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Characterizations of strong and statistical convergences

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Abstract. The two main objectives of the paper are to cast the concept of A-uniform integrability in the measure-theoretic sense leading to showing that a sequence is A-strongly convergent if and only if it is A-statistically convergent and it is A-uniformly integrable. The second aim is to give an almost all subsequence characterization of A-statistical convergence.

1. Introduction

Let $x = (x_k)$ be a given sequence of real (or complex) numbers. When the notational convenience requires, x_k will be denoted by $x(k)$. Let $A = (a_{nk})$ be an infinite matrix with nonnegative entries with each row adding up to one. The class of such summability methods will be denoted by $\mathbb{M}^+$, and such matrices form the most commonly used class of summability methods. Their primary use is to extend the notion of convergence for some properly divergent sequences. The main results of this paper deal with the interplay of the following three modes of convergence. Our approach is measure-theoretic, along the lines of Buck [2].

- A real sequence, x, is defined to be A-distributionally convergent to F, where

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F is a probability distribution on $\Re$, if

$$\lim_{n\to\infty} \sum_{k:x_k\le t} a_{nk} = F(t), \tag{1}$$

for all t at which F is continuous.

- A real (or complex) sequence, x, is defined to be A-statistically convergent to α if for any $\epsilon > 0$, we have

$$\lim_{n\to\infty} \sum_{k:|x_k-\alpha|\ge\epsilon} a_{nk} = 0. \tag{2}$$

The concepts of A-statistical convergence of a sequence, and the A-density of a subset E of nonnegative integers are related. We say [9] that E has A-density if

$$\delta_A(E) := \lim_{n\to\infty} \sum_{k\in E} a_{nk} \quad \text{exists.}$$

Of course a sequence x is A-statistically convergent to α if the set $E = \{k : |x_k - \alpha| \ge \epsilon\}$ has A-density zero, for every $\epsilon > 0$.

- A real (or complex) sequence, x, is defined to be A-strongly summable to α if

$$\lim_{n\to\infty} \sum_{k} |x_k - \alpha|\, a_{nk} = 0. \tag{3}$$

The concept of strong summability was introduced by Hardy and Littlewood [14] (cf. Boos [1]) in the context of summability of Fourier series. Statistical convergence has been studied in various contexts (see, for instance, Zygmund [30], Freedman and Sember [9]). It is well known (Freedman and Sember [9], Connor [6], Kolk [19], Fridy and Orhan [12]) that if a sequence x is A-strongly convergent to α then it is A-statistically convergent to α and that if a sequence x is bounded and A-statistically convergent to α then x is A-strongly convergent to α. For the converse result the assumption of boundedness of x is not totally essential. Some attempts to relax this assumption have been made (see Connor [6]). In the next section we will introduce the concept of an A-uniformly integrable sequence x, and show that x is A-strongly convergent if and only if it is A-statistically convergent and A-uniformly integrable. This provides an answer to a problem left open by Connor [6].

Section three contains a subsequence characterization of A-statistical convergence, which solves a problem left open in Miller [23].

Section four provides extensions of several results of the earlier sections when the entries of a sequence $x = (x_k)$ is replaced by the entries of a matrix $X = (x_{kn})$ in the defining equations (1), (2), and (3).

2. *A*-strong convergence and uniform integrability

It should be noted that both A-statistical convergence and A-strong convergence could be defined over arbitrary metric spaces. However, since the conclusions are based on some real continuous functions of the distance, without loss of any generality, we will assume that the sequence $x = (x_k)$ is real valued.

For the characterization of A-statistical and A-strong convergence we introduce the following concept for real sequences, however, again this concept can be analogously defined over metric spaces.

Definition 2.1. Let $A \in \mathbb{M}^+$. A sequence $x = (x_k)$ will be called A-uniformly integrable if

$$\lim_{c\to\infty} \sup_n \sum_{k:|x_k|>c} |x_k|\, a_{nk} = 0.$$

In the above definition the assumption of A having nonnegative entries can be dropped by using $|a_{nk}|$ instead, nor does one need that the row sums be one. For our later developments, however, the above definition will suffice.

To link up this notion of A-uniform integrability and to provide its various characterizations, we introduce the following notation that we will use throughout the paper. Define a sequence of functions $f_n : [0,1] \to \{0, 1, 2, \dots\}$, where

$$f_n(s) = k \quad \text{if and only if} \quad s \in \left[\sum_{j=0}^{k-1} a_{nj}, \sum_{j=0}^{k} a_{nj}\right),$$

for $k = 1, 2, \dots$ and $f_n(s) = 0$ if $0 \le s < a_{n0}$. Next define $g_n(s) := x(f_n(s))$. We have the following characterization of A-uniform integrability.

Lemma 2.1. *Let $x = (x_k)$ be a given real sequence and let $A \in \mathbb{M}^+$. Let $L^1([0,1], \lambda)$ be the space of Lebesgue integrable functions on $[0,1]$, and λ be the Lebesgue measure. The following are equivalent.*

- *The sequence x is A-uniformly integrable.*
- *The set $K = \{g_n, n \ge 1\}$ is uniformly integrable in L^1.*
- *The set $K = \{g_n, n \ge 1\}$ is relatively weakly compact in L^1.*
- *K is bounded in L^1 and the indefinite integrals of members of K are uniformly countable additive.*
- *There exists a function $\Phi : \Re \to \Re$ so that Φ is convex, even, $\Phi(0) = 0$, $\lim_{x\to\infty} \Phi(x)/x = \infty$ and*

$$\sup_n \sum_k \Phi(|x_k|)\, a_{nk} < \infty.$$

PROOF. A straight forward rewriting of the Lebesgue integrals concerning $g_n(s) = x(f_n(s))$ in L^1, shows that the first two parts are equivalent. The equivalence of the second through fourth parts is the well known Dunford–Pettis theorem [7]. The equivalence of the second and the last part is a result of de la Vallee Poussin. □

We may remark that relative weak compactness has been characterized in $L^1(B, \mu)$, when B is a Banach space, see ÜLGER [29] and DIESTEL et. al. [8]. However, we will not need this generality for the results of this paper.

All bounded sequences are A-uniformly integrable. One may easily construct unbounded sequences that are A-uniformly integrable when $A \in \mathbb{M}^+$ is regular and $\max_k a_{nk} \to 0$. In fact, if $A \in \mathbb{M}^+$ has null columns and

$$\liminf_k \big\{ \max_n a_{nk} \big\} = 0, \tag{4}$$

then we may construct unbounded A-uniformly integrable sequences as follows. Select column indices, $K(0) < K(1) < \dots$ so that

$$\max_n a_{n,K(m)} < 2^{-m}, \quad m = 0, 1, 2, \dots .$$

Then choose increasing row indices $N(j)$ such that if $n > N(m)$ and $k \le K(m)$ then $a_{nk} < 2^{-m}$. Now define a sequence y by $y_k := k + 1$ if $k = K(m)$ and zero otherwise. Then for all $n > N(m)$ note that

$$(Ay)_n = \sum_{j=0}^{\infty} (j+1)\, a_{n,K(j)} \le \sum_{j=0}^{m} (j+1)\, 2^{-m} + \sum_{j>m} (j+1)\, 2^{-j} \to 0.$$

Therefore, by the above de la Vallee Poussin characterization of A-uniform integrability, we see that the unbounded sequence x, defined by $x_k := \sqrt{y_k}$, is A-uniformly integrable. In fact, the condition (4) may also be dispensed with, by using the MAZUR–ORLICZ theorem [20], [21], [10] as long as A sums a nonconvergent sequence. The collection of all A-uniformly integrable sequences is a linear space.

The equivalence of A-strong convergence concept to the A-statistical convergence concept for bounded sequences has appeared in several independent sources such as ZYGMUND [30], HILL and SLEDD [16], FRIDY and ORHAN [12], CONNOR [6] and KOLK [19], some of which assume further conditions on the matrix A. Boundedness of the sequence x is, of course, unnecessarily restrictive. The following theorem shows that the natural condition for the sequence x, for characterizing A-strong convergence via A-statistical convergence, is A-uniform integrability. This provides an answer to a question left open by CONNOR [6].

Theorem 2.1. *Let $A \in \mathbb{M}^+$. Let $x = (x_k)$ be a real sequence. The following statements are equivalent.*

(i) *The sequence x is A-distributionally convergent to F and x is A-uniformly integrable, where F is the characteristic function of the half line $[0, \infty)$.*

(ii) *The sequence x is A-statistically convergent to 0 and x is A-uniformly integrable.*

(iii) *The sequence x is A-strongly convergent to 0.*

Proof. For any $\epsilon > 0$, we see that

$$\sum_{k:|x_k|>\epsilon} a_{nk} \leq \sum_{k:x_k \leq -\epsilon} a_{nk} + \sum_k a_{nk} - \sum_{k:x_k \leq \epsilon} a_{nk} \to 0 + 1 - 1 = 0.$$

This gives that (i) implies (ii). When (ii) holds, by Lemma 2.1 there exists a convex Φ so that $x/\Phi(x) \to 0$ as x gets large. For any $\epsilon > 0$, this gives an $L > 0$ so that $x/\Phi(x) < \epsilon$ for all $x > L$. Define a bounded sequence $y_k = x_k$ when $|x_k| \leq L$ and zero otherwise. Note that (y_k) is also A-statistically convergent to zero. Since it is bounded, (y_k) is A-strongly convergent to zero. Using $0/0$ to represent zero, we have

$$\limsup_n \sum_k |x_k|\, a_{nk} \leq \epsilon \sup_n \sum_k \Phi(|x_k|)\, a_{nk} + \limsup_n \sum_k |y_k|\, a_{nk}.$$

This implies that (ii) implies (iii). To show that (iii) implies (i), for any $t < 0$, we have

$$\sum_{k:x_k \leq t} a_{nk} \leq \frac{-1}{t} \sum_k |x_k|\, a_{nk} \to 0.$$

A similar argument shows that, for $t > 0$, $\sum_{k:x_k \leq t} a_{nk} \to 1$. Finally, to show that x must be A-uniformly integrable, let $\epsilon > 0$. There exists a positive integer N so that $\sum_k |x_k| a_{nk} < \epsilon$ for all $n \geq N$. Since $\sum_k |x_k| a_{nk} < \infty$, for each $n = 1, 2, \ldots, N-1$, choose a positive integer K large enough so that $\sum_{k>K} |x_k| a_{nk} < \epsilon$ for all $n < N$. Now, whenever $c > \max\{|x_1|, \ldots, |x_K|\}$, we see that

$$\sup_n \sum_{k:|x_k|>c} |x_k|\, a_{nk} < \epsilon.$$

This finishes the proof. □

We should point out that the spaces of A-uniformly integrable sequences are also extremely useful in characterizing the multiplier spaces, as well as characterizing A-statistical convergence by a single matrix summability method. For more on this direction, see Khan and Orhan [18].

3. *A*-statistical convergence of subsequences

To date subsequence characterization of A-statistical convergence is not fully understood if one identifies subsequences by the dyadic expansion of points in $(0,1]$ and the subsequences are "chosen" with respect to the Lebesgue measure over $(0,1]$. The only known example is for the Cesáro method, due to Miller [23]. We will give a way of identifying those summability methods A for which such a characterization will hold. As examples we will show that all the classical summability methods of convolution type (such as the Euler family, Borel matrix method, the circle family) have this property.

If $x=(x_k)$ is a sequence, a subsequence (x_{k_j}) will be identified by the dyadic expansions of points of $(0,1]$. Notationally,

$$S(x,\omega)=(x_{k_1},x_{k_2},\dots),$$

where ω has the dyadic expansion $(e_k(\omega))$ which takes value 1 over the positive integers $k_1,k_2,\dots$ and zero otherwise. In this section the phrase 'almost all subsequences' may be identified with a subset of the set of normal numbers that has Lebesgue measure one.

The following theorem gives a general class of summability methods for which the above mentioned characterization can be provided. The class of summability methods that we will consider in the theorem is defined as follows.

Definition 3.1. Let $A\in\mathbb{M}^+$. We will say that A has the "density translativity property" if for any subset E of positive integers with A-density zero, and for almost all subsequences of positive integers $(m_1,m_2,\dots)$,

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}a_{n,k}I(m_k\in E)=0.$$

(Here the notation $I(k\in E)$ stands for 1 if $k\in E$ and zero otherwise.)

As we will show below, besides the Cesáro method, most of the usual regular summability methods, such as Euler, Borel, circle and random walk family of methods have the above density translativity property. First we present the main result of this section when x is a complex sequence but the result goes, without change, to arbitrary metric space valued sequences.

Theorem 3.1. *Let $A\in\mathbb{M}^+$ be regular and let x be a sequence. Let $([0,1],\mathbb{B},\lambda)$ be the Lebesgue measure space. The summability method A has the density translativity property if and only if the following two statements are equivalent.*

- *The sequence x is A-statistically convergent to α.*
- $\lambda(\{\omega : S(x,\omega)$ *is A-statistically convergent to* $\alpha\}) = 1$.

Proof. Let A have the density translativity property and assume that the first statement is true. This gives that there exists a set E with $\delta_A(E) = 0$ and x is convergent to α over E^c. Miller [23] proves this result for triangles, however, one may extend this result to arbitrary regular methods in $\mathbb{M}^+$ by replacing A with a boundedly equivalent triangle in $\mathbb{M}^+$ (see also [19]).

For any $\epsilon > 0$, get M so that

$$|x_k - \alpha| < \epsilon, \quad k > M, \ k \notin E.$$

Next, for any subsequence $S(x,\omega)$, note that

$$\begin{aligned}
\sum_{k:|S_k(x,\omega)-\alpha|\geq\epsilon} a_{nk} &= \sum_{k\leq M:|S_k(x,\omega)-\alpha|\geq\epsilon} a_{nk} + \sum_{k>M:|S_k(x,\omega)-\alpha|\geq\epsilon} a_{nk} \\
&\leq \sum_{k\leq M} a_{nk} + \sum_{k>M:|S_k(x,\omega)-\alpha|\geq\epsilon} a_{nk} \\
&\leq \sum_{k\leq M} a_{nk} + \sum_{k>M:m_k\in E} a_{nk} \\
&\leq \sum_{k\leq M} a_{nk} + \sum_{k=0}^{\infty} a_{nk} \ I(m_k \in E).
\end{aligned}$$

The first sum goes to zero by the regularity of A, and the last sum goes to zero for almost all ω since A has the density translativity property.

On the other hand if ω is a normal number having dyadic expansion $(e_k(\omega))$ then ω', defined to have the dyadic expansion $(1-e_k(\omega))$, is also a normal number. The given information implies that there exists a pair ω, ω' of normal numbers with the property that both $S(x,\omega)$, $S(x,\omega')$ are A-statistically convergent to α. So, we see that x must be A-statistically convergent to α.

The converse follows easily by using the fact that for bounded sequences A-strong convergence is equivalent to A-statistical convergence. □

Example 3.1. In the following we outline some of the classical summability methods that have the density translativity property.

(1) It is easy to show that if A, B are equivalent over bounded sequences and one of them has the density translativity property then so does the other.

(2) The Cesáro method has the density translativity property. Indeed, let E have

Cesáro density zero. Note that if $\frac{m_n}{n} \to 2$, as n gets large, we have

$$\frac{1}{n}\sum_{k=1}^{n} I(m_k \in E) \leq \frac{m_n}{n}\frac{I(1\in E)+I(2\in E)+\cdots+I(m_n\in E)}{m_n} \to 0.$$

Using the result of item (1), and the fact that for bounded sequences all Cesáro (C,k) methods and the Abel method are equivalent and that (C,k) method is equivalent to Hölder (H,k) method, we notice that $(C,k),(H,k)$, for all $k \geq 1$ and the Abel method have the density translativity property.

(3) To show that all the regular circle family of methods have the density translativity property, we consider a slightly more general family of convolution methods with finite (positive) variance. Such a method, $A=(a_{nk})$ is constructed by the Cauchy products of a sequence $p=(p_k)$, $p_k \geq 0$, $\sum_k p_k = 1$. Namely, $a_{1k}=p_k$ and $a_{n+1,k}=(a_{n,\cdot} * p)_k$. Denote by $a=\sum_k kp_k$ and $\sigma^2 = \sum_k (k-a)^2 p_k$ as the mean and variance of p. For any $\epsilon \in (0,1)$, choose M large enough so that

$$\frac{1}{\sqrt{2\pi}}\int_{\Re\setminus[-M,M]} e^{-t^2/2}\,dt < \epsilon. \tag{5}$$

Now note that row maximum of any such method is of the order of $n^{-1/2}$. Therefore, for some constant K,

$$\begin{aligned}
\sum_{k=0}^{\infty} a_{nk} I(m_k\in E) &= \sum_{k:|k-na|>\sigma M\sqrt{n}}^{\infty} a_{nk} I(m_k \in E)\\
&\quad + \sum_{k:|k-na|\leq\sigma M\sqrt{n}}^{\infty} a_{nk} I(m_k \in E)\\
&\leq \sum_{k:|k-na|>\sigma M\sqrt{n}}^{\infty} a_{nk} + \frac{K}{\sqrt{n}} \sum_{k:|k-na|\leq\sigma M\sqrt{n}}^{\infty} I(m_k\in E)\\
&\leq \sum_{k:|k-na|>\sigma M\sqrt{n}}^{\infty} a_{nk} + \frac{K\sum_{\ell=1}^{2M\sigma\sqrt{n}} G_\ell^*}{\sqrt{n}} \frac{1}{\sum_{\ell=1}^{2M\sigma\sqrt{n}} G_\ell^*}\\
&\quad \times \sum_{k=m_{na-M\sigma\sqrt{n}}}^{m_{na-M\sigma\sqrt{n}}+\sum_{\ell=1}^{2M\sigma\sqrt{n}} G_\ell^*} I(k\in E),
\end{aligned}$$

where $G_\ell^* = G_{na-M\sigma\sqrt{n}+\ell}$. By the central limit theorem and (5) the first sum can be made smaller than 2ϵ for all large n. The fact that the expression

$\sum_{\ell=1}^{2M\sigma\sqrt{n}} G_\ell^*/\sqrt{n}$ converges is by a result of Chow [4] concerning delayed sums. To show that the remaining term goes to zero, we use the fact that the A-density of a set E is zero if and only if (for any $\delta > 0$)

$$\frac{1}{\sqrt{n}} \sum_{\ell \in [n, n+\delta\sqrt{n}\,)} I(\ell \in E) \to 0. \tag{6}$$

This gives that, for any $a > 0$,

$$\frac{1}{\sqrt{na}} \sum_{\ell \in [na, na+\delta\sqrt{an}\,)} I(\ell \in E) \to 0.$$

Now consider

$$\begin{aligned} &\frac{1}{\sqrt{na}} \sum_{\ell \in [na - M\sigma\sqrt{n}, na+M\sigma\sqrt{n}\,)} I(\ell \in E) \\ &\quad = \frac{1}{\sqrt{na}} \sum_{\ell \in [na - M\sigma\sqrt{n}, na)} I(\ell \in E) + \frac{1}{\sqrt{na}} \sum_{\ell \in [na, na+M\sigma\sqrt{n}\,)} I(\ell \in E). \end{aligned}$$

Here the second sum goes to zero. To show that the first sum also goes to zero, note that we need to see if $b_n := na - M\sigma\sqrt{n}$ has the property that $[b_n, b_n + \delta\sqrt{b_n}\,)$ contains the interval $[b_n, na)$, for some $\delta > 0$. This is indeed the case if we take $\delta = 2M\sigma/\sqrt{a}$.

We should remark here that analogous results for $(C,1)$-summability of almost all subsequences were studied by Buck and Pollard [3] and Tsuchikura [28]. The A-summability of almost all subsequences, for arbitrary regular matrices $A \in \mathbb{M}^+$, is still an open problem.

4. Summability of double arrays

In this section we point out that several of our earlier results carry over to the case when $X = (x_{kn})$ is a double array and we define A summability of X by

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} x_{kn}\, a_{nk} = \alpha. \tag{7}$$

The concepts of A-distributional convergence of X, A-statistical convergence of X and A-strong convergence of X are defined analogously by replacing the term x_k

by x_{kn} in our defining equations (1), (2) and (3). Lemma 2.1, and Theorem 2.1, continue to hold in this generality. The proofs, being identical to the sequence case, are omitted.

We should remark that there are other forms of summability concepts for double arrays, [13], [24], [25] and [26]. We will, however, limit our discussion to the form ((7)).

Summability of matrices, as defined in ((7)), commonly arises in approximation theory, probability and statistics contexts. For instance, central limit theorem of triangular arrays [5], [11]; limit theorems concerning random matrices [22]; limit theorems concerning order statistics [27]; and various positive linear approximation operators [15], [17], and [11]. In some of these contexts the four summability methods that play a prominent role are the Euler, logarithmic, Cesáro and Abel summability methods.

It is natural to ask how one should specify regularity in the context of summability of double arrays. A plausible definition is to say that A is "matrix-regular" if for any bounded matrix X that is Pringsheim convergent to α (which means that for any $\epsilon > 0$ there exists an N so that $|x_{kn} - \alpha| < \epsilon$ for all $n, k \geq N$) implies that (7) holds. However, one can easily show that A is "matrix-regular" if and only if A is regular in the classical sense. However, this similarity ends here, as the following remarks show, the classical comparison results in the sequence context fail to hold in this matrix context even at the very basic level.

For instance, consider the classical Bernstein polynomial operator of approximation theory,

$$B_n(f, r) := \sum_{k=0}^{n} f(k/n) \binom{n}{k} r^k (1-r)^{n-k}, \quad n \geq 1.$$

If we take $A = (e_{nk}(r))$, where $e_{nk}(r) := \binom{n}{k} r^k (1-r)^{n-k}$, to be the Euler method and $T = (t_{kn})$, where $t_{kn} = f(k/n)$, then the pointwise convergence of $B_n(f, r)$ to $f(r)$ is the same as the A summability of the matrix T according to Definition 7.

When $f \in C[0, 1]$ we see that the pointwise convergence of Bernstein polynomial is equivalent to saying that the nodes matrix $X = (x_{kn})$, where $x_{kn} = k/n$, is A-distributionally (or equivalently A-statistically) convergent to r. Since f is bounded, this is equivalent to saying that X is A-strongly convergent to r. The fact that this is always the case for any $f \in C[0, 1]$ is well known.

However, X is Cesáro-distributionally convergent to the uniform distribution, since

$$\sum_{k:(k/n) \leq t} \frac{1}{n} = \frac{[nt]}{n} \to t, \qquad \text{for } t \in (0, 1).$$

Hence, for double arrays, even boundedness of X does not imply that, for matrix summability, the Cesáro and Euler methods are consistent, let alone the question of whether one includes the other.

Another point of departure occurs when A-statistical convergence of a sequence is characterized by the existence of a set E with $\delta_A(E^c) = 0$ and x is convergent over E. This result fails to carry over to our definition of A-statistical convergence of a double array. A simple example demonstrating this fact is to take $X = I$ to be the identity matrix. No infinite submatrix of I is Pringsheim convergent to zero, however, I is A-statistically convergent to 0 when the diagonal terms of A go to zero.

References

[1] J. Boos, Classical and Modern Methods in Summability, *Oxford University Press*, 2000.

[2] R. C. Buck, Generalized asymptotic density, *Amer. J. Math.* **75** (1953), 335–346.

[3] R. C. Buck and H. Pollard, Convergence and summability properties of subsequences, *Bull. Amer. Math. Soc.* **49** (1943), 924–931.

[4] Y. S. Chow, Delayed sums and Borel summability of independent, identically distributed random variables, *Bull. Institute Mathematics Acad. Sinica* **1**(2) (1973), 207–220.

[5] K. L. Chung, A Course in Probability Theory, *Academic Press, New York*, 1974.

[6] J. Connor, On strong matrix summability with respect to a modulus and statistical convergence, *Canad. Math. Bull.* **32** (1989), 194–198.

[7] J. Diestel, Sequences and Series in Banach Spaces, *Springer Verlag, New York*, 1984.

[8] J. Diestel, W. M. Ruess and W. Schachermayer, Weak compactness in $L^1(\mu, X)$, *Proc. Amer. Math. Soc.* **118**(2) (1993), 447–453.

[9] A. R. Freedman and J. J. Sember, Densities and summability, *Pacific J. Math.* **95** (1981), 293–305.

[10] J. A. Fridy, A simple proof of the Mazur–Orlicz summability theorem, *Math. Proc. Camb. Phil. Soc.* **89** (1981), 391–392.

[11] J. A. Fridy, R. A. Goonatilake and M. K. and Khan, Berry–Esseen bounds of summability transforms, *Proc. Amer. Math. Soc.* **132** (2004), 273–282.

[12] J. A. Fridy and C. Orhan, Lacunary statistical convergence, *Pacific J. Math.* **160** (1993), 43–51.

[13] G. H. Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.* **19** (1917), 86–95.

[14] G. H. Hardy and J. E. Littlewood, Sur la série de Fourier d'une fonction à carré sommable, *C. R. Acad. Sci. Paris* **156** (1913), 1307–1309.

[15] G. H. Hardy and W. W. and Rogosinski, Fourier Series, *Cambridge University Press*, 1965.

[16] J. D. Hill and W. T. Sledd, Approximation in bounded summability fields, *Canad. J. Math.* **20** (1968), 410–315.

[17] M. K. Khan, B. Della Vecchia and A. Fassih, On the monotonicity of positive linear operators, *J. Approx. Theory* **92** (1998), 22–37.

[18] M. K. Khan and C. Orhan, Matrix characterization of A-statistical convergence, *J. Math. Anal. Appl.* **335** (2007), 406–417.

[19] E. Kolk, The statistical convergence in Banach spaces, *Tartu Ül. Toimetised* **928** (1991), 41–52.

[20] S. Mazur and W. Orlicz, Sur les méthodes lineáres de sommation, *C. R. Acad. Sci. Paris* **196** (1933), 32–34.

[21] S. Mazur and W. Orlicz, On linear methods of summability, *Studia Math.* **14** (1954), 129–160.

[22] M. L. Mehta, Random Matrices, 2nd ed., *Academic Press, San Diego*, 1991.

[23] H. I. Miller, A measure theoretic subsequence characterization of statistical convergence, *Trans. Amer. Math. Soc.* **347**(5) (1995), 1811–1819.

[24] F. Móricz, Statistical convergence of multiple sequences, *Arc. Math. (Basel)* **81** (2003), 82–89.

[25] F. Móricz and B. E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.* **104** (1988), 283–294.

[26] R. F. Patterson, Analogues of some fundamental theorems of summability theory, *Internat. J. Math. Math. Sci.* **23** (2000), 1–9.

[27] R. J. Serfling, Approximation Theorems of Mathematical Statistics, *John Wiley, New York*, 1980.

[28] T. Tsuchikura, Arithmetic means of subsequences, *Tôhoku Math. J.* (2) (1950), 188–191.

[29] A. Ülger, Weak compactness in $L^1(\mu, X)$, *Proc. Amer. Math. Soc.* **113**(1) (1991), 143–149.

[30] A. Zygmund, Trigonometric Series, Vol. II, *Cambridge*, 1959.

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