Publ. Math. Debrecen
76/1-2 (2010), 89–100

Almost periodic solutions in impulsive competitive systems with infinite delays

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Abstract. A general impulsive nonautonomous Lotka–Volterra system of integro-differential equations with infinite delay is considered. The impulses are realized at fixed moments of time. By means of piecewise continuous functions Lyapunov's functions we give new sufficient conditions for the global exponential stability of the unique positive almost periodic solutions.

1. Introduction

During the last two decades, traditional Lotka–Volterra competition system has been studied extensively [1], [2], [5], [6], [8], [10], [11], [16]–[18]. The model can be expressed as follow

$$\dot{u}_i(t) = u_i(t)\left[r_i(t) - \sum_{j=1}^{N} a_{ij}(t)u_j(t)\right], \quad 1 \le i \le N. \tag{1.1}$$

Many results concerned with the permanence, global asymptotic stability and the existence of positive periodic solutions of (1.1) are obtained.

It is well know that the time delay is quite a common for a natural population. Gopalsamy [5] studied the existence of periodic solutions of the equation

$$\dot{u}_i(t) = u_i(t)\left[r_i(t) - a_i(t)u_i(t) - \sum_{j=1}^{N}\int_{-\infty}^{t} k_i(t,s)h_{ij}(t)u_j(s)\,ds\right], \tag{1.2}$$

Mathematics Subject Classification: 34C27, 34D05, 34A37.
Key words and phrases: almost periodic solution, impulsive N-dimensional competitive systems, infinite delay.

$1 \leq i \leq N$, when the delay kernel $k_i(t,s) = k_i(t-s)$ is of convolution type.

Many popolation systems are subject to short-term perturbations during their evolution. Adequate mathematical models of such processes are the impulsive differntial equations which natural generalization of ordinary differential equations. The duration of the perturbations is negligible in comparison with the duration of the process considered,and can thought of momentary.

In recent years impulsive differential equations have been intensively researched (see the monographs of Samoilenko and Perestyuk [12] and Lakshmikantham et al. [9]). Recently, some qualitative properties (oscillation, asymptotic behavior and stability) are investigated by several authors (see [3], [7]).

On the most important parts of qualitative theory of the differential equations is the theory of almost periodic solutions. The main results related to the study of the existence of almost periodic solutions for impulsive dynamical systems are studied in [12], [13]–[15].

In this paper we shall investigate the existence of almost periodic solutions of more general equation from (1.2) with impulsive perturbations of the population density at fixed moments of time. Impulses can be considered as a control. The investigations in this paper are carried out by means of piecewise continuous functions which are modifications of classical Lyapunov's functions.

2. Preliminaries and basic results

Let $\Re^N$ be the N-dimensional Euclidean space with norm $\|u\| = \sum_{i=1}^{N} |u_i|$, $\Re^+ = [0, \infty)$.

$\mathbb{B} = \{\{\tau_k\} : \tau_k \in \Re,\ \tau_k < \tau_{k+1},\ k \in \mathbb{Z}\}$ – the set of all sequences unbounded and strictly increasing with distance $\rho\big(\{\tau_k^{(1)}\}, \{\tau_k^{(2)}\}\big)$.

$PC[\Re, \Re^N] = \{\varphi : \Re \to \Re^N,\ \varphi$ is piecewise continuous function with points of discontinuity of the first kind τ_k, $\{\tau_k\} \in \mathbb{B}$ at which $\varphi(\tau_k^-)$ and $\varphi(\tau_k^+)$ exist, and $\varphi(\tau_k^-) = \varphi(\tau_k)\}$.

$PC^1[\Re, \Re^N] = \{\varphi : \Re \to \Re,\ \varphi$ is continuously differentiable everywhere except the poins τ_k, $\{\tau_k\} \in \mathbb{B}$ at which $\dot\varphi(\tau_k^-)$ and $\dot\varphi(\tau_k^+)$ exist, and $\dot\varphi(\tau_k^-) = \dot\varphi(\tau_k)\}$.

Consider the impulsive nonautonomous competitive Lotka–Volterra system of integro-differential equations with infinite delay and fixed moments of impulsive perturbations

$$\begin{cases} \dot{u}_i(t) = u_i(t)\Big[r_i(t) - a_i(t)u_i(t) - \sum_{j=1}^{N}\int_{-\infty}^{t} k_i(t,s)h_{ij}(t,u_j(s))\,ds\Big], & t \neq \tau_k, \\ u_i(\tau_k^+) = u_i(\tau_k) + p_{ik}u_i(\tau_k) + c_i, & k \in \mathbb{Z}, \end{cases} \tag{2.1}$$

where $1 \leq i \leq N$, $N \geq 2$, $t \in \Re$ and:

(a) The functions $r_i(t)$, $a_i(t) \in C[\Re, \Re^+]$, $1 \leq i \leq N$ and $k_i(t,s)$, $h_{ij}(t,x) \in C[\Re \times \Re, \Re^+]$, $1 \leq i, j \leq N$;

(b) The constants $p_{ik} \in \Re$, $c_i \in \Re^+$, $1 \leq i \leq N$, $\{\tau_k\} \in \mathbb{B}$, $k \in \mathbb{Z}$.

Let $u^{t_0} : (-\infty, t_0] \to \Re^N$, $u^{t_0} = \mathrm{col}(u_1^{t_0}, u_2^{t_0}, \ldots, u_N^{t_0})$ is a continuous function. We denote by $u(t) = u(t; t_0, u^{t_0}) = \mathrm{col}(u_1(t; t_0, u^{t_0}), u_2(t; t_0, u^{t_0}), \ldots, u_N(t; t_0, u^{t_0}))$ the solution of system (2.1) satisfying the initial conditions

$$\begin{cases} u(s; t_0, u^{t_0}) = u^{t_0}(s), & s \in (-\infty, t_0], \\ u(t_0^+; t_0, u^{t_0}) = u(t_0). \end{cases} \tag{2.2}$$

Note that the solution $u(t) = u(t; t_0, u^{t_0})$ of problem (2.1), (2.2) is a piecewise continuous with points of discontinuity of the first kind at $\tau_k, k \in \mathbb{Z}$ at which it is left continuous, i.e. the following relations are satisfied:

$$u_i(\tau_k^-) = u_i(\tau_k),$$
$$u_i(\tau_k^+) = u_i(\tau_k) + p_{ik}u_i(\tau_k) + c_i, \quad k \in \mathbb{Z},\ 1 \leq i \leq N.$$

In our subsequent analysis, we will consider only initial functions that belong to a class of bounded continuous functions.

Let $BC = BC[(-\infty, t_0], \Re^N]$ be the set of all bounded continuous functions from $(-\infty, t_0]$ into $\Re^N$. Let $u^{t_0}(.) \in BC[(-\infty, t_0], \Re^N]$. If $u(t)$ is an $\Re^N$−valued function on $(-\infty, \beta)$, $\beta \leq \infty$, we define for each $t \in (-\infty, \beta)$, $u^t(.)$ to be the restriction of $u(s)$ given by $u^t(s) = u(t+s)$, $-\infty < s \leq t$, and the norm is defined by

$$\|u^t(.)\| = \sup_{-\infty < s \leq t} \|u(s)\|.$$

It is clear that $\|u(t)\| \leq \|u^t(.)\|$.

Since the solutions of (2.1) are piecewise continuous functions we adopt the following definitions for almost periodicity.

Let for $T, P \in \mathbb{B}$, $s(T \cup P) : \mathbb{B} \to \mathbb{B}$ is a map such that the set $s(T \cup P)$ forms a strictly increasing sequence and if $D \subset \Re$ and let for $\epsilon > 0$, $\theta_\epsilon(D) = \{t + \epsilon,\ t \in D\}$, $F_\epsilon(D) = \cap\{\theta_\epsilon(D)\}$.

By $\phi = (\varphi(t), T)$ we denote the element from the space $PC \times \mathbb{B}$, and for every sequence of real number $\{\alpha_n\}$, $n = 1, 2, \dots$ with $\theta_{\alpha_n}\phi$ denote the sets $\{\varphi(t + \alpha_n), T - \alpha_n\} \subset PC \times \mathbb{B}$, where $T - \alpha_n = \{\tau_k - \alpha_n,\ k \in \mathbb{Z},\ n = 1, 2, \dots\}$.

Definition 2.1 ([12]). The set of sequences $\{\tau_k^l\}$, $\tau_k^l = \tau_{k+l} - \tau_k$, $k \in \mathbb{Z}$, $l \in \mathbb{Z}$ is said to be *uniformly almost periodic* if for any $\epsilon > 0$ there exists a relatively dense set in $\Re$ of ϵ-almost periods common for all the sequence $\{\tau_k^l\}$.

Lemma 2.1 ([12]). *The set of sequences $\{\tau_k^l\}$ is uniformly almost periodic, if and only if from each infinite sequences of shifts $\{\tau_k - \alpha_n\}$, $k \in \mathbb{Z}$, $n = 1, 2, \dots,$ $\alpha_n \in \Re$ we can choose a subsequence, convergent in $\mathbb{B}$.*

Definition 2.2. The sequence $\{\phi_n\}$, $\phi_n = (\varphi_n(t), T_n) \in PC \times \mathbb{B}$ is convergent to ϕ, $\phi = (\varphi(t), T)$, $(\varphi(t), T) \in PC \times \mathbb{B}$ if and only if for any $\epsilon > 0$ there exists $n_0 > 0$ such that for $n \geq n_0$ it follows that

$$\rho(T, T_n) < \epsilon, \quad \|\varphi_n(t) - \varphi(t)\| < \epsilon$$

hold uniformly for $t \in \Re \setminus F_\epsilon(s(T_n \cup T))$.

Definition 2.3. The function $\varphi \in PC(\Re, \Re^N)$ is said to be *almost periodic piecewise continuous function* with points of discontinuity of the first kind from the set T if for every sequence of real numbers $\{\alpha'_m\}$ it follows that there exists a subsequence $\{\alpha_n\}$, $\alpha_n = \alpha'_{m_n}$ such that $\theta_{\alpha_n}\phi$ is compact in $PC \times \mathbb{B}$.

Definition 2.4 ([4]). The system (2.1) is said to be *globally exponentially stable* if for all $c > 0$, there exists $\gamma = \gamma(c) > 0$, $\|u^{t_0} - v^{t_0}\| \leq c$, then for all $t \geq t_0$,

$$\|u(t; t_0, u^{t_0}) - v(t; t_0, v^{t_0})\| < \gamma(\|u^{t_0} - v^{t_0}\|) \exp[-c(t - t_0)].$$

Definition 2.5. Suppose $u(t) = (u_1(t), u_2(t), \dots, u_n(t))$ is any one solution of system (2.1) then $u(t)$ is said to be a *strictly positive solution*, if for $1 \leq i \leq N$,

$$0 < \inf_{t\in\Re} u_i(t) \leq \sup_{t\in\Re} u_i(t) < \infty.$$

We will use piecewise continuous auxiliary functions which are analogues of the classical Lyapunov's functions and then consider the following sets:

$$G_k = \{[\tau_{k-1}, \tau_k) \times \Re^N,\ k \in \mathbb{Z}\};\ G = \cup_{k=-\infty}^{\infty} G_k;$$

$V_0 = \{V \in C[G, \Re^+]$, there exist the limits $V(\tau_k^-, u_0)$, $V(\tau_k^+, u_0)$, $u_0 \in \Re^N$, V is locally Lipschitz for $u \in \Re^N\}$.

Let $V \in V_0$. For any $(t,u) \in [\tau_{k-1}, \tau_k) \times \Re^N$ the right-hand derivative $D^+V(t,u(t))$ along the solution $u(t;t_0,u_0)$ of (2.1) is defined by

$$D^+V(t) = D^+V(t,u(t)) = \lim_{\delta \to 0^+} \inf \delta^{-1}\{V(t+\delta, u(t+\delta)) - V(t,u(t))\}.$$

Given a continuous function $g(t)$ which is defined on $\Re$, we set

$$g_M = \sup_{t\in\Re} g(t), \quad g_L = \inf_{t\in\Re} g(t).$$

Introduce the following assumptions:

H1. The functions $r_i(t)$, $a_i(t)$, $1 \le i \le N$ are almost periodic and $r_{iL} > 0$, $a_{iL} > 0$.

H2. The functions $k_i(t,s) > 0$, $i = 1,2,\ldots,N$ are uniformly continuous, almost periodic with respect to t on $\Re$, integrable with respect to s on $(-\infty, t_0]$ and there exist positive numbers μ_i such that

$$\int_{-\infty}^{t} k_i(t,s)\,ds \le \mu_i < \infty$$

for all $t \in \Re$, and $1 \le i \le N$.

H3. The functions $h_{ij}(t,x)$ are almost periodic with respect to t on $\Re$, non-decreasing with respect to $x \in \Re$ and and there exist positive almost periodic continuous functions $L_{ij}(t)$, $1 \le i \le N$, $1 \le j \le N$ such that

$$|h_{ij}(t,x) - h_{ij}(t,y)| \le L_{ij}(t)|x-y|, \quad x,y \in \Re.$$

H4. The sequences $\{p_{ik}\}$ are almost periodic and $-1 \le p_{ik} \le 0$, $1 \le i \le N$, $k \in \mathbb{Z}$.

H5. The set $\{\tau_k^l\}$, $\tau_k^l = \tau_{k+l} - \tau_k$, $k \in \mathbb{Z}$, $l \in \mathbb{Z}$ is uniformly almost periodic.

H6. $c^M < \infty$, $c^L > 0$, where $c^M = \max\{c_i\}$ and $c^L = \min\{c_i\}$ for $1 \le i \le N$.

Let the assumptions **H1**–**H6** be fulfilled and let $\{\alpha_m{}'\}$ be arbitrary sequence of real numbers. Then there exists a subsequence $\{\alpha_n\}$, $\alpha_n = \alpha_{m_n}{}'$ such that the sequences $\{r_i(t+\alpha_n)\}$, $\{a_i(t+\alpha_n)\}$, $\{k_i(t+\alpha_n, s)\}$, $\{h_{ij}(t+\alpha_n, x)\}$, convergent uniformly on $1 \le i,j \le N$, to the functions $\{r_i^\alpha(t)\}$, $\{a_i^\alpha(t)\}$, $\{k_i^\alpha(t,s)\}$, $\{h_{ij}^\alpha(t,x)\}$, and the set of sequences $\{\tau_k - \alpha_n\}$, $k \in \mathbb{Z}$ is convergent to the sequence τ_k^α uniformly with respect to $k \in \mathbb{Z}$ as $n \to \infty$.

Be $\{k_{n_i}\}$ we denote the sequence of integers such that the subsequence $\{\tau_{k_{n_i}}\}$ convergent to the sequence τ_k^α uniformly with respect to $k \in \mathbb{Z}$ as $i \to \infty$. From

H4 it follows that there exists a subsequence of the sequence $\{k_{n_i}\}$ such that the sequences $\{p_{ik_{n_i}}\}$, convergent uniformly to the limits denoted by p_{ik}^{α}.

Then for every sequence $\{\alpha'_m\}$ the system (2.1) moving to the system

$$\begin{cases} \dot{u}_i(t) = u_i(t)\Big[r_i^{\alpha}(t) - a_i^{\alpha}(t)u_i(t) \\ \qquad - \displaystyle\sum_{j=1}^{N}\int_{-\infty}^{t} k_i^{\alpha}(t,s)h_{ij}^{\alpha}(t,u_j(s))\,ds\Big], \quad t \neq \tau_k^{\alpha}, \\ u_i(\tau_k^{\alpha+}) = u_i(\tau_k^{\alpha}) + p_{ik}^{\alpha}u_i(\tau_k^{\alpha}) + c_i, \quad k \in \mathbb{Z}. \end{cases} \tag{2.3}$$

In the proofs of the main theorems we will use the following comparison results.

Lemma 2.2 ([3]). *Let the following conditions hold:*

1. *The hypotheses* **H1**–**H6** *hold.*
2. *There exist functions* $P_i, Q_i \in PC^1[[t_0,\infty),\Re]$ *such that* $P_i(t_0^+) \le u_i^{t_0}(s) \le Q_i(t_0^+)$, $s \le t_0$, $t_0 \in \Re$.

Then we have

$$P_i(t) \le u_i(t) \le Q_i(t) \tag{2.4}$$

for all $t \ge t_0$ *and* $1 \le i \le N$.

Lemma 2.3 ([3]). *Let the following conditions hold:*

1. *The hypotheses* **H1**–**H6** *hold.*
2. $u_i(t) = u_i(t; t_0, u_i^{t_0})$ *is a solution of* (2.1), (2.2) *such that*

$$u_i(s) = u_i^{t_0}(s) \ge 0, \quad \sup u_i^{t_0}(s) < \infty, \quad u_i^{t_0} > 0,\ 1 \le i \le N. \tag{2.5}$$

3. *For each* $1 \le i \le N$ *and* $k \in \mathbb{Z}$, $1 + p_{ik} > 0$.

Then:

1. $u_i(t) > 0$, $1 \le i \le N$, $t > t_0$.
2. *There exist positive constants* α_i *and* β_i *such that*

$$\alpha_i \le u_i(t) \le \beta_i,$$

for all $t > t_0$ *and* $1 \le i \le N$ *and if in addition*

$$0 < 1 + p_{ik} \le 1 \quad \text{and} \quad -p_{ik}\alpha_i < c_i < -g_{ik}\beta_i$$

then

$$\alpha_i \le u_i(t) \le \beta_i,$$

for all $t \in \Re$ *and* $1 \le i \le N$.

3. Main results

For the proof of the main results we consider system (2.3) and then discuss the almost periodic solutions of the system (2.1).

Lemma 3.1. *Let the following conditions hold:*

1. *The conditions* **H1**–**H6** *are satisfied.*
2. $\{\alpha_m'\}$ *be arbitrary sequence of real numbers.*
3. *For the systems* (2.3) *there exist strictly positive solutions.*

Then the system (2.1) *has a unique strictly positive almost periodic solution.*

Proof. For simplification, we write (2.1) in the form

$$\begin{cases} \dot{u} = F(t,u), & t \neq \tau_k, \\ u(\tau_k^+) = u(\tau_k) + P_k u(\tau_k), & k \in \mathbb{Z}. \end{cases} \tag{3.1}$$

In (3.1) from **H1**–**H3** it follows that $F(t,u)$ is an almost periodic function with respect to $t \in \mathbb{R}$ and $u \in S_\nu$, P_k is almost periodic sequence with respect to $k \in \mathbb{Z}$. Let $\phi(t)$ is a strictly positive solution of (3.1) and let the sequences of real numbers α' and β' are such that for they common subsequence $\alpha \subset \alpha'$, $\beta \subset \beta'$, we have $\theta_{\alpha+\beta}F(t,u) = \theta_\alpha \theta_\beta F(t,u)$.

Let $\theta_{\alpha+\beta}\phi(t)$, $\theta_\alpha\theta_\beta\phi(t)$ exist uniformly on compact set $\mathbb{R}\times B$ and are solutions of the following equation

$$\begin{cases} \dot{u} = F^{\alpha+\beta}(t,u), & t \neq \tau_k^{\alpha+\beta}, \\ u(\tau_k^{\alpha+\beta+}) = u(\tau_k^{\alpha+\beta}) + P_k^{\alpha+\beta} u(\tau_k^{\alpha+\beta}), & k \in \mathbb{Z}. \end{cases}$$

Therefore, $\theta_{\alpha+\beta}\phi(t) = \theta_\alpha\theta_\beta\phi(t)$, and thus according to Lemma 2, [16], it follows that $\phi(t)$ is an almost periodic solution of system (3.1).

The proof is complete. □

Let $u_i(t;t_0,u^{t_0})$ and $v_i(t;t_0,v^{t_0})$, $1 \le i \le N$, (t_0,u^{t_0}), $(t_0,v^{t_0}) \in \Re \times BC$ be any two solutions of (2.1) such that

$$u_i(s) = u_i^{t_0}(s) \ge 0, \quad \sup u_i^{t_0}(s)) < \infty,\ u_i^{t_0}(t_0) > 0.$$

$$v_i(s) = v_i^{t_0}(s) \ge 0, \quad \sup v_i^{t_0}(s)) < \infty,\ v_i^{t_0}(t_0) > 0.$$

Define a Lyapunov function

$$V(t,u(t),v(t)) = \sum_{i=1}^{N} V_i(t) = \sum_{i=1}^{N} \left| \ln \frac{u_i(t)}{v_i(t)} \right|. \tag{3.2}$$

By Mean Value Theorem it follows that for any closed interval contained in $t \in (\tau_{k-1}, \tau_k]$, $k \in \mathbb{Z}$ there exist positive numbers r and R such that for $1 \le i \le N$, $r \le u_i(t)$, $v_i(t) \le R$ and

$$\frac{1}{R}|u_i(t) - v_i(t)| \le |\ln u_i(t) - \ln v_i(t)| \le \frac{1}{r}|u_i(t) - v_i(t)|. \tag{3.3}$$

Theorem 3.1. *Assume that:*

1. *The conditions* **H1**–**H6** *are satisfied.*
2. *There exist nonnegative continuous functions* $\delta_i(t)$ *such that*

$$ra_i(t) - R\sum_{j=1}^{N} \mu_i L_{ij}(t) > \delta_i(t), \quad t \ne \tau_k,\ k \in \mathbb{Z}. \tag{3.4}$$

Then:

1. *For the system* (2.1) *there exists a unique strictly positive almost periodic solution.*
2. *If*

$$\int_{t_0}^{t} \delta(s)ds = c(t - t_0),$$

where $\delta(t) = \min(\delta_1(t), \delta_2(t), \dots, \delta_N(t))$, *then the almost periodic solution is globally exponentially stable.*

Proof. From construction of the system (2.3) it follows that the conditions **H1**–**H6** are hold for the functions and sequences in the right hand. If $u^\alpha(t)$ is solution of (2.3) then from Lemma 2.3 we have

$$0 < \inf_{t \ge t_0} u_i^\alpha(t) \le \sup_{t \ge t_0} u_i^\alpha(t) < \infty, \quad 1 \le i \le N. \tag{3.5}$$

Suppose that the system (2.3) has two arbitrary strictly positive solutions

$$u^\alpha = (u_1^\alpha(t), u_2^\alpha(t), \dots, u_n^\alpha(t)),\ v^\alpha = (v_1^\alpha(t), v_2^\alpha(t), \dots, v_n^\alpha(t)).$$

Consider the Lyapunov function

$$V^\alpha(t) = V^\alpha(t, u^\alpha(t), v^\alpha(t)) = \sum_{i=1}^{N} V_i^\alpha(t) = \sum_{i=1}^{N} \left| \ln \frac{u_i^\alpha(t)}{v_i^\alpha(t)} \right|.$$

Then for $1 \le l \le N$, $t \in \Re$, $t \ne \tau_k^\alpha$, $k \in \mathbb{Z}$ and hypotheses **H1** $-$ **H6** we have

$$D^+ V_l^\sigma(t) = \left(\frac{\dot{u}_l(t)}{u_l(t)} - \frac{\dot{v}_l(t)}{v_l(t)} \right) \mathrm{sgn}(u_l(t) - v_l(t)) \le \Big[-a_l^\sigma(t)|u_l(t)) - v_l(t))|$$
$$+ \sum_{j=1}^{N} \int_{-\infty}^{t} k_l^\sigma(t,s)|h_{lj}^\sigma(t, u_j(s)) - h_{lj}(t, v_j(s))|\, ds \Big].$$

From hypotheses **H1–H6**, we obtain

$$D^+V_l^\sigma(t) \le \left[-a_l^\sigma(t)|u_l(t)-v_l(t)| + \sum_{j=1}^{N}\int_{-\infty}^{t} k_l^\sigma(t,s)L_{lj}^\sigma(t)|u_j(s)-v_j(s)|\,ds\right]$$
$$\le \left[-a_l^\sigma(t)|u_l(t)-v_l(t)| + \sum_{j=1}^{N}\int_{-\infty}^{t} k_l^\sigma(t,s)L_{lj}^\sigma(t)|u_j(s)-v_j(s)|\,ds\right].$$

Thus in view of hypothesis (3.4) we obtain

$$D^+V^\sigma(t,u^\sigma(t),v^\sigma(t)) \le -\delta^\sigma(t)m^\sigma(t), \tag{3.6}$$

where $t \in \Re,\ t \ne \tau_k^\sigma$,

$$m^\sigma(t) = \sum_{i=1}^{N}|u_i^\sigma - v_i^\sigma|, \quad \delta^\sigma(t) = \min(\delta_1^\sigma(t),\delta_2^\sigma(t),\ldots,\delta_N^\sigma(t)).$$

On the other hand for $t = \tau_k^\sigma$ we have

$$V^\sigma(\tau_k^{\sigma+}) = \sum_{i=1}^{N}\left|\ln\frac{u_i^\sigma(\tau_k^{\sigma+})}{v_i^\sigma(\tau_k^{\sigma+})}\right| = \sum_{i=1}^{N}\left|\ln\frac{(1+p_{ik})u_i(\tau_k)+c_i}{(1+p_{ik})v_i(\tau_k)+c_i}\right|$$
$$\times\sum_{i=1}^{N}\left|\ln\frac{(1+p_{ik})R-p_{ik}R}{(1+p_{ik})r-p_{ik}r}\right| = \sum_{i=1}^{N}\left|\ln\frac{R}{r}\right|$$
$$= \sum_{i=1}^{N}\left|-\ln\frac{R}{r}\right| = \sum_{i=1}^{N}\left|\ln\frac{r}{R}\right| \le \sum_{i=1}^{N}\left|\ln\frac{u_i(\tau_k)}{v_i(\tau_k)}\right| = V^\sigma(\tau_k^\sigma). \tag{3.7}$$

From (3.6) and (3.7) it follows that for $t<0$

$$\int_t^0 \delta^\sigma(t)m^\sigma(t)dt \le V^\sigma(t) - V^\sigma(0)$$

and

$$\int_{-\infty}^0 |u_i^\sigma(s)-v_i^\sigma(s)|ds < \infty,$$

and then $u_i^\sigma(t)-v_i^\sigma(t)\to\ 0$ as $t\to-\infty$. From (3.8) and (3.9) it follows that for $t<0$

$$\int_t^0 \delta^\sigma(t)m^\sigma(t)dt \le V^\sigma(t) - V^\sigma(0)$$

and

$$\int_{-\infty}^{0} |u_i^\sigma(s) - v_i^\sigma(s)| ds < \infty,$$

then $u_i^\sigma(t) - v_i^\sigma(t) \to \ 0$ as $t \to -\infty$.

Let $\mu^\sigma = \inf_{t\in\Re}\{u_i^\sigma(t), v_i^\sigma(t),\, 1 \leq i \leq N\}$. From **H1-H6** and definition of V we have

$$V^\sigma(t) = \sum_{i=1}^{N} V_i^\sigma(t) \leq \sum_{l=1}^{N} \frac{1}{\mu^\sigma} |u_l^\sigma(t) - v_l^\sigma(t)|.$$

Hence $V(t) \to 0,\ t \to -\infty$.

We have that $V(t)$ is non increasing nonnegative function on $\Re$ and consequently

$$V^\sigma(t) = 0, \quad t \neq \tau_k^\sigma,\ t \in \Re. \tag{3.8}$$

From (3.6), (3.7) and (3.8) it follows that $u^\sigma \equiv v^\sigma$ for all $t \in \Re$ and $1 \leq i \leq N$. Then every equation from (2.3) has a unique strictly positive solution.

From Lemma 3.1 analogously it follows that system (2.1) has unique strictly positive almost periodic solution.

Let for the system (2.1) there exists another bounded strictly positive solution $v_i(t; t_0, v^{t_0}),\ 1 \leq i \leq N,\ (t_0, v^{t_0}) \in \Re \times BC$.

Now consider again the Lyapunov function $V(t) = V(t, u(t), v(t))$ and from (3.3) we obtain

$$V(t_0^+, u^{t_0+}, v^{t_0+}) = \sum_{l=1}^{N} \left| \ln \frac{u_l(t_0)}{v_l(t_0)} \right| \leq \frac{1}{r} \| u^{t_0} - v^{t_0} \|. \tag{3.9}$$

Then

$$D^+V(t, u(t), v(t)) \leq -\delta(t)m(t) \leq -\delta(t)RV(t, u(t), v(t)), \tag{3.10}$$

for $t \in \Re,\ t \neq \tau_k$.

On the other hand for $t \in \Re,\ t = \tau_k,\ k \in \mathbb{Z}$

$$V(\tau_k + 0) \leq V(\tau_k). \tag{3.11}$$

From (3.9), (3.10) and (3.11) it follows

$$V(t, u(t), v(t)) \leq V(t_0^+, u_0, v_0) \exp\Big\{ - R \int_{t_0}^{t} \delta(s) ds \Big\}. \tag{3.12}$$

Therefore, from (3.11), (3.12) and (3.12) we deduce the inequality

$$\sum_{i=1}^{N} |u_i(t) - v_i(t)| \leq \frac{R}{r} \|u_0 - v_0\| e^{-rc(t-t_0)},$$

$t \geq t_0$.

This shows that the system (E) is globally exponentially stable.

The proof of Theorem 3.1. is complete. □

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(Received July 18, 2008)