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Parameter-independent structure in periodic orbits of an iterated function system on the real line

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Abstract. For the iterated function system on $\mathbb{R}$ comprising the maps $f(x) = ax + 1$ and $g(x) = bx$, with $a > 0$ and $0 < b < 1$, we represent each n-cycle by the composition (or *word*) in f and g corresponding to the cycle's point of least magnitude (or *perigee*). These representations are partitioned into equivalence classes using simple combinatorial criteria. Associated with each n-cycle are n polynomials in a and b whose values at a special value of a are partially ordered. An example is given showing that, for fixed b, the perigee word of an n-cycle is a function of a; but the ordering of the polynomial values enables us to prove that the *maximal* perigee word in each equivalence class is independent of the parameters a and b.

1. Introduction

Let $\Psi(a, b)$ be the iterated function system on $\mathbb{R}$ comprising the maps

$$f(x) = ax + 1 \quad \text{and} \quad g(x) = bx, \tag{1}$$

where $a > 0$ and $0 < b < 1$. The dynamics of this system for $a > 1$ are considered in [2]. Here we present some combinatorial properties of Ψ's cycle structure which are independent of the parameters a and b; in particular, these properties hold whether or not f is a contraction.

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Given the maps f and g defined in (1), and a positive integer n, choose functions $t_i, 1 \leq i \leq n$, from the set $\{f, g\}$, and compose them by right-to-left concatenation. We call $w = t_n t_{n-1} \cdots t_1$ the *word* for the cycle point x_1 satisfying

$$t_n t_{n-1} \cdots t_1(x_1) = x_1.$$

Since the word g^n yields just the trivial cycle point $x_1 = 0$ for all $n \in \mathbb{N}$ and all $b \in (0, 1)$, we exclude words of this form in what follows. Let Σ^n be the set of n-letter words on the symbols f and g in which f appears at least once. The cyclic permutations, or *rotations*, of a word $w \in \Sigma^n$ yield the set of cycle points $\{x_1, x_2, \ldots, x_n\}$, in which the (not necessarily unique) point of least magnitude or *perigee* of the cycle [1] is generated by its corresponding *perigee word*. We use the following combinatorial properties of cycle words.

Definition 1. Given $w \in \Sigma^n$.

(a) The f-rank of w, denoted by r, is its number of fs. Note $r \geq 1$.

(b) The density, denoted by α, is the ratio $(n - r)/r$ of the number of gs to fs in w.

(c) The base markers in w are the fs indexed from left to right and from 1 to r in w.

(d) For $2 \leq i \leq r$, the ith gap d_i is the number of gs between base markers f_{i-1} and f_i, while d_1 is the number of gs to the left of f_1. The ordered r-tuple $D(w) = (d_1, d_2, \ldots, d_r)$ is the gaps vector of w.[1]

(e) The g-rank of base marker f_i is the number of gs to its left in w, and is denoted by q_i. Equivalently, it is the sum of gaps $d_1 + d_2 + \cdots + d_i$. The ordered r-tuple $Q(w) = (q_1, q_2, \ldots, q_r)$ is the cycle word code, or more briefly the code of w.

For example, $ggf_1gf_2f_3ggf_4f_5$ shows the base markers labeled for the word $w = ggfgffggff \in \Sigma^{10}$, for which $D(w) = (2, 1, 0, 2, 0)$ and $Q(w) = (2, 3, 3, 5, 5)$.

Given a and b, a word's length and code determine its cycle point.

Proposition 1. *For $n \in \mathbb{N}$ and $1 \leq r \leq n$, let $w \in \Sigma^n$ have code $(q_1, q_2, \ldots, q_r)$ and density α. Then*

$$x_1 = \frac{1}{1 - (ab^\alpha)^r} \sum_{i=1}^{r} b^{q_i} a^{i-1} \tag{2}$$

is the unique point satisfying $w(x_1) = x_1$.

[1] The gaps vector is the mirror image of the *encoding vector* defined in [4, pp. 38–39].

PROOF. We show that

$$w(x) = a^r b^{n-r} x + \sum_{i=1}^{r} b^{q_i} a^{i-1}, \tag{3}$$

from which the result follows. For $m \geq 0$ and $n > m$, define Σ_m^n to be the set of words w that can be written $w = t_n \dots t_{m+2} f g^m$, where $t_i \in \{f, g\}$, so that

$$\Sigma^n = \bigcup_{m<n} \Sigma_m^n.$$

For each $m \geq 0$ we prove by induction on n that, for all $n > m$,

$$w \in \Sigma_m^n \implies w(x) = a^r b^{n-r} x + \sum_{i=1}^{r} b^{q_i} a^{i-1}. \tag{4}$$

For the initial step, fix $m \geq 0$, use the base value $n = m + 1$, and let $w \in \Sigma_m^{m+1}$. Then $w = f g^m$, $r = 1$, $q_1 = 0$, and

$$f g^m(x) = a b^m x + 1 = a^r b^{(m+1)-r} x + \sum_{i=1}^{r} b^{q_i} a^{i-1}, \tag{5}$$

as required. For the inductive step, let $n > m$, and assume (4). Let $w = t_{n+1} \dots t_{m+2} f g^m \in \Sigma_m^{n+1}$, where $Q(w) = (q_1, \dots, q_r)$. Write $w = t_{n+1} w'$, where w' has length n. If $t_{n+1} = f$, then $q_1 = 0$, $Q(w') = (q_2, \dots, q_r)$, and

$$w(x) = a\left(a^{r-1} b^{n-(r-1)} x + \sum_{i=2}^{r} b^{q_i} a^{i-1}\right) + 1. \tag{6}$$

If $t_{n+1} = g$, on the other hand, then $Q(w') = (q_1 - 1, q_2 - 1, \dots, q_r - 1)$, and

$$w(x) = b\left(a^r b^{n-r} x + \sum_{i=1}^{r} b^{q_i - 1} a^{i-1}\right). \tag{7}$$

The induction hypothesis is confirmed, since both (6) and (7) reduce to

$$w(x) = a^r b^{(n+1)-r} x + \sum_{i=1}^{r} b^{q_i} a^{i-1}. \qquad \square$$

Note that, if $a = b^{-\alpha}$, we have division by 0 in equation (2), and the cycle point does not exist; we return to this important fact in Section 7.

2. Representing cycles by perigee words

In general, the correspondence between a cycle point word in Σ^n and its code is not a bijection. For instance, the words $ggfgf$, $ggfgfg$, $ggfgfgg$, and so on all have gaps vector $(2,1)$ and code $(2,3)$. We now define a subset F^n of Σ^n which contains all the perigee words, and in which every word is uniquely represented by its gaps vector, or, equivalently, by its code.

Definition 2. For each $n \in \mathbb{N}$ and for $1 \le r \le n$,

$$F^n = \{w \in \Sigma^n \mid w = t_n t_{n-1} \cdots t_2 t_1 \text{ with } t_1 = f\},$$

$$F_r^n = \{w \in F^n \mid w \text{ has } f\text{-rank } r\},$$

$$P_r^n = \{w \in F_r^n \mid w \text{ is a perigee word}\}.$$

F_r^n contains P_r^n because, if $t_n \ldots t_2 g(x_1) = x_1$, then $|x_2| = |g(x_1)| < |x_1|$; hence x_1 cannot be the perigee, and no word ending in g can be a perigee word. The sets P_r^n are equivalence classes imposed on the set of n-length perigee words by the relation "possesses r letters f" for $1 \le r \le n$. Table 1 shows the P_r^7 for $r = 1, \ldots, 7$ in $\Psi\left(\frac{5}{3}, \frac{1}{2}\right)$. By construction, the perigees in this table increase in absolute value within each P_r^n.

Note that, for every $w \in F_r^n$, the base marker f_r is always rightmost and its g-rank is always $n-r$, so all words in F_r^n have $q_r = n - r = r\alpha$. Furthermore, the allowable rotations of any $w \in F_r^n$ put each f in the rightmost position exactly once; consequently, w admits of r such rotations and r (not necessarily distinct) cycle points. Finally, we have the useful property that any allowable rotation of a word in F_r^n yields a corresponding cyclic permutation of its gaps vector. That is, for $w \in F_r^n$ and $1 \le i \le r$, w's ith gap d_i is the number of gs between base markers f_{i-1} and f_i, indices taken modulo r.[2] This follows because none of the gs counted by d_1 lie to the right of f_r for any word in F_r^n.

While there is an obvious bijection between words in F_r^n and their codes, we do not claim a bijection between words and cycle points. Distinct cycles need not be disjoint; for instance, in $\Psi(2, \frac{1}{2})$, the cycle words $ggggffgfggf$ and $gggfggffggf$ both yield the perigee $\frac{3}{7}$. However, disjoint cycles are not required here.

[2] When the residues modulo r are used as indices, we take them to be $\{1, \ldots, r\}$ rather than the usual $\{0, \ldots, r-1\}$.

set	perigee word	code	perigee	decimal
P_1^7	$ggggggf$	(6)	3/187	0.01604
P_2^7	$gggggff$	(5, 5)	24/263	0.09125
	$ggggfgf$	(4, 5)	33/263	0.12548
	$gggfggf$	(3, 5)	51/263	0.19392
P_3^7	$ggggfff$	(4, 4, 4)	147/307	0.47883
	$gggfgff$	(3, 4, 4)	174/307	0.56678
	$gggffgf$	(3, 3, 4)	219/307	0.71336
	$ggfggff$	(2, 4, 4)	228/307	0.74267
	$ggfgfgf$	(2, 3, 4)	273/307	0.88925
P_4^7	$gggffff$	(3, 3, 3, 3)	816/23	35.47826
	$ggfgfff$	(2, 3, 3, 3)	39	39.00000
	$ggffgff$	(2, 2, 3, 3)	1032/23	44.86957
	$gfggfff$	(1, 3, 3, 3)	1059/23	46.04348
	$gfgfgff$	(1, 2, 3, 3)	1194/23	51.91304
P_5^7	$ggfffff$	(2, 2, 2, 2, 2)	−4323/2153	−2.00790
	$gfgffff$	(1, 2, 2, 2, 2)	−4566/2153	−2.12076
	$gffgfff$	(1, 1, 2, 2, 2)	−4971/2153	−2.30887
P_6^7	$gffffff$	(1, 1, 1, 1, 1, 1)	−22344/14167	−1.57719
P_7^7	$fffffff$	(0, 0, 0, 0, 0, 0, 0)	−3/2	−1.50000

Table 1. The distinct perigee words of length 7, with their codes and perigee values, for $f := f(x) = \frac{5}{3}x + 1$ and $g := g(x) = \frac{x}{2}$.

3. Minimal and maximal perigees

Our main results show that the minimal and maximal perigee words in each equivalence class may be characterized purely combinatorially.

Theorem 1. *In $\Psi(a,b)$, the minimal perigee word with density α in P_r^n is $w_{\min} = g^{r\alpha} f^r$, whose code is the r-tuple*

$$Q(w_{\min}) = (r\alpha, r\alpha, \ldots, r\alpha).$$

Proof. Let $g^{r\alpha} f^r(y) = y$. Given any $w \neq g^{r\alpha} f^r \in F_r^n$, with $Q(w) = (q_1, \ldots, q_r)$, there exists an integer $j, 1 \le j \le r$, for which

$$q_i < r\alpha, \quad 1 \le i \le j,$$

$$q_i \le r\alpha, \quad j+1 \le i \le r.$$

By Proposition 1 with $0 < b < 1$, the cycle point x_1 for w satisfies

$$x_1 = \frac{1}{1-(ab^\alpha)^r}\sum_{i=1}^{r} b^{q_i}a^{i-1} > \frac{1}{1-(ab^\alpha)^r}\sum_{i=1}^{r} b^{r\alpha}a^{i-1} = y,$$

and the theorem follows. $\square$

Theorem 2 (Maximal Perigee Property)**.** *In $\Psi(a,b)$, the maximal perigee word $w_{\max} \in P_r^n$ with density α has code*

$$Q(w_{\max}) = (\lceil\alpha\rceil, \lceil 2\alpha\rceil, \ldots, \lceil r\alpha\rceil)\,,$$

where $\lceil\cdot\rceil$ is the ceiling function.

For example, $n = 7$ and $r = 4$ yield $\alpha = \frac{3}{4}$, and the maximal perigee word code in P_4^7 is

$$\left(\left\lceil\frac{3}{4}\right\rceil, \left\lceil\frac{2\cdot 3}{4}\right\rceil, \left\lceil\frac{3\cdot 3}{4}\right\rceil, \left\lceil\frac{4\cdot 3}{4}\right\rceil\right) = (1, 2, 3, 3),$$

as in Table 1. (The result holds trivially when $\alpha = 0$, that is, when $r = n$.)

An inquiry into general n-cycles follows, culminating in a proof of Theorem 2. In Section 4 we show how to use a word's code to calculate the code of any rotation. A lemma of Chisala (Section 5) implies that, among a word's rotated codes, there are at least two "extremal" codes: one which is *superdiagonal* and one *subdiagonal*. In Section 6 we introduce the *deviation vector* and *maximum deviation* of a word, and prove that the deviation vectors of a word's rotations become cyclic permutations of each other under a particular vertical translation; a special case involving subdiagonal words is crucial later.

Section 7 gives the name *code function* to the polynomial part of equation (2), along with an example showing that, for a given value of the parameter a and for fixed b, the perigee of a cycle corresponds to the minimal code function. Although no cycle points exist when $a = b^{-\alpha}$, the code functions for a word and its rotations are well-defined for this value of a, and in Section 8 we show that the code function values at $a = b^{-\alpha}$ are partially ordered, with the smallest and largest values corresponding to super- and subdiagonal codes, respectively. Section 9 establishes upper and lower bounds for super- and subdiagonal code function values at $a = b^{-\alpha}$, respectively, over all words in F_r^n, and we prove that the only superdiagonal word in F_r^n whose maximum deviation is less than 1 is the word whose code is $(\lceil\alpha\rceil, \lceil 2\alpha\rceil, \ldots, \lceil r\alpha\rceil)$. Finally, Section 10 employs these results to prove the Maximal Perigee Property.

We are less interested here with the cycle point values x_i than with their combinatorial representations, largely because the interesting aspects of the representations occur when the parameter values preclude the existence of the points.

4. Rotations of codes

We begin by showing how codes change when the words they belong to are rotated. We use the following notation, where i and $j \in \{1, \ldots, r\}$. Given $w \in F_r^n$ with $D(w) = (d_1, \ldots, d_r)$, w_i is the rotation of w whose gaps vector begins with d_i. (Hence $w_1 = w$.) $D(w_i)$ is abbreviated D_i, and likewise $Q_i = Q(w_i)$. x_i is the cycle point for w_i; that is, $w_i(x_i) = x_i$. Lastly, we write $y \sim z$ if y is a cyclic permutation of z. (For a given w_1, we have $w_i \sim w_j$ and $D_i \sim D_j$, but $Q_i \sim Q_j$ only for $w = f^n$).

Proposition 2. *If $w_1 \in F_r^n$ has code $Q_1 = (q_1, q_2, \ldots, q_r)$, then the rotation w_i, $1 \leq i \leq r$, has code $Q_i = ({q_1}', {q_2}', \ldots, {q_r}')$, where*

$$ {q_j}' = \begin{cases} q_{i+j-1} - q_{i-1}, & j \in \{1, \ldots, r-i+1\} \\ q_r + q_{i+j-r-1} - q_{i-1}, & j \in \{r-i+2, \ldots, r\} \end{cases}, \tag{8} $$

and where we define $q_0 = d_0 = 0$.

Proof sketch. The gaps vector for w_i is

$$ D_i = (d_i, d_{i+1}, \ldots, d_r, d_1, \ldots, d_{i-1}), $$

so $D_i = ({d_1}', {d_2}', \ldots, {d_r}')$ has

$$ {d_j}' = \begin{cases} d_{i+j-1}, & j \in \{1, \ldots, r-i+1\} \\ d_{i+j-r-1}, & j \in \{r-i+2, \ldots, r\} \end{cases}. $$

The formulas (8) then follow directly. The definitions $q_0 = d_0 = 0$ preserve identity when $i = 1$. □

The transformation in Proposition 2 may also be expressed as

$$ {q_{j-i+1}}' = q_j - q_{i-1}, \quad 1 \leq j \leq r, \tag{9} $$

where arithmetic is performed modulo $n - r$, and the indices are calculated modulo r. The formulas (8) will be used in the final proof of the Maximal Perigee Property (Section 10), while (9) will be applied in Section 6.

5. Chisala's Lemma, and sub- and superdiagonal codes

We now invoke a modified lemma of Chisala [3] to define two kinds of words which are "extremal," in the sense that all the partial averages of the g-ranks are either no less than or no greater than the word's density.

Lemma 1. *Given a sequence $D = (d_1, \ldots, d_r)$ of real numbers and a sequence $M = (m_1, \ldots, m_r)$ of weights, let $A = \sum_{i=1}^{r} d_i m_i / \sum_{i=1}^{r} m_i$ be the weighted average.*

(a) (Chisala 1994) *There exists a cyclic permutation σ on the indices such that for every $k \in \{1, \ldots, r\}$, the partial weighted averages $\sum_{i=1}^{k} d_{\sigma(i)} m_{\sigma(i)} / \sum_{i=1}^{k} m_{\sigma(i)}$ are bounded above by A.*

(b) *There exists a cyclic permutation τ on the indices such that for every $k \in \{1, \ldots, r\}$, the partial weighted averages $\sum_{i=1}^{k} d_{\tau(i)} m_{\tau(i)} / \sum_{i=1}^{k} m_{\tau(i)}$ are bounded below by A.*

The proof of part (b) follows from part (a) by considering $(-d_1, \ldots, -d_r)$.

We call a word $w \in F_r^n$, its gaps vector, and its code *subdiagonal* if the r-tuple

$$\left(\frac{q_1}{1}, \frac{q_2}{2}, \ldots, \frac{q_r}{r}\right) \tag{10}$$

satisfies part (a) of Chisala's Lemma (with $A = \alpha$); or, equivalently, if $q_i \le i\alpha$ for $1 \le i \le r$. Similarly, w is *superdiagonal* if (10) satisfies part (b) of Chisala's Lemma, or, equivalently, if $q_i \ge i\alpha$. Note that the word with code $(\alpha, 2\alpha, \ldots, r\alpha)$, where necessarily $\alpha \in \{0, 1, 2, \ldots\}$, is both sub- and superdiagonal.

For example, if $w_1 \in F_5^{15}$ has gaps vector $D_1 = (1, 4, 2, 0, 3)$ and density $\alpha = 2$, then w_3 is a subdiagonal rotation with $D_3 = (2, 0, 3, 1, 4)$ and $Q_3 = (2, 2, 5, 6, 10)$, while w_2 is superdiagonal with $D_2 = (4, 2, 0, 3, 1)$ and $Q_2 = (4, 6, 6, 9, 10)$.

6. Deviations

We measure and compare a word's rotations using the maximum signed difference $q_i - i\alpha$.

Definition 3. Let $w_1 \in F_r^n$ have density α and code $Q_1 = (q_1, \ldots, q_r)$. The deviation vector of w_1 is

$$\Delta_1 = (q_1 - \alpha, q_2 - 2\alpha, \ldots, q_r - r\alpha),$$

and $q_j - j\alpha$ is the jth deviation, $1 \le j \le r$. The quantity

$$h(w_1) = \max_{1 \le j \le r} (q_j - j\alpha).$$

is the maximum deviation for w_1. For any $w_i \sim w_1$ we write $h_i = h(w_i)$.

Let $(y)^j$ be the j-tuple $(y, y, \ldots, y)$. Given a word and one of its rotations, we now show that special vertical translations of their deviation vectors produce two new vectors which are again cyclic permutations of each other.

Proposition 3. *Given* $w_1 \in F_r^n$*, and adding termwise,*

$$\Delta_1 + (h_i)^r \sim \Delta_i + (h_1)^r$$

for any $w_i \sim w_1$*.*

Proof. Suppose $h_1 = q_k - k\alpha$ for some $k \in \{1, \dots, r\}$. Using equation (9), we find that $h_i = (q_k - q_{i-1}) - (k - i + 1)\alpha$. Writing $(y_m)_{m=1}^r$ for the r-tuple $(y_1, y_2, \dots, y_r)$, we then have

$$\begin{aligned} \Delta_1 + (h_i)^r &= \big(q_m - m\alpha\big)_{m=1}^r + \big(q_k - q_{i-1} - (k - i + 1)\alpha\big)_{m=1}^r \\ &= \big(q_m + q_k - q_{i-1} - (m + k - i + 1)\alpha\big)_{m=1}^r, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \Delta_i + (h_1)^r &= \big(q_m - q_{i-1} - (m - i + 1)\alpha\big)_{m=i \bmod r}^{(i+r-1) \bmod r} + \big(q_k - k\alpha\big)_{m=i \bmod r}^{(i+r-1) \bmod r} \\ &= \big(q_m + q_k - q_{i-1} - (m + k - i + 1)\alpha\big)_{m=i \bmod r}^{(i+r-1) \bmod r}. \end{aligned} \tag{12}$$

Equations (11) and (12) are identical, save for the limits on the index m, which cycles once through the numbers $1, \dots, r$ in both cases. □

As an example, take $w_1 \in F_5^{12}$ having $D_1 = (1, 1, 3, 2, 0)$ and $\alpha = \frac{7}{5}$. Then $Q_1 = (1, 2, 5, 7, 7)$, $\Delta_1 = \big(-\frac{2}{5}, -\frac{4}{5}, \frac{4}{5}, \frac{7}{5}, 0\big)$, and $h_1 = \frac{7}{5}$. We also have $Q_3 = (3, 5, 5, 6, 7)$, $\Delta_3 = \big(\frac{8}{5}, \frac{11}{5}, \frac{4}{5}, \frac{2}{5}, 0\big)$, and $h_3 = \frac{11}{5}$. We see that

$$\Delta_1 + (h_3)^5 = \left(\frac{9}{5}, \frac{7}{5}, 3, \frac{18}{5}, \frac{11}{5}\right) \sim \left(3, \frac{18}{5}, \frac{11}{5}, \frac{9}{5}, \frac{7}{5}\right) = \Delta_3 + (h_1)^5.$$

For rotations of a given w_1, the largest maximum deviation occurs for a superdiagonal rotation, while the smallest is attained when every difference $q_j - j\alpha$ is at most 0, namely when the rotation is subdiagonal. (Note that the only superdiagonal code with maximum deviation 0 is $(\alpha, 2\alpha, \dots, r\alpha)$, where necessarily $\alpha \in \{0, 1, 2, \dots\}$.) Proposition 3 for subdiagonal rotations merits special mention.

Corollary 1. *Let* w_m *be a subdiagonal rotation of* $w_1 \in F_r^n$*. Then* $\Delta_1 \sim \Delta_m + (h_1)^r$*.*

With this corollary we set the stage for making subdiagonal rotations the standard against which all other rotations are measured; this will be developed further in Section 8.

7. Code functions

We now apply these properties of codes and deviation vectors to the polynomial part of the rational function of Proposition 1. Here, a is a parameter to be varied through nonnegative values, and we make extensive use of the quantity

$$\lambda = b^{-\alpha}.$$

Let $S = (s_1, \ldots, s_j)$ be any sequence of nonnegative integers, and define $u : S \times \mathbb{R} \to \mathbb{R}$ by

$$u(S, x) = \sum_{i=1}^{j} b^{s_i} x^{i-1}. \tag{13}$$

Definition 4. For $w_1 \in F_r^n$, let $w_i \sim w_1$ have code Q_i. We call $u(Q_i, a)$ (also written $u_i(a)$ or u_i) the ith code function of w_1.

Code functions will be called sub- or superdiagonal in accordance with their corresponding cycle point words.

Figure 1 shows the code functions corresponding to the ten rotations of the word $w_1 \in F_{10}^{30}$ whose gaps vector is $D_1 = (4, 2, 5, 1, 1, 0, 1, 2, 3, 1)$ and whose density is $\alpha = 2$. Here $b = \frac{1}{2}$. The code functions for the perigee words are drawn in bold lines, and the upper sections of the curves are shown with compressed vertical scale for clarity. The codes for w_1's rotations are listed in Table 2.

D_1	(4,	2,	5,	1,	1,	0,	1,	2,	3,	1)	h_i	$u_i(\lambda)$
Q_1	(4,	6,	11,	12,	13,	13,	14,	16,	19,	20)	5	151/128
Q_2	(2,	7,	8,	9,	9,	10,	12,	15,	16,	20)	3	151/32
Q_3	(5,	6,	7,	7,	8,	10,	13,	14,	18,	20)	3	151/32
Q_4	(1,	2,	2,	3,	5,	8,	9,	13,	15,	20)	0	151/4
Q_5	(1,	1,	2,	4,	7,	8,	12,	14,	19,	20)	1	151/8
Q_6	(0,	1,	3,	6,	7,	11,	13,	18,	19,	20)	2	151/16
Q_7	(1,	3,	6,	7,	11,	13,	18,	19,	20,	20)	4	151/64
Q_8	(2,	5,	6,	10,	12,	17,	18,	19,	19,	20)	5	151/128
Q_9	(3,	4,	8,	10,	15,	16,	17,	17,	18,	20)	5	151/128
Q_{10}	(1,	5,	7,	12,	13,	14,	14,	15,	17,	20)	4	151/64

Table 2. Gaps vector D_1 and codes Q_i for the code functions of Figure 1, with maximum deviations h_i and the code function values at $a = \lambda$.

We see in this example that, for fixed b, the minimal code function (and thus the perigee word) depends on a, and this is true in general. Proof of the following formal statement is left to the reader.

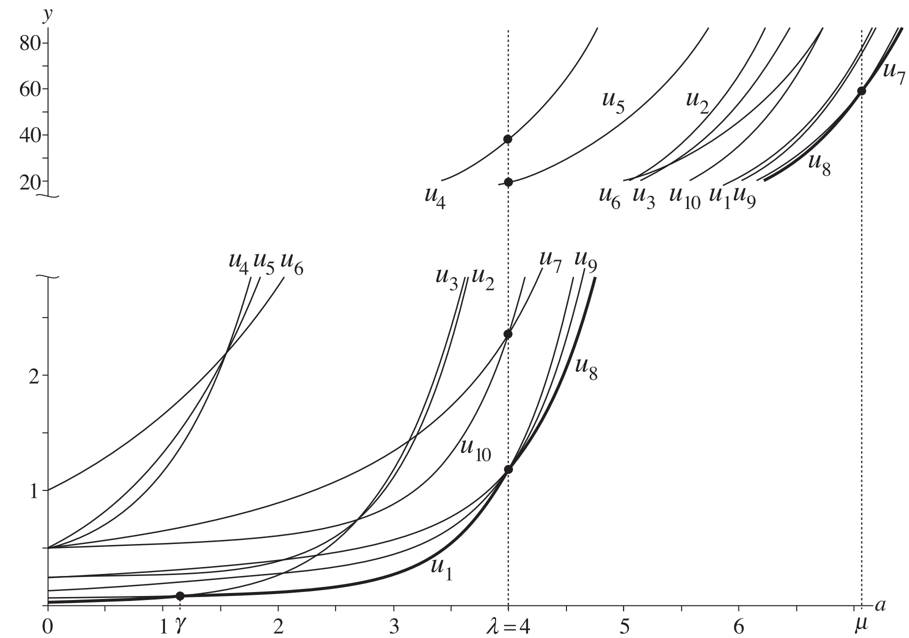

Figure 1. The code functions of the ten rotations of the word whose gaps vector is $D_1 = (4, 2, 5, 1, 1, 0, 1, 2, 3, 1)$, where $b = \frac{1}{2}$.

Proposition 4. *In $\Psi(a_0, b)$ with $a_0 \in \mathbb{R}_+ \backslash \{\lambda\}$, w_i is a perigee word precisely when $u_i(a_0)$ is minimal over all $i \in \{1, \ldots, r\}$.*

8. Ordering of the code functions at $a = \lambda$

Although every rotation w_i yields an undefined cycle point at $a = \lambda$, the code function values $u_i(\lambda)$ are finite and, it turns out, in a convenient order. As the reader may surmise from the last two columns of Table 2, there is an elegant relationship between h_i, $u_i(\lambda)$, and subdiagonal rotations.

Theorem 3. *In $\Psi(a, b)$, let $w_1 \in F_r^n$ have code function u_1, maximum deviation h_1, and density α. Let $w_m \sim w_1$ be subdiagonal with code function u_m. Then*

$$u_1(\lambda) = b^{h_1} u_m(\lambda). \tag{14}$$

This formula says that, in a given n-cycle, a code function's value at $a = \lambda$ is a multiple of *any subdiagonal code function's value at* λ, where the multiplier is a monotone function of the maximum deviation. (If w_1 is itself subdiagonal, then $h_1 = 0$, and (14) holds trivially.)

Proof. We employ the function $u(S, x)$ from equation (13), using various values for x and sequences S. Begin with

$$u_1(\lambda) = u(Q_1, \lambda) = \sum_{i=1}^{r} b^{q_i}(b^{-\alpha})^{i-1} = b^{\alpha} \sum_{i=1}^{r} b^{q_i - i\alpha} = b^{\alpha} u(\Delta_1, 1). \tag{15}$$

Because w_m is subdiagonal, we know from Corollary 1 that Δ_1 is simply a cyclic permutation of $\Delta_m + (h_1)^r$. Thus $u(\Delta_1, 1) = u(\Delta_m + (h_1)^r, 1)$. Substitution in (15) yields

$$\begin{aligned} u(Q_1, \lambda) &= b^{\alpha} u(\Delta_m + (h_1)^r, 1) = b^{\alpha} b^{h_1} u(\Delta_m, 1) \\ &= b^{h_1} u(Q_m, \lambda) = b^{h_1} u_m(\lambda). \end{aligned} \qquad \square$$

If $h_i < h_1$, and w_m is subdiagonal, then by Theorem 3 we have $u_i(\lambda) = b^{h_i} u_m \lambda > b^{h_1} u_m \lambda = u_1(\lambda)$. Indeed, we can say

Corollary 2. *Given $w_i \sim w_1 \in F_r^n$.*

(a) *If $h_i < h_1$, then $u_i(\lambda) > u_1(\lambda)$.*

(b) *If $h_i = h_1$, then $u_i(\lambda) = u_1(\lambda)$.*

It follows that the points $u_i(\lambda)$ are partially ordered. Since subdiagonal rotations have the smallest maximum deviation ($h_i = 0$), and superdiagonal rotations have the largest, Corollary 2 implies that the subdiagonal code functions intersect the line $a = \lambda$ at the highest point, while the superdiagonal code functions meet the line at the lowest point. This is illustrated in Figure 1, where u_4 is subdiagonal and u_1, u_8, and u_9 are superdiagonal. (This example was constructed to show that, additionally, the perigee word need not be superdiagonal in intervals not containing λ, as shown by w_3 and w_7 on $(0, \gamma)$ and (μ, ∞), respectively.)

9. Code function bounds

Two final lemmas are needed to prove the Maximal Perigee Property; the first establishes upper and lower bounds, respectively, on super- and subdiagonal code functions at $a = \lambda$ over all words in F_r^n.

Lemma 2. *Let $w_1 \in F_r^n$ have density α and sub- and superdiagonal rotations w_m and w_k, respectively. Then*

$$u_k(\lambda) \leq r b^{\alpha} \leq u_m(\lambda). \tag{16}$$

Proof. If w_m is subdiagonal with code $(q_{m,1}, q_{m,2}, \dots, q_{m,r})$, then $q_{m,i} - i\alpha \le 0$, and hence $b^{q_{m,i}-i\alpha} \ge 1$ for $i \in \{1, \dots, r\}$. Thus

$$u_m(\lambda) = \sum_{i=1}^{r} b^{q_{m,i}} \lambda^{i-1} = b^{\alpha} \sum_{i=1}^{r} b^{q_{m,i}-i\alpha} \ge rb^{\alpha}.$$

The derivation for the left-hand inequality in (16) is similar. □

Lemma 3. *The only superdiagonal word in F_r^n whose maximum deviation is less than* 1 *is the word whose code is* $(\lceil\alpha\rceil, \lceil 2\alpha\rceil, \dots, \lceil r\alpha\rceil)$.

Proof. For any $w_1 \in F_r^n$, every g-rank q_i is an integer. If $\alpha \notin \{0, 1, 2, \dots\}$, then w_1's superdiagonality and the condition $0 < \max(q_i - i\alpha) < 1$ imply that $i\alpha \le q_i < i\alpha + 1$ for each $i \in \{1, \dots, r\}$, except for at least one i for which $i\alpha < q_i < i\alpha + 1$. But the only such integers are $q_i = \lceil i\alpha\rceil$. As noted at the end of Section 6, the only superdiagonal code of maximum deviation 0 has $\alpha \in \{0, 1, 2, \dots\}$; we thus have $i\alpha = \lceil i\alpha\rceil$ for each i, and again the lemma holds. □

10. Proof of the Maximal Perigee Property

To prove the Maximal Perigee Property, we show that, for the particular value $a = \lambda$, the maximal perigee word $w_{\max} \in P_r^n$ has code $\big(\lceil\alpha\rceil, \lceil 2\alpha\rceil, \dots, \lceil r\alpha\rceil\big)$. It will then follow that this $w_{\max}$ is unique and minimal over all $a \in \mathbb{R}_+$.

Since the case $r = n$ is trivially true, assume $\alpha \ne 0$. By Corollary 2, the minimal code function among a given word's rotations is superdiagonal at $a = \lambda$. To obtain the *largest* such minimal function, we seek a superdiagonal $w_1 \in F_r^n$ for which the nonnegative quantity

$$rb^{\alpha} - u_1(\lambda) \tag{17}$$

from Lemma 2 is minimized. Using Theorem 3, we may write this as

$$0 \le rb^{\alpha} - b^{h_1} u_m(\lambda), \tag{18}$$

where w_m is a subdiagonal rotation of w_1. From Lemma 2 we also have $u_m(\lambda) \ge rb^{\alpha}$. Therefore

$$-b^{h_1} u_m(\lambda) \le -b^{h_1} rb^{\alpha},$$

and this, combined with equations (17) and (18), yields

$$0 \le rb^{\alpha} - u_1(\lambda) = rb^{\alpha} - b^{h_1} u_m(\lambda) \le rb^{\alpha} - rb^{\alpha+h_1}.$$

or, more simply,

$$rb^{\alpha+h_1} \leq u_1(\lambda) \leq rb^{\alpha}.$$

The smallest possible h_1 minimizes the range of $u_1(\lambda)$. Therefore the terms of the desired superdiagonal Q_1 are the r integers on or above, and closest to, the line $y = \alpha x$; that is, $Q_1 = (\lceil\alpha\rceil, \lceil 2\alpha\rceil, \ldots, \lceil r\alpha\rceil)$. Furthermore, Lemma 3 implies that u_1 is unique; it is the only code function whose value at $a = \lambda$ lies in the interval $(rb^{\alpha+1}, rb^{\alpha}]$. We conclude that the Maximal Perigee Property holds at $a = \lambda$; that is, at this one value of a, $w_1 = w_{\max}$ and $Q(w_1) = Q(w_{\max}) = (\lceil\alpha\rceil, \lceil 2\alpha\rceil, \ldots, \lceil r\alpha\rceil)$.

We now prove that this same u_1 is the maximal minimum code function for all $a \in \mathbb{R}_+$. Write $Q_1 = (q_{1,1}, q_{1,2}, \ldots, q_{1,r})$, where $q_{1,j} = \lceil j\alpha\rceil$. We find the rotated code $Q_i = (q_{i,1}, q_{i,2}, \ldots, q_{i,r}), i \in \{1, \ldots, r\}$, using Proposition 2:

$$q_{i,j} = \begin{cases} \lceil (i+j-1)\alpha\rceil - \lceil (i-1)\alpha\rceil, & j \in \{1, \ldots, r-i+1\} \\ \lceil r\alpha\rceil + \lceil (i+j-r-1)\alpha\rceil - \lceil (i-1)\alpha\rceil, & j \in \{r-i+2, \ldots, r\} \end{cases}$$

But the latter case reduces as follows:

$$\begin{aligned} \lceil r\alpha\rceil + \lceil (i+j-r-1)\alpha\rceil - \lceil (i-1)\alpha\rceil = n - r + \lceil (i+j-1)\alpha\rceil \\ - (n-r) - \lceil (i-1)\alpha\rceil = \lceil (i+j-1)\alpha\rceil - \lceil (i-1)\alpha\rceil, \end{aligned}$$

so in fact

$$q_{i,j} = \lceil (i+j-1)\alpha\rceil - \lceil (i-1)\alpha\rceil \tag{19}$$

for all $j \in \{1, \ldots, r\}$. Because $\lceil x\rceil + \lceil y\rceil \geq \lceil x+y\rceil$ for nonnegative real numbers x and y, equation (19) allows us to write

$$\begin{aligned} \lceil j\alpha\rceil + \lceil (i-1)\alpha\rceil &\geq \lceil (i+j-1)\alpha\rceil \\ \lceil j\alpha\rceil &\geq \lceil (i+j-1)\alpha\rceil - \lceil (i-1)\alpha\rceil \\ q_{1,j} &\geq q_{i,j}. \end{aligned} \tag{20}$$

Observe, however, that equality cannot hold in (20) for all $j \in \{1, \ldots, r\}$; if it did, we would have $Q_1 \sim Q_i$, which is possible only when $w_1 = f^n$ and $\alpha = 0$. Therefore, $b^{q_{1,j}} \leq b^{q_{i,j}}$ for $j \in \{1, \ldots, r\}$, except for at least one j' for which $b^{q_{1,j'}} < b^{q_{i,j'}}$. Thus

$$u_1(a) = \sum_{j=1}^{r} b^{q_{1,j}} a^{j-1} < \sum_{j=1}^{r} b^{q_{i,j}} a^{j-1} = u_i(a)$$

for $i \in \{2, \ldots, r\}$. By Proposition 4, it follows that x_1 is less than any other cycle point x_i for any nonnegative a. We conclude that $w_1 = w_{\max}$ for all $a \in \mathbb{R}_+$, and the proof of the Maximal Perigee Property is complete.

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