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A study of chaos for processes under small perturbations

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Abstract. The method of isolating segments was developed to prove topological chaos (positive topological entropy) for Poincaré maps of time-periodic processes. The authors recently proved that it may also be used to verify distributional chaos, when the obtained semi-conjugacy covers a periodic point exactly one to one (so the solution giving raise to this preimage is also periodic). When we make a small perturbation of dynamics then usually the same isolating segments may be used (and as a result the same semi-conjugacy but possibly on a different set is obtained), however the periodic solution may be destroyed (then we have infinite set in the preimage), or if we are lucky, it may bifurcate to a finite number of periodic solutions.

In this article we cover the case when two periodic solutions are continued from the previous one. We prove that in this case distributional chaos survives. Homoclinic and heteroclinic connections between these two solutions are also discussed.

1. Introduction

The name *chaos* in the context of discrete dynamical systems was first used in 1975 by T. Li and J. Yorke in [8]. In this article we are concerned with the definition of *distributional chaos* which extends approach of Li and Yorke and was introduced in 1994 by Schweizer and Smítal [15]. The motivation for this definition was the fact that it is equivalent to positive topological entropy in the case of interval maps; however it was later realized that it is no longer true when dimension of the space is different than one (see [17] or [10] for the summary on these relations), i.e. both notions are not related in general.

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In practice, it is hard to prove that a dynamical system exhibits distributional chaos. First of all, we have to predict long time behavior of trajectories, much more complex than in the case of sensitive dependence on initial conditions. Furthermore, this notion does not transfer via semi-conjugacy [13] which makes the task even harder (the usual tool to prove topological chaos, i.e. positive topological entropy, is to apply semi-conjugacy arguments). There are many techniques in the theory of dynamical systems which help to construct a semi-conjugacy with symbolic dynamical systems and usually it is also possible to say something more about the properties of the factor map. In [13] we gave sufficient conditions to prove distributional chaos when the factor map is constructed in a neighborhood of unique fixed point. Unfortunately, when we start to study a family of maps (indexed by a continuous parameter) then usually the fixed point does not survive and bifurcation occurs. This happens even in the simplest case of logistic map of the compact interval (when the parameter increases, a fixed point explodes becoming period 2 orbit). When a fixed point transforms to a periodic point or a few fixed points (which is a result of a small perturbation applied to the system), it seems intuitively that the dynamics is richer than before. Then, if the map was chaotic before the perturbation it should remain chaotic (of course, in practice, it may happen that repelling point becomes attracting so the dynamics may change rapidly).

In this article we are concerned with the chaos of a Poincaré map induced by time-periodic local process (nonautonomous ordinary differential equation). We provide an example of local process with the property that stationary trajectory is continued (after a small perturbation) to two periodic trajectories (fixed point of the Poincaré map explodes becoming at least 2 fixed points). Then results of [12] or [13] may not be applied because there are two trajectories which remain close to the previous position of the stationary one (possibly with a heteroclinic connection). Constructed semi-conjugacy which survives after a (small) perturbation is applied, however the set on which it is defined may "grow" a lot (explosion similar to the case of stationary trajectory may occur). Our idea is to present a method which may help in that (and similar) situation. We also show how to extend the method of isolating segments for proving the existence of heteroclinic solutions.

We deal with the case of only two periodic solution bifurcating from the trivial one since the problem of the greater number of them meets with some difficulties (see Remark 6).

In our considerations, we investigate the equation

$$\dot{z} = \left(1 + e^{i\kappa t} |z|^2\right) \overline{z}^2 - N e^{-i\frac{\pi}{3}} \tag{1.1}$$

when the parameter value N is between 0 and 0.01 and $\kappa \in (0, 0.18]$. When $N = 0$ then we may apply results of [13], so in that case we know that the induced local process exhibits uniform distributional chaos. We will use topological tools like isolating segments and symbolic factors together with an analysis of the vector field. Combination of these will allow us to answer what happens with the dynamics when N increases. More specifically, it is possible to prove that distributional chaos survives the perturbation and that numerous homoclinic and heteroclinic solutions appear.

The article is organized as follows. In Section 2 we recall all definitions and basic facts used in the further parts of the article. Section 3 develops all the tools we need to study the equation (1.1). Our idea is to present them in a way that they may be applied to other similar situations. Next, in Section 4 we present theorems on chaos and existence homoclinic and heteroclinic solutions of (1.1) together with a sketchy description of the proof. Formal proof extends the capacity of this publication and it will be presented in another article [14], which ensure that our conclusions (about chaos in considered local process) contained in Section 4 are mathematically rigorous.

2. Basic notions

2.1. Topological dynamics. Let (X, f) be a dynamical system on a compact metric space. By *positive orbit* of x we mean the set

$$\mathrm{Orb}^+(x, f) = \left\{x, f(x), f^2(x), \dots\right\}.$$

If additionally f is a homeomorphism, we may define its *negative orbit* and *(full) orbit* by, respectively

$$\mathrm{Orb}^-(x, f) = \left\{x, f^{-1}(x), f^{-2}(x), \dots\right\}, \quad \mathrm{Orb}(x, f) = \mathrm{Orb}^-(x, f) \cup \mathrm{Orb}^+(x, f).$$

A point $y \in X$ is an *ω-limit point* (*α-limit point*) of a point x if it is an accumulation point of the sequence $x, f(x), f^2(x), \dots$ (resp. $x, f^{-1}(x), f^{-2}(x), \dots$). The set of all ω-limit points (α-limit points) of x is called *ω-limit set* (resp. *α-limit set*) of x and denoted $\omega_f(x)$ (resp $\alpha_f(x)$). A point $p \in X$ is said to be *periodic* if $f^n(p) = p$ for some $n \geq 1$. The set of all periodic points for f is denoted by $\mathrm{Per}(f)$.

Let (X, f), (Y, g) be dynamical systems on compact metric spaces. A continuous map $\Phi : X \to Y$ is called a *semiconjugacy* (or a *factor map*) between f and g if Φ is surjective and $\Phi \circ f = g \circ \Phi$.

The specification property was introduced by Bowen in [3] for the first time (see [16] or [4] for further examples of maps with specification property and their basic characteristic). We recall the definition below, however we will use the terminology introduced in [2].

Definition 1. Let (X, ρ) be a compact metric space. We say that $f \in C(X)$ has the *weak specification property* (briefly *WSP*) if, for any $\delta > 0$, there is a positive integer N_δ such that for any points $y_1, y_2 \in X$ and any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2$ satisfying $j_2 - k_1 \geq N_\delta$ there is a point x in X such that, for $m = 1, 2$ and all integers i with $j_m \leq i \leq k_m$, the following condition holds:

$$\rho(f^i(x), f^i(y_m)) \quad < \quad \delta. \tag{2.1}$$

2.2. Shift spaces. Let $\mathcal{A} = \{0, 1, \ldots, n-1\}$. We denote

$$\Sigma_n = \mathcal{A}^{\mathbb{Z}}, \quad \Sigma_n^+ = \mathcal{A}^{\mathbb{N}}.$$

By *a word*, we mean any element of a free monoid $\mathcal{A}^*$ with the set of generators equal to $\mathcal{A}$. If $x \in \Sigma_n$ and $i < j$ then by $x_{[i,j]}$ we mean a sequence $x_i, x_{i+1}, \ldots, x_j$. We may naturally identify $x_{[i,j]}$ with the word $x_{[i,j]} = x_i x_{i+1} \ldots x_j \in \mathcal{A}^*$. It is also very convenient to denote $x_{[i,j)} = x_{[i,j-1]}$. The same way we may define $x_{[i,j]}$ and $x_{[i,j)}$ for $x \in \Sigma_n^+$ with the only difference that $i \geq 0$.

We introduce a metric ρ in Σ_n^+ by

$$\rho(x, y) = 2^{-k}, \text{ where } k = \min\left\{m \geq 0 : x_{[0,m]} \neq y_{[0,m]}\right\}.$$

In the case of Σ_n we use the condition $x_{[-m,m]} \neq y_{[-m,m]}$.

If $a_1 \ldots a_m \in \mathcal{A}^*$ then we define so called *cylinder set*:

$$[a_1 \ldots a_m] = \left\{x \in \Sigma_n^+ : x_{[0,m)} = a_1 \ldots a_m\right\}.$$

It is well known that cylinder sets form a neighborhood basis for the space Σ_n^+. One can use a similar concept of cylinder set in the case of Σ_n, however in that case the sequence $a_1 \ldots a_m$ is centered over x_0.

By the 0^∞ we denote the element $x \in \Sigma_n$ such that $x_i = 0$ for all $i \in \mathbb{Z}$ (the same for Σ_n^+ with the only difference that $i \geq 0$). The usual map on Σ_n and Σ_n^+

is the shift map σ defined by $\sigma(x)_i = x_{i+1}$ for all i. Dynamical systems (Σ_n, σ) and (Σ_n^+, σ) are called full two-sided and full one-sided shift over n symbols.

If $X \subset \Sigma_n$ is closed and invariant (i.e. $\sigma(X) \subset X$) then we say that X is a shift (the same definition in the one-sided case). There are many equivalent ways to define shifts, e.g. X is shift iff there exists a set (of forbidden words) $\mathcal{F} \subset \mathcal{A}^*$ such that $X = X_{\mathcal{F}}$ where

$$X_{\mathcal{F}} = \left\{x \in \Sigma_n : x_{[i,j]} \notin \mathcal{F} \text{ for every } i \leq j\right\}.$$

One of the most important classes of shifts is the class of shifts of finite type. It contains all shifts which can be defined by finite sets of forbidden words. Equivalently, $X \subset \Sigma_n$ is a shift of finite type if there is an integer $m > 0$ and $M \subset \mathcal{A}^m$ such that

$$x \in X \iff x_{[i,i+m)} \in M \text{ for all } i \in \mathbb{Z}$$

The same way one-sided shifts of finite type are defined. The class of shifts of finite type coincides with the class of shifts such that σ is expansive and has shadowing (see books [7], [9] for more details). A shift which may be obtained as a factor of a shift of finite type is called a *sofic shift*.

Another way to define shifts of finite type and sofic shifts is to use directed graphs and labeled directed graphs respectively, called their presentations (elements of shift are identified with bi-infinite paths on graph). The reader not familiar with this approach is once again referred to [7] or [9].

2.3. Dynamical systems and Ważewski sets. Let X be a topological space and W be its subset. Denote by $\operatorname{cl} W$ the closure of W. The following definitions come from [19]. Let D be an open subset of $\mathbb{R} \times X$. By a *local flow* on X we mean a continuous map $\phi : D \longrightarrow X$, such that three conditions are satisfied:

i) $I_x = \{t \in \mathbb{R} : (t, x) \in D\}$ is an open interval (α_x, ω_x) containing 0, for every $x \in X$,

ii) $\phi(0, x) = x$, for every $x \in X$,

iii) $\phi(s + t, x) = \phi(t, \phi(s, x))$, for every $x \in X$ and $s, t \in \mathbb{R}$ such that $s \in I_x$ and $t \in I_{\phi(s,x)}$.

In the sequel we write $\phi_t(x)$ instead of $\phi(t, x)$. We distinguish three subsets of W given by

$$W^- = \{x \in W : \phi([0, t] \times \{x\}) \not\subset W, \text{ for every } t > 0\},$$

$$W^+ = \{x \in W : \phi([-t, 0] \times \{x\}) \not\subset W, \text{ for every } t > 0\},$$

$$W^* = \{x \in W : \phi(t, x) \notin W, \text{ for some } t > 0\}.$$

It is easy to see that $W^- \subset W^*$. We call W^- the *exit set of* W, and W^+ the *entrance set of* W. We call W a *Ważewski set* provided

(1) if $x \in W$, $t > 0$, and $\phi([0,t] \times \{x\}) \subset \operatorname{cl} W$ then $\phi([0,t] \times \{x\}) \subset W$,

(2) W^- is closed relative to W^*.

2.4. Processes. Let X be a topological space and $\Omega \subset \mathbb{R}\times\mathbb{R}\times X$ be an open set.

By a *local process* on X we mean a continuous map $\varphi : \Omega \longrightarrow X$, such that the following three conditions are satisfied:

i) $\forall \sigma \in \mathbb{R}$, $x \in X$, $\{t \in \mathbb{R} : (\sigma, t, x) \in \Omega\}$ is an open interval containing 0,

ii) $\forall \sigma \in \mathbb{R}$, $\varphi(\sigma, 0, \cdot) = \mathrm{id}_X$,

iii) $\forall x \in X$, $\sigma, s \in \mathbb{R}$, $t \in \mathbb{R}$ if $(\sigma, s, x) \in \Omega$, $(\sigma + s, t, \varphi(\sigma, s, x)) \in \Omega$ then $(\sigma, s + t, x) \in \Omega$ and $\varphi(\sigma, s + t, x) = \varphi(\sigma + s, t, \varphi(\sigma, s, x))$.

For abbreviation, we write $\varphi_{(\sigma,t)}(x)$ instead of $\varphi(\sigma, t, x)$.

Given a local process φ on X one can define a local flow ϕ on $\mathbb{R} \times X$ by

$$\phi(t, (\sigma, x)) = (t + \sigma, \varphi(\sigma, t, x)).$$

Let M be a smooth manifold and let $v : \mathbb{R}\times M \longrightarrow TM$ be a time-dependent vector field. We assume that v is regular enough to guarantee that for every $(t_0, x_0) \in \mathbb{R} \times M$ the Cauchy problem

$$\dot{x} = v(t, x), \tag{2.2}$$

$$x(t_0) = x_0 \tag{2.3}$$

has a unique solution. Then the equation (2.2) generates a local process φ on M by $\varphi_{(t_0,t)}(x_0) = x(t_0, x_0, t + t_0)$, where $x(t_0, x_0, \cdot)$ is the solution of the Cauchy problem (2.2), (2.3).

Let T be a positive number. We assume that v is T-periodic in t. It follows that the local process φ is T-periodic, i.e.,

$$\forall \sigma, t \in \mathbb{R}\ \varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)},$$

hence there is a one-to-one correspondence between T-periodic solutions of (2.2) and fixed points of the Poincaré map $P_T = \varphi_{(0,T)}$.

2.5. Periodic isolating segments. Let X be a topological space and T be a positive number. We assume that φ is a T-periodic local process on X.

For any set $Z \subset \mathbb{R} \times X$ and $a, b, t \in \mathbb{R}$, $a < b$ we define

$$Z_t = \{x \in X : (t, x) \in Z\},$$
$$Z_{[a,b]} = \{(t, x) \in Z : t \in [a, b]\}.$$

Let $\pi_1 : \mathbb{R} \times X \longrightarrow \mathbb{R}$ and $\pi_2 : \mathbb{R} \times X \longrightarrow X$ be projections on, time and space variable respectively.

A compact set $W \subset [a, b] \times X$ is called an *isolating segment over* $[a, b]$ *for* φ if it is ENR (Euclidean neighborhood retract – cf. [5]) and there are $W^{--}, W^{++} \subset W$ compact ENR's (called, respectively, the *proper exit set* and *proper entrance set*) such that

(1) $\partial W = W^- \cup W^+$,

(2) $W^- = W^{--} \cup (\{b\} \times W_b)$, $W^+ = W^{++} \cup (\{a\} \times W_a)$,

(3) there exists homeomorphism $h : [a, b] \times W_a \longrightarrow W$ such that $\pi_1 \circ h = \pi_1$ and $h([a, b] \times W_a^{--}) = W^{--}$, $h([a, b] \times W_a^{++}) = W^{++}$.

Every isolating segment is also a Ważewski set (for the local flow associated to a process φ). We say that an isolating segment W over $[a, b]$ is *$(b - a)$-periodic* (or simply *periodic*) if $W_a = W_b$, $W_a^{--} = W_b^{--}$ and $W_a^{++} = W_b^{++}$. Let $T > 0$. Given the set $Z \subset [0, T] \times X$ such that $Z_0 = Z_T$ we define its infinite catenation by

$$Z^\infty = \left\{(t, z) \in \mathbb{R} \times X : z \in Z_{t \bmod T}\right\}.$$

2.6. Continuation method. Let X be a metric space. We denote by ρ the corresponding distance on $\mathbb{R} \times X$. Let φ be a local process on X, $T > 0$ and W, U be two subsets of $\mathbb{R} \times X$. We consider the following conditions (see [20], [23]):

(G1) W and U are T-periodic segments for φ which satisfy

$$U \subset W, \quad (U_0, U_0^{--}) = (W_0, W_0^{--}), \tag{2.4}$$

(G2) there exists $\eta > 0$ such that for every $(t, w) \in W^{--}$ and $(t, z) \in U^{--}$ there exists $\tau_0 > 0$ such that for $0 < \tau < \tau_0$ holds $(t + \tau, \varphi(t, \tau, w)) \notin W$, $\rho((t + \tau_0, \varphi_{(t,\tau_0)}(w)), W) > \eta$ and $(t + \tau, \varphi(t, \tau, z)) \notin U$, $\rho((t + \tau_0, \varphi_{(t,\tau_0)}(z)), U) > \eta$.

Let K be a positive integer and let $E[1], \ldots, E[K]$ be disjoint closed subsets of the essential exit set U^{--} which are T-periodic, i.e. $E[l]_0 = E[l]_T$, and such that

$$U^{--} = \bigcup_{l=1}^{K} E[l].$$

(In applications we will use the decomposition of U^{--} into connected components). For $n \in \mathbb{N}$, $D \subset W_0$ and every finite sequence $c = (c_0, \ldots, c_{n-1}) \in \{0, 1, \ldots, K\}^{\{0,1,\ldots,n-1\}}$ we define D_c as a set of points satisfying the following conditions:

(H1) $\varphi_{(0,lT)}(x) \in D$ for $l \in \{0, 1, \ldots, n\}$,

(H2) $\varphi_{(0,lT+t)}(x) \in W_t \setminus W_t^{--}$ for $t \in [0, T]$ and $l \in \{0, 1, \ldots, n-1\}$,

(H3) for each $l = 0, 1, \ldots, n-1$, if $c_l = 0$, then $\varphi_{(0,lT+t)}(x) \in U_t \setminus U_t^{--}$ for $t \in (0, T)$,

(H4) for each $l = 0, 1, \ldots, n-1$, if $c_l > 0$, then $\varphi_{(0,lT)}(x)$ leaves U in time less than T through $E[c_l]$.

Let $\Omega \subset \mathbb{R} \times \mathbb{R} \times X$ be open and

$$[0, 1] \times \Omega \ni (\lambda, \sigma, t, x) \mapsto \varphi^\lambda_{(\sigma,t)}(x) \in X$$

be a continuous family of T-periodic local processes on X. We say that the conditions (G1) and (G2) are satisfied *uniformly* (with respect to λ) if they are satisfied with φ replaced by φ^λ and the same η in (G2) is valid for all $\lambda \in [0, 1]$.

We write D_c^λ for the set defined by the conditions (H1)–(H4) for the local process φ^λ.

The following theorem plays the crucial role in the method of continuation.

Theorem 2 (see [20], [22]). *Let φ^λ be a continuous family of T-periodic local processes such that (G1) and (G2) hold uniformly. Then for every $n > 0$ and every finite sequence $c = (c_0, \ldots, c_{n-1}) \in \{0, 1, \ldots, K\}^{\{0,1,\ldots,n-1\}}$ the fixed point indices* $\operatorname{ind}\left(\varphi^\lambda_{(0,nT)} |_{(W_0 \setminus W_0^{--})_c^\lambda}\right)$ *are correctly defined and equal each to the other (i.e. do not depend on $\lambda \in [0, 1]$).*

2.7. Distributional chaos. Let $\mathbb{N}$ denotes the set of positive integers and let f be a continuous self map of a compact metric space (X, ρ). We define a function $\xi_f : X \times X \times \mathbb{R} \times \mathbb{N} \longrightarrow \mathbb{N}$ by:

$$\xi_f(x, y, t, n) = \#\left\{i : \rho(f^i(x), f^i(y)) < t,\ 0 \le i < n\right\}$$

where $\#A$ denotes the cardinality of the set A. By the means of ξ_f we define the following two functions:

$$F_{xy}(f, t) = \liminf_{n\to\infty} \frac{1}{n}\xi_f(x, y, t, n), \quad F^*_{xy}(f, t) = \limsup_{n\to\infty} \frac{1}{n}\xi_f(x, y, t, n).$$

For brevity, we often write ξ, $F_{xy}(t)$, $F^*_{xy}(t)$ instead of ξ_f, $F_{xy}(f, t)$, $F^*_{xy}(f, t)$ respectively.

Both functions F_{xy} and F^*_{xy} are nondecreasing, $F_{xy}(t) = F^*_{xy}(t) = 0$ for $t < 0$ and $F_{xy}(t) = F^*_{xy}(t) = 1$ for $t > \operatorname{diam} X$. Functions F_{xy} and F^*_{xy} are called *lower* and *upper distribution* functions, respectively.

Definition 3. A pair of points $(x, y) \in X \times X$ is called *distributionally chaotic (of type 1)* if

(1) $F_{xy}(s) = 0$ for some $s > 0$,

(2) $F^*_{xy}(t) = 1$ for all $t > 0$.

A set containing at least two points is called *distributionally scrambled set of type 1* (or *d-scrambled set* for short) if any pair of its distinct points is distributionally chaotic.

A map f is *distributionally chaotic (DC1)* if it has an uncountable d-scrambled set. Distributional chaos is said to be *uniform* if a constant s from condition (1) may be chosen the same for all the pairs of distinct points of d-scrambled set.

We remark here that the definition of distributional chaos was introduced to extend approach proposed by Li and Yorke in their famous paper [8]. Then it is clear why we use the name d-scrambled set. Namely each d-scrambled set is also scrambled set as defined by Li and Yorke.

We also should mention that our notation is a slightly different compared to that introduced by Schweizer and Smítal (founders of distributional chaos) in [15]. It is mainly because the definition of distributional chaos passed a very long journey since its introduction (even its name changed as it was originally called strong chaos). The definition we present is one of the strongest possibilities [1] and is usually called distributional chaos of type 1 to be distinguished from other two weaker definitions – DC2 and DC3.

Definition 4. We say that a T-periodic local process φ on M is *(uniform) distributionally chaotic* if there exists compact set $\Lambda \subset M$ invariant for the Poincaré map $P_T = \varphi_{(0,T)}$ such that $P_T|_\Lambda$ is (uniform) distributionally chaotic.

We say that the equation (2.2) is *(uniform) distributionally chaotic* if it generates a local process which is (uniform) distributionally chaotic.

3. Main tools

Lemma 5. *Let $S \subset \{0,1\}^{\mathbb{N}}$ be uncountable. There exists an uncountable set $A \subset S$ such that*

$$\#\{i : a_i = b_i\} = +\infty, \quad \#\{i : a_i \neq b_i\} = +\infty$$

for all $a, b \in A$, $a \neq b$.

Proof. We define a relation $\sim$ on $S \times S$ by the formula

$$a \sim b \iff \#\{i : a_i = b_i\} < +\infty \text{ or } \#\{i : a_i \neq b_i\} < +\infty.$$

It is easy to verify that $\sim$ is an equivalence relation and that $[a]_\sim$ is at most countable for any a. We define A to consist of exactly one representative of each equivalence class $[a]_\sim$, $a \in S$. The quotient space $S/\sim$ is uncountable, thus so is A. □

Remark 6. If $S = \{0, 1, 2, \ldots, n-1\}^{\mathbb{N}}$ then a set A with the above properties can be constructed easily (e.g. by application of Lemma 5). But in the case of arbitrary uncountable subset $S \subset \{0, 1, 2, \ldots, n-1\}^{\mathbb{N}}$ the authors don't know any method of construction of the set A; surely in that case the proof of Lemma 5 doesn't work since the set $[a]_\sim$ can be uncountable for some $a \in S$.

It can be shown that the relation

$$aRb \iff \#\{i : a_i \neq b_i\} < +\infty$$

is an equivalence relation such that $[a]_R$ is at most countable for any $a \in S$. So we can construct an uncountable subset A such that every distinct points of A differ on infinitely many indices. By pigeonhole principle, among every $n+1$ distinct elements of A, at least two ones coincide on infinitely many indices. Still some arguments are missing to have a proof of Lemma 5 (for $n > 2$), however the above observation allow us to reformulate the problem in the following way: Let be given a simple graph with an uncountably many vertices. Among every $n+1$ vertices, there are at least two ones connected by an edge. *Does it imply the existence of a full subgraph with uncountably many vertices?*
This observation was suggested to us by Michał Misiurewicz.

Lemma 7. *Let $(X, d), (Y, \rho)$ be compact metric spaces, let $f \in C(X)$, $g \in C(Y)$ and let $\Phi : X \to Y$ be a semi-conjugacy. Suppose that $F_{xy}(g, s) = 0$ for some $s > 0$. In that case there exists $t > 0$ (depending only on s) such that $F_{pq}(f, t) = 0$ for every $p \in \Phi^{-1}(x)$, $q \in \Phi^{-1}(y)$.*

Proof. The proof is straightforward. Namely, if $t > 0$ is an s-modulus of continuity for Φ (note that Φ is uniformly continuous) then $\xi_f(p, q, t, n) \leq \xi_g(x, y, s, n)$ and the proof is finished. □

Theorem 8. *Let $(X, d), (Y, \rho)$ be compact metric spaces and let $f \in C(X)$, $g \in C(Y)$. Let $\Phi : X \to Y$ be a semi-conjugacy such that $\#\Phi^{-1}(y) \leq 2$ for some $y \in \mathrm{Per}(g)$. If g is surjective and has WSP then f is distributionally chaotic and distributional chaos is uniform.*

PROOF. It is well known (see e.g. [21, Lemma 6]) that a DC1 pair for some iterate f^k is also a DC1 pair for f. Then we may assume that y is a fixed point of g. Let $\Phi^{-1}(y) = \{z_1, z_2\}$ (we do not assume that $z_1 \neq z_2$) and let $\lambda = d(z_1, z_2)$. The map g has WSP and is onto, thus (by [11]) there exists a d-scrambled set S for g and increasing sequences m_i, s_i such that for any $x \in S$ the following condition holds:

$$\rho(g^j(x), y) < \frac{1}{i} \quad \text{for } m_i \leq j \leq m_i + s_i, \tag{3.1}$$

where

$$\lim_{i\to\infty} \frac{s_i}{m_i + s_i} = 1. \tag{3.2}$$

By continuity arguments, for every integer i and points $x \in S$, $v \in \Phi^{-1}(x)$ there exists $k_i^v \in \{1, 2\}$ such that

$$d(f^j(v), z_{k_i^v}) < \delta_i \quad \text{for } m_i \leq j \leq m_i + s_i, \tag{3.3}$$

where δ_i depends only on i and

$$\lim_{i\to\infty} \delta_i = 0. \tag{3.4}$$

Let $\tilde{S} \subset \Phi^{-1}(S)$ be an uncountable set with the property that $\Phi(p) \neq \Phi(q)$ for any distinct $p, q \in \tilde{S}$. We will show that $\tilde{S}$ contains a d-scrambled set for f.

For any $p \in \tilde{S}$ we define a sequence $c_p = k_1^p, k_2^p, \ldots$. We have two possibilities:

(1) There exists an uncountable set $A \subset \tilde{S}$ such that $c_p = c_q$ for any $p, q \in A$. In that case we fix $\hat{S} = A$.

(2) For any $p \in \tilde{S}$ the set $\{q \in \tilde{S} : c_p = c_q\}$ is at most countable. In that case, $\{c_p : p \in \tilde{S}\}$ is uncountable and we may apply Lemma 5 obtaining a set $\hat{S} \subset \tilde{S}$ such that for any distinct $p, q \in \hat{S}$ it holds that

$$\#\{i : k_i^p = k_i^q\} = +\infty, \quad \#\{i : k_i^p \neq k_i^q\} = +\infty.$$

We have just proved that there always exists a set $\hat{S} \subset \tilde{S}$ such that the equality $\#\{i : k_i^p = k_i^q\} = +\infty$ holds for any $p, q \in \hat{S}$. By (3.3), we obtain

$$d(f^j(p), f^j(q)) < 2\delta_{\mu_i} \quad \text{for all } m_{\mu_i} \leq j \leq m_{\mu_i} + s_{\mu_i} \tag{3.5}$$

where $\{\mu_i\}_{i=1}^{\infty} \subset \mathbb{N}$ is some strictly increasing sequence depending only on p and q. Equalities (3.2) and (3.4) imply that $F_{pq}^*(t) = 1$ for any parameter value $t > 0$. By Lemma 7, the proof is finished. □

Now we deal with the homoclinic solutions.

Proposition 9. *Let $(X,d),(Y,\rho)$ be compact metric spaces and $f\in C(X)$, $g\in C(Y)$. Let $\Phi : X\to Y$ be a semi-conjugacy such that $\Phi^{-1}(y)=\{p_1,\dots,p_k\}\subset \mathrm{Per}(f)$ for some $y\in \mathrm{Per}(g)$ and $k\in\mathbb{N}$. Then, for any $z\in Y$ with the property $\omega_g(z)=\alpha_g(z)=Orb(y,g)$ and for any $q\in\Phi^{-1}(z)$ there exist $u,v\in\Phi^{-1}(y)$ such that $\alpha_f(q)=Orb(u,f)$ and $\omega_f(q)=Orb(v,f)$.*

Proof. Let us fix $\lambda>0$ such that $3\lambda<d(s,r)$ for all $s,r\in\bigcup_{i=1}^k \mathrm{Orb}(p_i,f)$, $s\neq r$. Let $\delta>0$ be a λ modulus of continuity of f ($\delta<\lambda$). Let us fix any $z\in Y$ satisfying $\omega_g(z)=\alpha_g(z)=\mathrm{Orb}(y,g)$ and fix any $q\in\Phi^{-1}(z)$. For every $\varepsilon>0$ the set $\Phi^{-1}(\bar B(\varepsilon,\mathrm{Orb}(y,g)))$ is compact so

$$\bigcap_{\varepsilon>0}\Phi^{-1}(\bar B(\varepsilon,\mathrm{Orb}(y,g)))=\bigcup_{i=1}^k \mathrm{Orb}(p_i,f).$$

This implies that there exists an integer $N>0$ such that for any $j\geq N$ there exists $l(j)\in\{1,\dots,k\}$ with the property $d(f^j(q),f^j(p_{l(j)}))<\delta$. Thus

$$d(f^{j+1}(p_{l(j)}),f^{j+1}(q))<\lambda$$

so

$$\begin{aligned}d(f^{j+1}(p_{l(j)}),f^{j+1}(p_{l(j+1)}))&\leq d(f^{j+1}(p_{l(j)}),f^{j+1}(q))\\&+d(f^{j+1}(q),f^{j+1}(p_{l(j+1)}))<\lambda+\delta<3\lambda\end{aligned}$$

which immediately implies that $l(j+1)=l(j)$. Combining this observation with the inclusion $\omega_f(q)\subset\bigcup_{i=1}^k \mathrm{Orb}(p_i,f)$ we obtain that $\omega_f(q)=\mathrm{Orb}(p_{l(N)},f)$. It is enough to fix $v=p_{l(N)}$. The case of α_f is similar. □

Corollary 10. *Let (X,d), (Y,ρ) be compact metric spaces and let $f\in C(X)$, $g\in C(Y)$. Let $\Phi: X\to Y$ be a semi-conjugacy such that $\Phi^{-1}(y)=\{p_1,p_2\}\subset \mathrm{Per}(f)$ for some $y\in\mathrm{Per}(g)$. Additionally assume that g is surjective and has WSP. Then there is an infinite set $A\subset X$ and $z\in A$ the following equalities hold $\alpha_f(z)=p_i$ and $\omega_f(z)=p_j$ where $i,j\in\{1,2\}$ and $i=j$ (homoclinic connection) or $i\neq j$ (heteroclinic connection).*

4. Applications

In this section we investigate the equation (1.1) which is a pertubation of the well known (see [23])

$$\dot z=\left(1+e^{i\kappa t}\,|z|^2\right)\overline{z}^2 \tag{4.1}$$

We state the main theorem of this section

Theorem 11. *Let the inequalities*

$$0 < \kappa \leq 0.18, \tag{4.2}$$

$$0 \leq N \leq 0.01 \tag{4.3}$$

be satisfied. Then the equation (1.1) *is uniform distributionally chaotic.*

Remark 12. The pertubation has the form $-Ne^{-2\alpha}$. It is possible to state similar theorems for every $\alpha \in (0, \pi)$. We chose $\alpha = \frac{\pi}{6}$ to simplify technical aspects of calculations since the choice of α affects the range of κ and N in (4.2) and (4.3).

It is also possible to investigate homo- and heteroclinic solutions of (1.1).

Theorem 13. *Let the inequalities* (4.2) *and*

$$0 < N \leq 0.01 \tag{4.4}$$

be satisfied. In that case, the set of solutions of (1.1) *which are*

- *homoclinic to* ψ_1,
- *homoclinic to* ψ_2,
- *heteroclinic from* ψ_1 *to* ψ_2,
- *heteroclinic from* ψ_2 *to* ψ_1

is infinite (for each of the listed possibilities) where ψ_1, ψ_2 *are the periodic solutions which bifurcate from the trivial one for* $N = 0$ *such that* $\Re[\psi_1] > 0$ *and* $\Re[\psi_2] < 0$.

The detailed proofs of Theorem 11 and 13 are quite technical and performed in many steps. For that reason we present below a general explanation why these theorems hold, while detailed arguments are forwarded to another article [14], which is sequel to the present one. Sketchy description below should help the reader to understand the main idea hidden behind the proofs (and calculations).

Idea of the proof of Theorem 11. For $N = 0$ there is a trivial periodic solution $\psi \equiv 0$ of the equation (1.1). It was proved in [23] that there exist a compact subset $\widetilde{I}$ of the Poincaré section $\mathbb{C}$ and semiconjugacy $g : \widetilde{I} \longrightarrow g(\widetilde{I}) \subset \Sigma_4$ of $\sigma|_{g(\widetilde{I})}$ and $\varphi_{(0,T)}$ such that $\Pi \subset g(\widetilde{I})$ where Π is a sofic shift which presentation is given in Figure 1.

The set $\widetilde{I}$ is the set of all initial conditions such that the solutions passing through them stay inside the infinite catenation W^∞ of the segment W (see Figure 2 (a)). The trivial solution is the only one which is completely contained

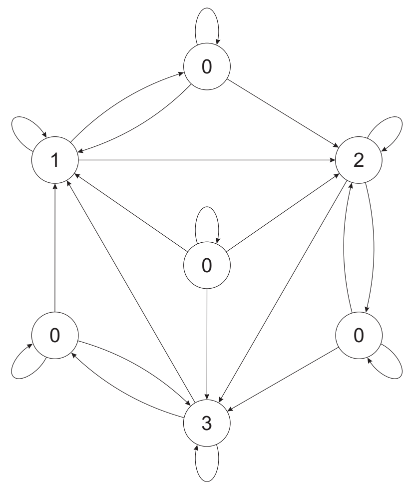

Figure 1. Presentation of the sofic shift Π.

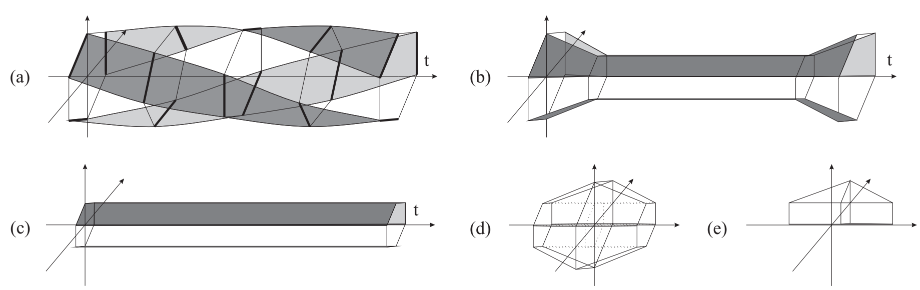

Figure 2. Isolating segments (a) W, (b) U, (c) $V(\xi)$ and sets (d) Z and (e) $\widehat{Z}$. Sets W^{--}, U^{--}, $V(\xi)^{--}$ are marked in grey.

in U^∞ (see Figure 2 (b)), thus it is coded by $0^\infty \in \Pi$. The existence of distributional chaos follows by the results of [13].

By the continuation method (Theorem 2), a conjugacy with Π can also be obtained with N satisfying (4.4). The only difference is that the trivial solution bifurcates into two solutions ψ_1 and ψ_2. Moreover, ψ_1 and ψ_2 are the only solutions which are fully contained in U^∞ so they are both coded by $0^\infty \in \Pi$. The existence of distributional chaos follows now by Theorem 8. □

Idea of the proof of Theorem 13. By Corollary 10 we see that there must be homoclinic or heteroclinic connections. Theorem 13 specifies that all possible connections are present in the system. The proof relies on the fact, that the segment W can be partitioned in a way, such that trajectory can visit (or leave) prescribed regions only if its symbolic itinerary contains specific sequence of symbols. It is the main tool to control the future and the past behavior of the orbit (in terms of solutions ψ_1 and ψ_2 it must approach). This is the main technique of the proof, which involves a deep analysis of the vector field. □

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