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On *R*-quadratic Einstein Finsler space

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Abstract. In this paper we investigate Finsler spaces that some characteristic tensor of them depends only on position. Einstein Finsler space as spaces whose Ricci scalar curvature depends only on position and R-quadratic spaces are considered. We prove that any R-quadratic non Ricci flat Einstein Finsler space must be Riemannian.

1. Introduction

Let (M, F) be a Finsler manifold. Given a non-zero vector $y \in T_xM$ at a point $x \in M$, F induces an inner product g_y on T_xM so that $F^2(y) = g_y(y, y)$. The second variation of geodesics gives rise to a family of linear maps $R_y : T_xM \to T_xM$, at any point $y \in T_xM$. Each R_y is self-adjoint with respect to g_y and satisfies $R_y(y) = 0$. R_y is called the Riemann curvature in the direction y. There are many Finsler metrics whose Riemann curvature in every direction is quadratic. In 2001, Shen called such Finsler metrics *R-quadratic* Finsler metrics [8]. This new family of Finsler metrics contains Berwald metrics and R-flat metrics. Indeed a Finsler metric is R-quadratic if and only if the h-curvature of Berwald connection depends on position only in the sense of Bácsó–Matsumoto [3].

The Einstein metrics comprise a major focus in differential geometry. These metrics are more general than those with constant curvature. The well-known Ricci tensor was introduced in 1904 by G. Ricci. Nine years later Ricci's work was used to formulate the Einstein's theory of gravitation. Riemannian metric whose Ricci is proportional to the metric have been studied extensively. They are called Einstein manifold. In the Lorantzian case, they are important in General

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Relativity. Indeed the Einstein equation in the vacuum is given by Ric = 0. Our definition of the Ricci scalar function in Finsler geometry is

$$\mathrm{Ric}(x,y) := \frac{1}{F^2} R^i_i$$

where R^i_k are coefficients of Riemann curvature. We obtain the Ricci tensor from the Ricci scalar function as follows:

$$\mathrm{Ric}_{ij} = \left(\frac{1}{2}F^2\,\mathrm{Ric}(x,y)\right)_{y^i y^j}.$$

This definition, due to Akbar-Zadeh, is motivated by the fact that, when F arises from any Riemannian metric a, the curvature tensor depends on x alone. A Finsler metric is said to be an Einstein metric if the Ricci scalar function is a function of x alone, equivalently

$$\mathrm{Ric}_{ij} = \mathrm{Ric}(x) g_{ij},$$

Going one step further, if $\mathrm{Ric}(x)$ does not depend on the location x either, F is said to be Ricci-constant. In [6], [7] Einstein (α,β) metrics and projectively relation between them are studied.

Finsler spaces that some characteristic tensor of them depends only on position, are investigated by Bácsó–Matsumoto [3], [4]. In this paper, we consider the common part of R-quadratic and Einstein Finsler space and by using of projective Weyl tensor prove that any R-quadratic non Ricci flat Einstein Finsler space must be Riemannian.

Throughout this paper, we make use of *Einstein* convention, that is, repeated indices with one upper index and one lower index denote summation over their range. We also set the *Berwald connection* on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " ; " and " . " respectively.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM as the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ as the tangent bundle of M. Each element of TM has the form (x,y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM/\{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x,y) = x$. The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber π^*_vTM at $v \in TM_0$ is just T_xM, where $\pi(v) = x$. Then

$$\pi^*TM = \{(x,y,v) \mid y \in T_xM_0,\ v \in T_xM\}$$

A Finsler metric on a manifold M is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on TM_0,

(ii) $F(x, \lambda y) = \lambda F(x, y) \ \ \lambda > 0$,

(iii) For any tangent vector $y \in T_x M$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$g_{ij}(x, y) = \left[\frac{1}{2}F^2\right]_{y^i y^j}$$

is positive definite.

Let F be a Finsler metric on M. For a non-zero vector $y \in T_p M$, F induces an inner product g_y on $T_p M$ by

$$g_y(u, v) := g_{ij}(x, y)u^i v^j = \frac{1}{2}[F^2]_{y^i y^j}(x, y)u^i v^j.$$

Here $x = (x^i)$ denotes the coordinates of $p \in M$ and $(x, y) = (x^i, y^i)$ denotes the local coordinates of $y \in T_p M$. The geodesics are characterized by

$$\frac{d^2 c^i}{dt^2} + 2G^i\big(\dot{c}(t)\big) = 0,$$

where $G^i := \frac{1}{2}g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}$ are called the *geodesic coefficients* of F.

Definition 2.1. A Finsler metric F on a manifold M is called *Berwald metric* if in a standard local coordinate system (x^i, y^i) in TM, the spray coefficients G^i are quadratic in $y \in T_x M$ for all $x \in M$.

The Riemann curvature $R_y = R^i_k dx^k \otimes \frac{\partial}{\partial x^i}|_p : T_p M \to T_p M$ is defined by

$$R^i_k(y) := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{1}$$

The Riemann curvature has the following properties [2]. For any non-zero vector $y, u, v \in T_p M$,

$$R_y(y) = 0, \quad g_y(R_y(u), v) = g_y(u, R_y(v)),$$

$$R^i_{kl} = \frac{1}{3}\left\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\right\}.$$

$$R^i{}_{jkl}(x, y) := \frac{1}{3}\frac{\partial}{\partial y^j}\left\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\right\} \tag{2}$$

where $R^i{}_{jkl}$ is the Riemann curvature of Berwald connection.

Definition 2.2. A Finsler metric is called *R-quadratic* if R_y is quadratic in y, namely, in local coordinates, $R^i_k(y)$ are quadratic in $y \in T_xM$.

From (2), we have:

$$R^i_k = R^i{}_{jkl}(x,y)y^jy^l.$$

So we can conclude that R^i_k is quadratic in $y \in T_xM$ if and only if $H^i{}_{jkl}$ are functions of position alone.

The relation between Landsberg metrics and R-quadratic metrics was studied by Shen. He proved the following

Theorem 2.1 ([8]). *Every compact R-quadratic Finsler metric is a Landsberg metric.*

It is obvious that every Berwald metric is quadratic in $y \in T_xM$, i.e, every Berwald space is a R-quadratic space. Hence on compact Finsler manifolds we have

$$\{\text{Berwald metrics}\} \subset \{R\text{-quadratic metrics}\} \subset \{\text{Landsberg metrics}\}.$$

For a two-dimensional plane $P \subset T_pM$ and a non-zero vector $y \in T_pM$, the flag curvature $K(P,y)$ is defined by

$$K(P,y) := \frac{g_y(u, R_y(u))}{g_y(y,y)g_y(u,u) - g_y(y,u)^2},$$

where $P = \text{span}\{y,u\}$. F is said to be of scalar curvature $K = \lambda(y)$ if for any $y \in T_pM$, the flag curvature $K(P,y) = \lambda(y)$ is independent of P containing $y \in T_pM$, that is equivalent to the following system in a local coordinate system (x^i, y^i) in TM,

$$R^i_k = \lambda\, F^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}.$$

If λ is a constant, then F is said to be of constant flag curvature.
The Ricci scalar function of F is given by

$$\text{Ric}(x,y) := \frac{1}{F^2}R^i_i. \tag{3}$$

Therefore, the Ricci scalar function is positive homogeneous of degree 0 in y. This means that $\text{Ric}(x,y)$ depends on the direction of the flag pole y but not its length. The Ricci tensor of a Finsler metric F is defined by

$$\text{Ric}_{ij} := \frac{\partial^2}{\partial y^i \partial y^j}\left\{\frac{1}{2}R^k_k\right\}. \tag{4}$$

It is obvious that the Ricci tensor defined by (4), is symmetric and can be a good generalization of Ricci tensor in Riemannian case. Ricci-flat manifolds are manifolds whose Ricci tensor vanishes. In physics, Riemannian Ricci-flat manifolds are important because they represent vacuum solutions to Einstein's equations. Ricci-flat manifolds are special cases of Einstein manifolds which are defined as follows:

Definition 2.3. A Finsler metric is said to be an *Einstein metric* if the Ricci scalar function is a function of x alone, equivalently

$$\mathrm{Ric}_{ij} = \mathrm{Ric}(x) g_{ij},$$

We want to consider projectively related Finsler metrics – those having the same geodesics as set of points. Rapcsák proved the following important lemma

Lemma 2.2 ([5], Rapcsák)**.** *Let (M, F) be a Finsler space. A Finsler metric $\tilde{F}$ is pointwise projective to F if and only if*

$$\frac{\partial \tilde{F}_{;k}}{\partial y^l} y^k - \tilde{F}_{;l} = 0.$$

In this case,

$$\tilde{G}^i = G^i + P y^i \tag{5}$$

with

$$P = \frac{\tilde{F}_{;k} y^k}{2\tilde{F}}. \tag{6}$$

Let F and $\tilde{F}$ be Finsler metrics on an n-dimensional manifold M. Assume that $\tilde{F}$ is pointwise projective to F. Plugging (5) into (1) yields

$$\tilde{R}^i_k = R^i_k + \Xi \delta^i_k + \tau_k y^i, \tag{7}$$

$$\tilde{R}^m_m = R^m_m + (n-1)\Xi(y) \tag{8}$$

where

$$\Xi(y) := P^2 - P_{;k} y^k, \qquad \tau_y(u) := 3(P_{;k} - P P_{.k}) + \Xi_{.k}.$$

Let

$$A^i_k := R^i_k - \frac{1}{n-1} R^m_m \delta^i_k.$$

Then the Weyl curvature tensor W^i_k, is defined by

$$W^i_k := A^i_k - \frac{1}{n+1} \frac{\partial A^m_k}{\partial y^m} y^i. \tag{9}$$

The Weyl curvature tensor W^i_k, satisfying

$$W^i_k y^k = 0, \quad W^i_i = 0$$

Suppose that Finsler metric $\tilde{F}$ is projective to F, i.e. $\tilde{G}^i = G^i + Py^i$. An important fact is that the Weyl curvature W^i_k of G is equal to the Weyl curvature $\widetilde{W}^i_k$ of $\tilde{G}$. Thus the Weyl curvature is a projective invariant. It is easy to see that a Finsler metric is of scalar flag curvature if and only if the Weyl tensor vanishes. We denote by $W(x)$ the spaces that the Weyl tensor of them depends only on position. In [3], Bácsó and Matsumoto showed that the spaces of h-curvature depends only on position and Douglas spaces are subset of the spaces that the Weyl tensor depends only on position.

In [9], Shen investigated projectively Einstein Finsler manifolds. He considered Einstein metrics as a *constant* Ricci scalar and described all Einstein metrics which are pointwise projective to the given an Einstein metric. He proved following theorem:

Theorem 2.3 ([10])**.** *Let F and $\tilde{F}$ be Einstein metrics on a closed n-manifold M with*

$$\mathrm{Ric} = (n-1)\lambda, \quad \widetilde{\mathrm{Ric}} = (n-1)\tilde{\lambda},$$

where $\lambda, \tilde{\lambda} \in \{-1, 0, 1\}$. Suppose that $\tilde{F}$ is pointwise projectively related to F. Then λ and $\tilde{\lambda}$ have the same sign. More details are given below.

(i) *If $\lambda = 1 = \tilde{\lambda}$, then along any unit speed geodesic $c(t)$ of F*

$$\tilde{F}(\dot{c}(t)) = \frac{2}{\big(a^2 - 1/a^2 - b^2\big)\cos(2t) + 2ab\sin(2t) + \big(a^2 + 1/a^2 + b^2\big)},$$

where $a > 0$ and $-\infty < b < \infty$ are constants. Thus, for any unit speed geodesic segment c of F with length of π, it is also a geodesic segment of $\tilde{F}$ (as a point set) with length of π.

(ii) *If $\lambda = 0 = \tilde{\lambda}$, then along any geodesic $c(t)$ of F or $\tilde{F}$,*

$$\frac{F(\dot{c}(t))}{\tilde{F}(\dot{c}(t))} = \textit{constant}.$$

(iii) *If $\lambda = -1 = \tilde{\lambda}$, then*

$$\tilde{F} = F.$$

3. Einstein Finsler space

In this section, we prove that any R-quadratic Einstein Finsler space with non zero Ricci scalar is Riemannian. From (3) and (9) we can conclude that

$$W^i_k := R^i_k - \frac{1}{n-1}\,\mathrm{Ric}(x)\delta^i_k F^2 - \frac{1}{n+1}\left\{R^m_{k.m} - \frac{2}{n-1}\,\mathrm{Ric}(x)y_k\right\}y^i \tag{10}$$

Theorem 3.1. *Let F^n be an R-quadratic Einstein Finsler space with non zero Ricci scalar. Then F^n must be Riemannian.*

Proof. F^n is Einstein space, so from (10), we get

$$W^i_{mkn}y^m y^n = R^i_{mkn}y^m y^n - \frac{1}{n+1}\left\{(R^h_{mkn}y^m y^n)_{.h} - \frac{2}{n-1}\,\mathrm{Ric}(x)y_k\right\}y^i - \frac{1}{n-1}\,\mathrm{Ric}(x)\delta^i_k F^2 \tag{11}$$

By assumption F^n is R-quadratic, i.e. R^i_{mkn} depends only on position, x, and by Proposition 4 in [3] F^n is W-quadratic. So we can get

$$\frac{\partial^3}{\partial y^a \partial y^b \partial y^c}\{R^i_{mkn}y^m y^n\} = 0$$

and

$$\frac{\partial^3}{\partial y^a \partial y^b \partial y^c}\{W^i_{mkn}y^m y^n\} = 0$$

By differentiation of (11) with respect to y^a, y^b and y^c

$$2(n+1)\,\mathrm{Ric}(x)\delta^i_k C_{abc} - \mathrm{Ric}(x)\{2C_{kab.c}y^i + 2C_{kab}\delta^i_c + 2C_{kac}\delta^i_b + 2C_{kbc}\delta^i_a\} = 0$$

By multiplying g^{ik}, we will have

$$2(n-1)(n+2)\,\mathrm{Ric}(x)C_{abc} = 0$$

By assumption F^n is non Ricci flat, so it is Riemannian. □

From above theorem, we can conclude as follows:

Corollary 3.1. *The Weyl curvature tensor of any R-quadratic Einstein Finsler space is written as follows:*

$$W^i_k := R^i_k - \frac{1}{n+1}R^m_{k.m}y^i$$

In the scalar flag curvature, Theorem 3.1 is reduced to the famous Numata theorem in compact case. Because in this case, we have $\mathrm{Ric}(x,y) = (n-1)K(x,y)$ and we know that any R-quadratic Finsler metric on compact manifold is Landsberg.

In [11], the following theorem and example have been shown:

Theorem 3.2 ([11]). *Let (M,F) be an n-dimensional ($n > 2$) R-quadratic Finsler manifold. Suppose that F is of scalar flag curvature. Then F is of constant flag curvature.*

In the scalar flag curvature spaces, any Finsler metrics of non constant flag curvature are neither R-quadratic nor Einstein.

Example 1. Let

$$F := |y| + \frac{\langle x,y\rangle}{\sqrt{1+|x|^2}}, \quad y \in T_x\mathbb{R}^n \simeq \mathbb{R}^n$$

where $|.|$ and $\langle\,,\rangle$ denote the Euclidian norm and inner product on $\mathbb{R}^n$ respectively. F is indeed a Randers metric on the whole of $\mathbb{R}^n$ and it is a projectively flat Randers metric on $\mathbb{R}^n$ i.e., the spray coefficients are in the form $G^i = Py^i$, for a scalar function on TM_0 given by

$$P = c\left(|y| - \frac{\langle x,y\rangle}{\sqrt{1+|x|^2}}\right),$$

where $c = 1/2(\sqrt{1+|x|^2}\,)$. Since F is projectively flat, it is of scalar flag curvature. The flag curvature of F given by

$$K = \frac{3}{4(1+|x|^2)}.\frac{|y|\sqrt{1+|x|^2} - \langle x,y\rangle}{|y|\sqrt{1+|x|^2} + \langle x,y\rangle}.$$

Then by Theorem 3.2, this Randers metric is not R-quadratic.

The above example shows that R-quadratic metrics are not invariant under projective change, because it is projectively related to an R-quadratic metric, i.e. Euclidean metric. This is also true for Einstein case, i.e. in general case, not only Einstein Finsler metrics but also Ricci flat metrics are not projectively isolated. For example, locally Minkowski metrics on a torus T^n are pointwise projective to the standard flat Riemann metric on T^n. In fact, these are the only flat metrics on T^n. From (8) we can get a corollary as follows:

Corollary 3.2. *Let F^n be an Einstein Finsler spaces projectively related with $\bar{F}^n$. Then $\bar{F}^n$ is Einstein Finsler space if and only if*

$$\left(\frac{\Xi}{\bar{F}^2}\right)_{.k} = 0$$

Until now, there are very few results on Ricci-flat spaces so the non-trivial problem is to study Ricci-flat metrics and projectively relation between them. On R-quadratic spaces, we will get following theorem:

Theorem 3.3. *Let F^n be R-quadratic Einstein Finsler spaces projectively related with $\bar{F}^n$. Suppose that $\bar{F}^n$ is R-quadratic, then $\bar{F}^n$ is Einstein if and only if*

$$y^k P_{;k} = P^2$$

Proof. Suppose $y^k P_{;k} = P^2$, so $\overline{\mathrm{Ric}} = \mathrm{Ric}$. We know that $\mathrm{Ric} = 0$ by Theorem 3.1. So $\overline{\mathrm{Ric}} = 0$ and $\bar{F}^n$. Vice versa, if $\bar{F}^n$ is Einstein then $\overline{\mathrm{Ric}} = 0$ and so $y^k P_{;k} = P^2$.

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