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Common fixed points for Ćirić type f-weak contraction with applications

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Abstract. We introduce a new conception of Ćirić type f-weakly contractive mappings and the existence of common fixed points is established for Ćirić type f-weakly contractive mapping T. As an application, the existence of solution of variational inequalities is obtained. Our results unify and improve several recent results existing in the current literature.

1. Introduction and preliminaries

Let (X,d) be a metric space. A map $T : X \to X$ is called to be *weakly contractive* [1], [31] if, for each $x, y \in X$,

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)),$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous function from the right such that φ is positive on $(0,\infty)$ and $\varphi(0) = 0$.

We will say that a mapping $T : X \to X$ is *f-weakly contractive* if, for each $x, y \in X$,

$$d(Tx,Ty) \leq d(fx,fy) - \varphi(d(fx,fy)), \tag{1}$$

where $f : X \to X$ is a self-mapping and $\varphi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous function from the right such that φ is positive on $(0,\infty)$ and $\phi(0) = 0$.

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If $\varphi(t) = (1-k)t$, $0 < k < 1$, then a f-weakly contractive mapping is called a *f-contraction.* Note that if $f = I$ and φ is continuous non-decreasing, then the definition of f-weakly contractive mapping is the same as it appeared in [1], [31]. Further if $f = I$ and $\varphi(t) = (1-k)t$, $0 < k < 1$, then a f-weakly contractive mapping is called a *contraction.* Also note that if $f = I$ and φ is lower semicontinuous from the right then $\psi(t) = t - \varphi(t)$ is upper semicontinuous from the right and condition (1) is replaced by

$$d(Tx, Ty) \le \psi(d(x, y)). \tag{2}$$

Therefore, f-weakly contractive maps for which φ is lower semicontinuous from the right are of Boyd and Wong [9] type. Further, if we define $k(t) = 1 - \frac{\varphi(t)}{t}$ for $t > 0$ and $k(0) = 0$ together with $f = I$, then condition (1) is replaced by

$$d(Tx, Ty) \le k(d(x, y))d(x, y). \tag{3}$$

Therefore f-weakly contractive maps are closely related to maps of Reich [30] type, which are also generally researched by Bae [4] and Mizoguchi and Takahashi [23].

The set of fixed points of T we shall denote by $F(T)$. A point $x \in X$ is a coincidence point (common fixed point) of f and T if $fx = Tx$ ($x = fx = Tx$). The set of coincidence points of f and T is denoted by $C(f, T)$. The pair $\{f, T\}$ is called (1) *commuting* [19] if $Tfx = fTx$ for all $x \in X$, (2) *compatible* (see [20], [21]) if $\lim_n d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n fx_n = t$ for some t in X; (3) *weakly compatible* [20] if they commute at their coincidence points, that is, if $fTx = Tfx$ whenever $fx = Tx$; (4) *R-weakly commuting* [25] if there exists an $R > 0$ such that $d(fTx, Tfx) \le R\, d(Tx, fx)$ for all $x \in X$; (5) *pointwise R-weakly commuting* if for given $x \in X$, there exists an $R > 0$ such that $d(fTx, Tfx) \le R\, d(Tx, fx)$ holds.

It was proved in [26] that pointwise R-weak commutativity is equivalent to commutativity at coincidence points; that is, f and T are pointwise R-weakly commuting if and only if they are weakly compatible.

We denote by $\mathbb{N}$ and $cl(M)$, the set of positive integers and the closure of a set M in X, respectively.

The concept of the weakly contractive mapping is defined by Alber and Guerre-Delabriere [1] in 1997. Actually, the authors in [1] proved the existence of fixed points for single-valued weakly contractive mapping on Hilbert spaces. In 2001, Rhoades ([31], Theorem 2) proved the very interesting fixed point theorem which is one of generalizations of Banach's Contraction Mapping

Principle, because the weakly contractions contains contractions as the special cases ($\varphi(t) = (1-k)t$), and also showed that most results of [1] are still true for any Banach space. In fact, weakly contractive mappings are closely related to maps of Boyd and Wong type ones [9] and Reich's type ones [30] (see [32], [34]).

In this paper, we introduce a new conception of Ćirić type f-weakly contractive mappings, and consequently establish the common fixed point results for weakly compatible Ćirić type f-weakly contractive mappings. As applications, we establish common fixed point results for a Banach operator pair and the existence of solution of variational inequalities is obtained. Our results improve and extend the recent common fixed point results of Al-Thagafi and Shahzad [2], [3], Beg and Abbas [5], Chen and Li [10], Ćirić [11], Das and Naik [15], Jungck [19], Jungck and Hussain [21], O'Regan and Hussain [21], Pant [25], Pathak and Hussain [28], and Song [32]

2. Common fixed point results

The Banach Contraction Mapping Principle states that if (X, d) is a complete metric space, K is a nonempty closed subset of X and $T : K \to K$ is a self-mapping satisfying $d(Tx, Ty) \le \lambda d(x, y)$ for all $x, y \in K$, where $0 < \lambda < 1$, then T has a unique fixed point, say z in K, and the Picard iterations $\{T^n x\}$ converge to z for all $x \in K$. Ćirić [11] introduced and studied self-mappings on K satisfying

$$d(Tx, Ty) \le \lambda m(x, y),$$

where $0 < \lambda < 1$ and

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Further investigations were developed by Berinde [7], Ćirić [12], Jungck [20], Jungck and Hussain [21], O'Regan and Hussain [24] and many other mathematicians(see [12] and references therein). Application of the contraction and generalized contraction principle for self-mappings are well known (c.f. [6], [12], [27], [28]).

We begin with the following result.

Theorem 2.1. *Let K be a subset of a metric space (X, d) and let f and T be a self-mappings of K. Assume that $clT(K) \subset f(K)$, $clT(K)$ is complete, f and T satisfy the following condition:*

$$d(Tx, Ty) \le M(x, y) - \varphi(M(x, y)) \tag{4}$$

for all $x, y \in K$, *where*

$$M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\} \qquad (5)$$

and $\varphi : [0,+\infty) \to [0,+\infty)$ *is a real function such that*

(i) $\varphi(t) > 0$ *for all* $t > 0$,

(ii) $\lim_{s\to t+} \varphi(s) > 0$ *for all* $t > 0$,

(iii) $t - \varphi(t)$*is non-decreasing,*

(iv) $\lim_{t\to\infty} \varphi(t) = +\infty$.

Then T *and* f *have a unique coincidence point in* K. *If, in addition,* (f,T) *is weakly compatible, then* $K \cap F(T) \cap F(f)$ *is a singleton.*

Proof. Let $x_0 \in K$ be arbitrarily. As $T(K) \subset f(K)$, one can choose x_1 in K, such that $fx_1 = Tx_0$. Consider now Tx_1. Since $Tx_1 \in f(K)$, there exists x_2 in K such that $fx_2 = Tx_1$. By induction, we construct a sequence $\{x_n\}$ of points in K such that

$$fx_{n+1} = Tx_n \quad \text{for } n \in \{0,1,2,3,\dots\}.$$

Denote

$$O(x_0,n) = \{Tx_0, Tx_1, Tx_2, \dots, Tx_n\}, \qquad (6)$$

$$O(x_0) = \{Tx_0, Tx_1, Tx_2, \dots, Tx_n, \dots\}. \qquad (7)$$

First we shall show that for any given $x_0 \in K$, the set $O(x_0)$ is bounded.

Let n be any fixed positive integer. We shall show that

$$\delta_n(x_0) = \operatorname{diam}(\{Tx_0, Tx_1, Tx_2, \dots, Tx_n\}) = d(Tx_0, Tx_k), \qquad (8)$$

where $k = k(n) \le n$ is a positive integer. Suppose, to the contrary, that there are positive integers $i = i(n) \ge 1$ and $j = j(n) \ge 1$ such that

$$\delta_n(x_0) = d(Tx_i, Tx_j). \qquad (9)$$

Without loss of generality we may suppose that $i < j$.

Assume that $\delta_n(x_0) > 0$ and that $i \ge 1$. Then $Tx_{i-1} \in O(x_0,n)$. Since $Ix_{n+1} = Tx_n$, from (5) with $x = x_i$ and $y = x_j$ we have

$$\begin{aligned} M(x_i,x_j) &= \max\{d(fx_i,fx_j), d(fx_i,Tx_i), d(fy_j,Tx_j), d(fx_i,Tx_j), d(fx_j,Tx_i)\} \\ &= \max\{d(Tx_{i-1},Tx_{j-1}), d(Tx_{i-1},Tx_i), d(Tx_{j-1},Tx_j), d(Tx_{i-1},Tx_j), \\ &\qquad d(Tx_{j-1},Tx_i)\} \le \delta_n(x_0). \end{aligned}$$

Thus from (4), (iii) and (i) we have

$$\delta_n(x_0) = d(Tx_i, Tx_j) \leq M(x_i, x_j) - \varphi(M(x_i, x_j))$$
$$\leq \delta_n(x_0) - \varphi(\delta_n(x_0)) < \delta_n(x_0),$$

a contradiction. Therefore, our assumption (9) is wrong. Thus (8) holds.

Since by the triangle inequality,

$$d(Tx_0, Tx_k) \leq d(Tx_0, Tx_1) + d(Tx_1, Tx_k),$$

from (8),

$$\delta_n(x_0) \leq d(Tx_0, Tx_1) + d(Tx_1, Tx_k). \tag{10}$$

Since from (4),

$$d(Tx_1, Tx_k) \leq M(x_1, x_k) - \varphi(M(x_1, x_k)),$$

and as $M(x_1, x_k) \leq \delta_n(x_0)$, from (iii) we have

$$d(Tx_1, Tx_k) \leq \delta_n(x_0) - \varphi(\delta_n(x_0)).$$

Now, by (10),

$$\delta_n(x_0) \leq d(Tx_0, Tx_1) + \delta_n(x_0) - \varphi(\delta_n(x_0)).$$

Hence

$$\varphi(\delta_n(x_0)) \leq d(Tx_0, Tx_1). \tag{11}$$

Since the sequence $\{\delta_n(x_0)\}$ is non-decreasing, there exists $\lim \delta_n(x_0)$. Suppose that $\lim \delta_n(x_0) = +\infty$. Then (iv) implies that the left-hand side of (11) becomes unbounded when n tends to infinity, but the right-hand side is bounded, a contradiction. Therefore, $\lim_{n\to\infty} \delta_n(x_0) = \delta(x_0) < +\infty$, that is,

$$\delta(x_0) = \operatorname{diam}(\{Tx_0, Tx_1, Tx_2, \dots, Tx_n, \dots\}) < +\infty. \tag{12}$$

Now we show that $\{Tx_n\}$ is a Cauchy sequence. Set

$$\delta(x_n) = \operatorname{diam}(\{Tx_n, Tx_{n+1}, \dots\})$$

$(n = 0, 1, 2, \dots)$. Since $\delta(x_n) \leq \delta(x_0)$, then by (12) we conclude that $\{\delta(x_n)\}$ is a sequence of finite nonnegative numbers. Since $\delta(x_{n+1}) \leq \delta(x_n)$, it follows that $\{\delta(x_n)\}$ converges to some $\delta \geq 0$ and $\delta \leq \delta(x_n)$ for all $n \geq 0$. We shall prove that $\delta = 0$. Let n be arbitrary and let r, s be any positive integers such that

$r, s \geq n+1$. Then $Tx_{r-1}, Tx_{s-1} \in \{Tx_n, Tx_{n+1}, \dots\}$ and hence we conclude that $M(x_r, x_s) \leq \delta(x_n)$. From (4),

$$d(Tx_r, Tx_s) \leq M(x_r, x_s) - \varphi(M(x_r, x_s)),$$

and then by (iii),

$$d(Tx_r, Tx_s) \leq \delta(x_n) - \varphi(\delta(x_n)).$$

Hence we get

$$\sup\{d(Tx_r, Tx_s) : r \geq n+1;\ s \geq n+1\} \leq \delta(x_n) - \varphi(\delta(x_n)).$$

Therefore,

$$\delta(x_{n+1}) = \sup\{d(Tx_r, Tx_s) : r \geq n+1;\ s \geq n+1\} \leq \delta(x_n) - \varphi(\delta(x_n)).$$

Hence, as $\delta \leq \delta(x_n)$ for all $n \geq 0$,

$$\delta \leq \delta(x_n) - \varphi(\delta(x_n)). \tag{13}$$

Suppose that $\delta > 0$. Then letting n tends to infinity in (13) we get

$$\delta \leq \delta - \lim_{n\to\infty} \varphi(\delta(x_n)) = \delta - \lim_{\delta(x_n)\to\delta+} \varphi(\delta(x_n)).$$

Hence we have

$$\lim_{\delta(x_n)\to\delta+} \varphi(\delta(x_n)) \leq 0,$$

a contradiction with (ii). Therefore, $\delta = 0$. Thus, we have proved that

$$\lim_{n\to\infty} \operatorname{diam}(\{Tx_n, Tx_{n+1}, \dots\}) = 0.$$

Hence we conclude that $\{Tx_n\}$ is a Cauchy sequence. By the completeness of $clT(K)$ there is some $u \in clT(K)$ such that

$$u = \lim_{n\to\infty} Tx_n.$$

As $clT(K) \subset f(K)$, there is some z in K such that

$$fz = u.$$

We show that $Tz = fz$. Suppose, by way of contradiction, that $d(Tz, fz) > 0$. Since $fx_{n+1} = Tx_n$, from (4) with $x = z$ and $y = x_{n+1}$ we have

$$\begin{aligned} d(fz, Tz) &\le d(fz, Tx_{n+1}) + d(Tz, Tx_{n+1}) \\ &\le d(fz, Tx_{n+1}) + M(z, x_{n+1}) - \varphi(M(z, x_{n+1})), \end{aligned} \tag{14}$$

where

$$\begin{aligned} &M(z, x_{n+1}) \\ &\quad = \max\{d(fz, fx_{n+1}), d(fz, Tz), d(fx_{n+1}, Tx_{n+1}), d(fz, Tx_{n+1}), d(fx_{n+1}, Tz)\} \\ &\quad = \max\{d(fz, Tx_n), d(fz, Tz), d(Tx_n, Tx_{n+1}), d(fz, Tx_{n+1}), d(Tx_n, Tz)\}. \end{aligned}$$

Since $\lim_{n\to\infty} Tx_n = fz$, for large enough n we have:

$$M(z, x_{n+1}) = \max\{d(fz, Tz), d(Tx_n, Tz)\}.$$

If $M(z, x_{n+1}) = d(fz, Tz)$, then from (14) and (iii) we get

$$d(fz, Tz) \le d(fz, Tx_n) + d(fz, Tz) - \varphi(d(fz, Tz)).$$

Letting n tends to infinity we get

$$d(fz, Tz) \le d(fz, Tz) - \varphi(d(fz, Tz)).$$

Thus we have

$$0 < d(fz, Tz) \le d(fz, Tz) - \varphi(d(fz, Tz)) < d(fz, Tz),$$

a contradiction.

If $M(z, x_{n_i+1}) = d(Tx_{n_i}, Tz)$, then from (14) and (iii) we get

$$d(fz, Tz) \le d(fz, Tx_n) + d(Tx_{n_i}, Tz) - \varphi(d(Tx_{n_i}, Tz)).$$

Letting i tends to infinity, by (ii) we get, as $d(Tx_{n_i}, Tz) \to d(fz, Tz)+$,

$$d(fz, Tz) < d(fz, Tz),$$

a contradiction. Thus our assumption $d(fz, Tz) > 0$ is wrong. Therefore $d(fz, Tz) = 0$. Hence $fz = Tz$, that is, z is a coincidence point of T and f.

We now show that Tz is a common fixed point of f and T. Since f and T are weakly compatible and $fz = Tz$, we obtain by the definition of weak compatibility that $fTz = Tfz$. Thus we have $TTz = Tfz = fTz$ and

$$d(TTz, Tz) \leq M(Tz, z) - \varphi(M(Tz, z)),$$

where

$$\begin{aligned} M(Tz, z) &= \max\{d(fTz, fz), d(fTz, TTz), d(fz, Tz), d(fTz, Tz), d(fz, TTz)\} \\ &= d(TTz, Tz). \end{aligned}$$

Thus

$$d(TTz, Tz) \leq d(TTz, Tz) - \varphi(d(TTz, Tz)).$$

Hence $d(TTz, Tz) = 0$ and hence $TTz = Tz$. Therefore $Tz = TTz = fTz$. This implies that $w = Tz$ is a common fixed point of T and f. Hence $K \cap F(T) \cap F(f)$ is a singleton. □

Corollary 2.2. *Let K be a nonempty subset of a metric space (X, d) and let T be a self-map of K. Assume that $clT(K) \subset K$, $clT(K)$ is complete, and T satisfies the following condition:*

$$d(Tx, Ty) \leq m(x, y) - \varphi(m(x, y))$$

for all $x, y \in K$, where

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

and $\varphi : [0, +\infty) \to [0, +\infty)$ is a real function satisfying conditions (i)–(iv) *in Theorem 2.1. Then T has a unique fixed point.*

Proof. Taking $f(t) = t$ in the proof of Theorem 2.1 we obtain Corollary 2.2. □

Corollary 2.3. *Let K be a nonempty subset of a metric space (X, d) and let f and T be a self-mappings of K. Assume that $clT(K) \subset f(K)$ and $clT(K)$ is complete. If T satisfies the following inequality for all $x, y \in K$,*

$$d(Tx, Ty) \leq \psi(M(x, y)) \tag{15}$$

where $M(x, y)$ is defined by (5) *and $\psi : [0, \infty) \to [0, \infty)$ is a real function such that*

(a) $\psi(t) < t$ *for all* $t > 0$,

(b) $\lim_{s \to t+} \psi(s) < t$ *for all* $t > 0$,

(c) $\psi(t)$ *is non-decreasing,*

(d) $\lim_{t\to\infty}(t-\psi(t))=\infty$.

Then $K\cap F(T)\cap F(f)$ *is a singleton.*

Proof. Set $\varphi(t)=t-\psi(t)$, then inequality (15) implies

$$d(Tx,Ty)\leq M(x,y)-\varphi(M(x,y)),$$

and also $\varphi:[0,\infty)\to[0,\infty)$ is a real function satisfying conditions (i)–(iv) in Theorem 2.1. The result follows from Theorem 2.1. □

Corollary 2.4. *Let K be a nonempty subset of a metric space (X,d) and let f and T be a self-mappings of K. Assume that $clT(K)\subset f(K)$ and $clT(K)$ is complete. If T satisfies the following inequality for all $x,y\in M$,*

$$d(Tx,Ty)\leq\alpha(M(x,y))M(x,y) \tag{16}$$

where $\alpha:[0,\infty)\to(0,1)$ *is a real function such that*

(a) $\lim_{s\to t+}\alpha(s)<1$ *for all* $t>0$,

(b) $\alpha(t)$ *is non-decreasing,*

(c) $\lim_{t\to\infty}\alpha(t)<1$.

Then $K\cap F(T)\cap F(f)$ *is a singleton.*

Proof. Set $\varphi(t)=(1-\alpha(t))t$, then inequality (16) implies

$$d(Tx,Ty)\leq M(x,y)-\varphi(M(x,y)),$$

where $\varphi:[0,\infty)\to[0,\infty)$ is a real function satisfying conditions (i)–(iv) in Theorem 2.1. The result now follows from Theorem 2.1. □

In Theorem 2.1, if $\varphi(t)=(1-k)t$ for a constant k with $0<k<1$, then we get:

Corollary 2.5 ([18], [21], Theorem 2.1)**.** *Let K be a subset of a metric space (X,d), and f and T be weakly compatible self-maps of K. Assume that $clT(K)\subset f(K)$, $clT(K)$ is complete, and T and f satisfy for all $x,y\in K$ and $0<k<1$,*

$$d(Tx,Ty)\leq k\max\{d(fx,fy),d(fx,Tx),d(fy,Ty),d(fx,Ty),d(fy,Tx)\}. \tag{17}$$

Then $K\cap F(f)\cap F(T)$ *is a singleton.*

Corollary 2.6 ([24], Theorem 2.1). *Let KM be a closed subset of a metric space (X, d), and let f, T be pointwise R-weakly commuting self-maps of K. Assume that either $T(K) \subset f(K)$ and $f(K)$ is closed or $cl(T(K)) \subset f(K)$. If $cl(T(K))$ is complete, T is f-continuous and T and f satisfy* (17). *Then $K \cap F(f) \cap F(T)$ is a singleton.*

Corollary 2.7 (Das and Naik [15]). *Let (X, d) be a complete metric space, $T, f : X \to X$ satisfy* (17). *Suppose that T, f are commuting maps, f is continuous and $T(X) \subset f(X)$. Then T and f have a unique common fixed point in X.*

In Theorem 2.1, if $\varphi(t) = (1 - k)t$ for a constant k with $0 < k < 1$, and $M(x, y) = d(fx, fy)$, then we get:

Corollary 2.8 ([2], Theorem 2.1). *Let K be a subset of a metric space (X, d), and f and T be weakly compatible self-maps of K. Assume that $clT(K) \subset f(K)$, $clT(K)$ is complete, and T is f-contraction. Then $K \cap F(f) \cap F(T)$ is a singleton.*

Corollary 2.9 (Jungck [19]). *Let (X, d) be a complete metric space, $T, f : X \to X$ be self-maps of X. Suppose that T is f-contraction, T, f are commuting maps, f is continuous and $T(X) \subset f(X)$. Then T and f have a unique common fixed point in X.*

Remark 2.10.

(1) Theorem 2.1 extends Theorem 1 due to Berinde [7], Theorems 2.1 and 2.5 due to Beg and Abbas [5] and Theorem 3.1 due to Song [32].

(2) In Corollary 2.2, if $\varphi(t) = (1 - k)t$ for a constant k with $0 < k < 1$, then we get the main result of Ćirić [11].

(3) Corollary 2.3 extends Theorem 1 due to Pant [25] to weakly compatible maps with more general contractive condition and generalizes main result of Boyd and Wong [9].

Recently, Chen and Li [10] introduced the class of Banach operator pairs, as a new class of noncommuting maps and it has been further studied by Al-Thagafi and Shahzad [3], Hussain [17] and Pathak and Hussain [28]. The pair (T, f) is called a *Banach operator pair*, if the set $F(f)$ is T-invariant, namely $T(F(f)) \subseteq F(f)$. Obviously, commuting pair (T, f) is a Banach operator pair but converse is not true in general, see [10], [28]. If (T, f) is a Banach operator pair, then (f, T) need not be a Banach operator pair (cf. Example 1 [10]). It is important to note that the class of Banach operator pairs is different from that of weakly compatible maps as is clear from the following example (see also [10], [28]).

Example 2.11. Consider $K=\mathbb{R}^2$ with the norm $\|(x,y)\|=|x|+|y|, (x,y)\in K$. Define T and f on K as follows:

$$T(x,y)=\left(x^3+x-1,\ \frac{\sqrt[3]{x^2+y^3-1}}{3}\right),$$
$$f(x,y)=\left(x^3+x-1,\ \sqrt[3]{x^2+y^3-1}\right).$$

Then

$$F(T)=\{(1,0)\};\quad F(f)=\{(1,y):y\in R^1\};$$
$$C(f,T)=\left\{(x,y):y=\sqrt[3]{1-x^2},\ x\in R^1\right\};$$
$$T(F(f))=\{T(1,y):y\in R^1\}=\left\{\left(1,\frac{y}{3}\right):y\in R^1\right\}\subseteq\{(1,y):y\in R^1\}=F(f).$$

Thus, (T,f) is a Banach operator pair. It is easy to see that T and f do not commute on the set $C(f,T)$, so T and f are not compatible.

As an application of Corollary 2.2, we obtain the following general result for Banach operators.

Theorem 2.12. *Let K be a nonempty subset of a metric space (X,d), and T, f be self-maps of K. Assume that $F(f)$ is nonempty, $cl(T(F(f)))\subseteq F(f)$, $cl(T(K))$ is complete, and T, f satisfy inequality* (4), *where $\varphi:[0,+\infty)\to[0,+\infty)$ is a real function satisfying conditions* (i)–(iv) *in Theorem 2.1. Then $K\cap F(T)\cap F(f)$ is a singleton.*

Proof. $cl(T(F(f)))$ being subset of $cl(T(K))$ is complete and $cl(T(F(f)))\subseteq F(f)$. Notice that $M(x,y)$ coincides with $m(x,y)$ on $F(f)$, hence for all $x,y\in F(f)$, we have by (4),

$$d(Tx,Ty)\le M(x,y)-\varphi(M(x,y))=m(x,y)-\varphi(m(x,y)).\qquad\square$$

By Corollary 2.2, T has a unique fixed point z in $F(f)$ and consequently, $K\cap F(T)\cap F(f)$ is a singleton.

Corollary 2.13. *Let K be a nonempty subset of a metric space (X,d), and (T,f) be a Banach operator pair on K. Assume that $cl(T(K))$ is complete, $F(f)$*

is nonempty and closed and T, f *satisfy (4) where* $\varphi : [0, +\infty) \to [0, +\infty)$ *is a real function satisfying conditions* (i)–(iv) *in Theorem 2.1. Then* $K \cap F(T) \cap F(f)$ *is a singleton.*

Corollary 2.14. *Let* K *be a nonempty subset of a metric space* (X, d)*, and* T, f *be self-maps of* K*. Assume that* $F(f)$ *is nonempty,* $clT(F(f)) \subseteq F(f)$, $cl(T(K))$ *is complete. If* T *satisfies the following inequality for all* $x, y \in K$,

$$d(Tx, Ty) \leq \psi(M(x, y))$$

where $\psi : [0, \infty) \to [0, \infty)$ *is a real function satisfying conditions* (a)–(d) *in Corollary 2.3. Then* $K \cap F(T) \cap F(f)$ *is a singleton.*

Proof. Set $\varphi(t) = t - \psi(t)$, then as in the proof of Corollary 2.3, T, f satisfy (4) and $\varphi : [0, \infty) \to [0, \infty)$ satisfies conditions (i)–(iv) in Theorem 2.1. The result follows from Theorem 2.12. □

In Theorem 2.12 and Corollary 2.13, if $\varphi(t) = (1 - k)t$ for a constant k with $0 < k < 1$, then we obtain the following results which extend and improve Lemma 3.1 of Chen and Li [10], Lemma 2.1 in [28], and provide the conclusions about common fixed points of Theorem 2.1 for the different classes of maps.

Corollary 2.15 ([3], Theorem 3.2)**.** *Let* K *be a nonempty subset of a metric space* (X, d)*, and* T, f *be self-maps of* K*. Assume that* $F(f)$ *is nonempty,* $cl(T(F(f))) \subseteq F(f)$, $cl(T(K))$ *is complete, and* T, f *satisfy* (17)*. Then* $M \cap F(T) \cap F(f)$ *is a singleton.*

Corollary 2.16. *Let* K *be a nonempty subset of a metric space* (X, d)*, and* (T, f) *be a Banach operator pair on* K*. Assume that* $cl(T(K))$ *is complete,* T, f *satisfy* (17) *and* $F(f)$ *is nonempty and closed. Then* $K \cap F(T) \cap F(f)$ *is a singleton.*

The following example shows that the contractive condition (4) is substantially more general than the condition (17), even if (X, d) is compact and convex Euclidean space.

Example 2.17. Let $K = \left[0, \frac{1}{2}\right]$ be the closed interval with usual metric and let $f, T : K \to K$ and $\varphi : [0, +\infty) \to [0, +\infty)$ be mappings defined as follows:

$$f(x) = x^2 \qquad \text{for all } 0 \leq x \leq \frac{1}{2},$$

$$T(x) = x^2 - x^4, \quad \text{for all } 0 \leq x \leq \frac{1}{2},$$

$$\varphi(t) = t^2, \qquad \text{for } 0 \le t \le \frac{1}{2},$$

$$\varphi(t) = \frac{1}{2}t, \qquad \text{for } t > \frac{1}{2}.$$

Let x, y in K be arbitrary. Without loss of generality we may suppose that $x \le y$. Then we have

$$\begin{aligned} &M(x,y) \\ &\quad = \max\{d(f(x), f(y)), d(f(x), T(x)), d(f(y), T(y)), d(f(x), T(y)), d(f(y), T(x))\} \\ &\quad = d(f(y), T(x)), \\ &d(f(y), T(x)) = y^2 - x^2(1-x^2). \end{aligned}$$

Since $y^2 \ge y^2 - x^2(1-x^2))$ for all $x \in \left[0, \frac{1}{2}\right]$, it follows that

$$-y^4 \le -(y^2 - x^2(1-x^2))^2.$$

Thus we have

$$\begin{aligned} d(T(x), T(y)) &= y^2 - y^4 - x^2 + x^4 = (y^2 - x^2(1-x^2)) - y^4 \\ &\le (y^2 - x^2(1-x^2)) - (y^2 - x^2(1-x^2))^2 \\ &= d(f(y), T(x)) - [d(f(y), T(x))]^2 = M(x,y) - \varphi(M(x,y)). \end{aligned}$$

Therefore, f and T satisfy (4). Also it is easy to see that the mapping $\varphi(t)$ satisfies all hypotheses (i)-(iv) in Theorem 2.1. Thus we can apply our Theorem 2.1 and Corollaries 2.2, 2.3 and 2.4, Theorem 2.12 and Corollaries 2.13 and 2.14. On the other hand, for any fixed k; $0 < k < 1$, we have, for $x = 0$ and each $y \in X$ with $0 < y < \sqrt{1-k}$,

$$d(T(0), T(y)) = y^2 - y^4 = (1-y^2)y^2 > k \cdot y^2 = k \cdot d(f(y), T(0)) = k \cdot M(0,y).$$

Thus, T does not satisfy (17). Therefore, the Theorems of Jungck and Hussain [21], Al-Thagafi and Shahzad [2], Jungck [19], Das and Naik [15], and Ćirić [11], as well as the Theorem of Al-Thagafi and Shahzad [3], can not be applied.

3. An application to variational inequalities

In this section, we apply Corollary 2.15 to show the existence of solution of variational inequalities as in the works of Belbas and Mayergoyz [6]

Pathak [27]. Variational inequalities arise in optimal stochastic control as well as in other problems in mathematical physics, for examples, deformation of elastic bodies stretched over solid obstacles, elastic-plastic torsion etc. [16], [28]. The iterative method for solutions of discrete variational inequalities are suitable for implementation on parallel computers with single instruction, multiple-data architecture, particularly on massively parallel processors.

The variational inequality problem is to find a function u such that

$$\max\{Lu - f, u - \phi\} = 0 \text{ on } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{18}$$

where Ω is a nonempty open bounded subset of $\mathbb{R}^N$ for some $q \in \Omega$ with smooth boundary such that $0 \in cl(\Omega)$, L is an elliptic operator defined on Ω by

$$L = -a_{ij}(x)\partial^2/\partial x_i \partial x_j + b_i(x)\partial/\partial x_i + c(x).I_N,$$

where summation with respect to repeated indices is implied, $c(x) \geq 0, [a_{ij}(x)]$ is a strictly positive definite matrix, uniformly in x, for $x \in \overline{\Omega}, f$ and ϕ are smooth functions defined in Ω and ϕ satisfies the condition: $\phi(x) \geq 0$ for $x \in \partial\Omega$.

The corresponding problem of stochastic optimal control can be described as follows: $L - cI$ is the generator of a diffusion process in $\mathbb{R}^N$, c is a discount factor, f is the continuous cost, and ϕ represents the cost incurred by stopping the process. The boundary condition "$u = 0$ on $\partial\Omega$" expresses the fact that stopping takes place either prior or at the time that the diffusion process exits from Ω.

A problem related to (18) is the two-obstacle variational inequality. Given two smooth functions ϕ and μ defined on $\overline{\Omega}$ such that $\phi \leq \mu$ on $\Omega, \phi \leq 0 \leq \mu$ on $\partial\Omega$, the corresponding variational inequality is as follows:

$$\max\{\min[Lu - f, u - \phi], u - \mu\} = 0 \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{19}$$

Let A be an $N \times N$ matrix corresponding to the finite difference discretization of the operator L. We shall make the following assumptions about the matrix A:

$$A_{ii} = 1, \sum_{j:j\neq i} A_{ij} > -1, A_{ij} < 0 \quad \text{for } i \neq j. \tag{20}$$

These assumptions are related to the definition of "M-matrices"; matrices arising from the finite difference discretization of continuous elliptic operators will have the property (20) under the appropriate conditions and Q denotes the set of all discretized vectors in Ω (see [8], [28], [33]). Note that the matrix A is an M-matrix if and only if every off-diagonal entry of A is nonpositive.

Let $B = I_N - A$. Then the corresponding properties for the B-matrices are:

$$B_{ii} = 0, \sum_{j:j\neq i} B_{ij} < 1, B_{ij} > 0 \quad \text{for } i \neq j. \tag{21}$$

Let $q = \max_i \sum_j B_{ij}$ and A^* be an $N \times N$ matrix such that $A^*_{ii} = 1 - q$ and $A^*_{ij} = -q$ for $i \neq j$. Then we have $B^* = I_N - A^*$.

Now, we show the existence of iterative solutions of variational inequalities:

Consider the following discrete variational inequalities as mentioned above:

$$\begin{aligned}\max[\min\{A(x - A^*.d(Ix, Tx)) - f, x - A^*.d(Ix, Tx) - \phi\},\\ x - A^*.\operatorname{dist}(Ix, Tx) - \mu] = 0,\end{aligned} \tag{22}$$

where T, I are mappings from $\mathbb{R}^N$ into itself implicitly defined by

$$\begin{aligned}Tx = \min k\big[\max\{BIx + A(1 - B^*).\operatorname{dist}(Ix, Tx) + f,\\ (1 - B^*).\operatorname{dist}(Ix, Tx) + \phi\}, (1 - B^*).\operatorname{dist}(Ix, Tx) + \mu\big]\end{aligned} \tag{23}$$

for all $x \in cl(Q)$, $0 < k < 1$ such that the following condition holds:
(i) $cl(T(\Omega))$ is complete, $F(I)$ is nonempty and $clT(F(I)) \subseteq F(I)$.
Then (22) is equivalent to the common fixed point problem:

$$x = Tx = Ix. \tag{24}$$

In a two-person game we determine the best strategies for each player on the basis of maxmin and minmax criterion of optimality. This criterion will be well stated as follows:
A player lists his/her worst possible outcomes and then he/she chooses that strategy which corresponds to the best of these worst outcomes. Here, the problem (22) exhibits the situation in which two players are trying to control a diffusion process; the first player is trying to maximize a cost functional, and the second player is trying to minimize a similar functional. The first player is called the maximizing player and the second one the minimizing player. Here, f represents the continuous rate of cost for both players, ϕ is the stopping cost for the maximizing player, and μ is the stopping cost for the minimizing player. This problem is fixed by inducting a pair of maps (T, I) under the constrained condition (i) as stated above.

Theorem 3.1. *Under the assumptions* (20) *and* (21), *a solution for* (24) *exists.*

PROOF. Let $(Ty)_i = k(1 - B^*_{ij}).[d(Iy_i, Ty_i) + \mu_i]$ for any $y \in cl(Q)$ and any $i, j = 1, 2, \ldots, N$. Now, for any $x \in cl(Q)$, since $(Tx)_i \leq k(1 - B^*_{ij}).[\mathrm{dist}(Ix_i, Tx_i) + \mu_i]$, we have

$$\begin{aligned}(Tx)_i - (Ty)_i &\leq k(1 - B^*_{ij}).\{d(Ix_i, Tx_i) - d(Iy_i, Ty_i)\}\\ &\leq k \max\{d(Ix_i, Tx_i),\ d(Iy_i, Ty_i)\}\\ &\leq k \max\{d(Ix_i, Tx_i),\ d(Iy_i, Ty_i), d(Ix_i, Ty_i), d(Iy_i, Tx_i)\}. \qquad (25)\end{aligned}$$

If

$$(Ty)_i = \max k\{B_{ij} Iy_j + (1 - B^*_{ij}).d(Iy_i, Ty_i) + f_i, (1 - B^*_{ij}).d(Iy_i, Ty_i) + \phi_i\},$$

i.e. if the maximizing player succeeds to maximize a cost functional in his/her strategy which corresponds to the best of N worst outcomes from his/her list, then the game would be one sided. In this situation, we introduce the one sided operator:

$$T^+x = \max k\{BIx + A(1 - B^*).d(Ix, Tx) + f, (1 - B^*).d(Ix, Tx) + \phi\}.$$

Therefore, we have

$$(Ty)_i = (T^+y)_i.$$

Now, if $(Tx)_i = k[B_{ij} Ix_j + A_{ij}(1 - B^*_{ij}).d(Ix_i, Tx_i) + f_i]$, then since $(Ty)_i \geq k[B_{ij} Iy_j + A_{ij}(1 - B^*_{ij}).d(Iy_i, Ty_i) + f_i]$, by using (20), we have

$$\begin{aligned}(T^+x)_i - (T^+y)_i &\leq k[B_{ij}.\|Ix_i - Iy_i\| + (1 - B^*_{ij}). \max\{d(Ix_i, Tx_i), d(Iy_i, Ty_i)\}]\\ &\leq k[B_{ij}.\|Ix_i - Iy_i\|\\ &\quad + (1 - B^*_{ij}). \max\{d(Ix_i, Tx_i), d(Iy_i, Ty_i), d(Ix_i, Ty_i),\ d(Iy_i, Tx_i)\}]. \qquad (26)\end{aligned}$$

If $(Tx)_i = k(1 - B^*_{ij}).d(Ix_i, Tx_i) + \phi_i$ then since

$$(Ty)_i \geq k(1 - B^*_{ij}).d(Iy_i, Ty_i) + \phi_i,$$

we have

$$\begin{aligned}(Tx)_i - (Ty)_i &\leq k(1 - B^*_{ij}). \max\{d(Ix_i, Tx_i), d(Iy_i, Ty_i)\}\\ &\leq k(1 - B^*_{ij}). \max\{d(Ix_i, Tx_i), d(Iy_i, Ty_i), d(Ix_i, Ty_i]),\ d(Iy_i, Tx_i)\}. \qquad (27)\end{aligned}$$

Hence, from (25)–(27), we have

$$\begin{aligned}(Tx)_i - (Ty)_i \leq k[q.\|Ix - Iy\| + (1 - q). \max\{&d(Ix, Tx), d(Iy, Ty),\\ &d(Ix, Ty),\ d(Iy, Tx)\}]. \qquad (28)\end{aligned}$$

Since x and y are arbitrarily chosen, we have

$$(Ty)_i - (Tx)_i \leq k[q.\|Ix - Iy\| + (1-q).\max\{d(Ix, Tx), d(Iy, Ty), \\ d(Ix, Ty),\ d(Iy, Tx)\}]. \quad (29)$$

From (28) and (29), it follows that

$$\begin{aligned}\|Tx - Ty\| &\leq k[q.\|Ix - Iy\| + (1-q).\max\{d(Ix, Tx), d(Iy, Ty), \\ &\qquad d(Ix, Ty),\ d(Iy, Tx)\}] \\ &\leq k[\max\{\|Ix - Iy\|, \max\{d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\}\}]\end{aligned}$$

that is,

$$\|Tx - Ty\| \leq k \max\big\{\|Ix - Iy\|, \|Ix - Tx\|, \|Iy - Ty\|, \|Ix - Ty\|, \|Iy - Tx\|\big\}.$$

Hence the condition (17) is satisfied. Therefore, Corollary 2.15 ensures the existence of a solution of (24). This completes the proof. □

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