *The operator algebra generated by the system* A *is isometrically isomorphic to the polydisc algebra if and only if the point* $(1, 1, \dots, 1)$ *is in the support of the Reinhardt measure.*

Now, we will continue a few examples.

Example 1. If $w_{I,j} = 1$ for all I and j, then the system $A = (A_1, A_2, \dots, A_N)$ with weights $w_{I,j}$ is in Ω'. $\beta_I = 1$ and $\lim_{|I|\to\infty} \sqrt[|I|]{\beta_I} = 1$.

Example 2. The system $A = (A_1, A_2, \dots, A_N)$ with weights

$$w_{I,j} = \begin{cases} 1/2, & i_1 = i_2 = \cdots = i_N = 0 \\ 1, & \text{otherwise} \end{cases}$$

is in Ω'. $\beta_I = \frac{1}{2}$ and $\lim_{|I|\to\infty} \sqrt[|I|]{\beta_I} = 1$.

Example 3. Let $N = 2$. The system $A = (A_1, A_2)$ with weights

$$w_{(i_1,i_2),1} = \sqrt{\frac{i_1 + 1}{i_1 + i_2 + 2}}, \quad w_{(i_1,i_2),2} = \sqrt{\frac{i_2 + 1}{i_1 + i_2 + 2}} \quad \text{for all } I = (i_1, i_2)$$

is in Ω but it is not in Ω'. (In fact $A \in \Omega'$ in [2]. Intersection of the sets Ω' and Ω' in [2] is empty). $\beta_I = \sqrt{\frac{i_1! \cdot i_2!}{(i_+ i_2 + 1)!}}$ and $\lim_{|I|\to\infty} \sqrt[|I|]{\beta_I} \neq 1$.

Acknowledgment. We thank the referees for clarifying some details and improving the presentation of this work.

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(Received May 19, 2008; revised October 3, 2009)