On continuous solutions of n-th order polynomial-like iterative equations

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Abstract. Many efforts have been made to present all continuous solutions of the iterative equation $\sum_{i=0}^{n} \lambda_i f^i(x) = c$ but for general n only the case of $c = 0$ was considered and no nonhyperbolic cases were discussed. In this paper we first prove that all continuous solutions are decided totally by those real characteristic roots, which not only gives a method to lower the order when complex characteristic roots are involved but also partly answers the question raised in Remark 8 in [Aequationes Math. 2004, 67: 80–105]. Then we find all continuous solutions of the equation with $c = 0$ in the case of smallest characteristic root being 1. Furthermore, we prove that in the case of all characteristic roots being 1 the equation with $c \neq 0$ has no continuous real solutions when n is even.

1. Introduction

Since C. BABBAGE [1] solved the functional equation $f^n(x) = x$, where $x \in \mathbb{R}$, $n \geq 2$ is an integer and f^n denotes the n-th iterate of the unknown function f, i.e., $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$, more and more attentions have been paid to the iterative root problem and the more general forms of iterative equation, where iteration of the unknown function is included as the main operation ([2], [6]). Among those forms an interesting one is the linear combination of iterates

$$\lambda_n f^n(x) + \lambda_{n-1} f^{n-1}(x) + \cdots + \lambda_1 f(x) + \lambda_0 x = F(x), \tag{1}$$

called the *polynomial-like iterative equation*, where $\lambda_i \in \mathbb{R}$.

Although many results (e.g. [5], [13], [15], [17], [18]) on existence of solutions are given for equation (1) with some nonlinear F, it is of special interests to find all continuous solutions of (1), even with constant F, i.e.,

$$\lambda_n f^n(x) + \lambda_{n-1} f^{n-1}(x) + \cdots + \lambda_1 f(x) + \lambda_0 x = c,$$

where c is a real constant. Since c is arbitrary, with out loss of generality, we may put $\lambda_n = 1$ in this equation, i.e,

$$f^n(x) + \lambda_{n-1} f^{n-1}(x) + \cdots + \lambda_1 f(x) + \lambda_0 x = c, \tag{2}$$

In 1974, motivated by Euler's equation $f(x + f(x)) = f(x)$ (see [14]), NABEYA ([10]) discussed the generalized equation $f(p+qx+rf(x)) = a+bx+cf(x)$, which actually can be transformed into the form (2) for $n = 2$. He presented all iterates f^k in the form of linear combination of f^0 and f^1 and discussed eigenvalues of the difference equation satisfied by the sequences of coefficients in the combination so as to formulate all continuous solutions. Later, less results were given until MATKOWSKI [7] indicated in 1989 that *when $\lambda_0 \neq 0$ the problem of determining all the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of (2) is still open, even for $n = 3$.*

Since 90's the iterative equation (2) with $c = 0$, i.e.,

$$f^n(x) + \lambda_{n-1} f^{n-1}(x) + \cdots + \lambda_1 f(x) + \lambda_0 x = 0, \tag{3}$$

again attracts interests of research [4], [8], [9], [12], [16]. In 1997, considering those solutions $f(x) = rx$ of linear form as *characteristic solution*, MATKOWSKI and ZHANG ([8]) discussed all cases of the characteristic root r and gave all continuous solutions of (3) for $n = 2$. As in [9], [16], substituting the linear function $f(x) = rx$ in equation (3) we see that the characteristic root r are determined by the polynomial equation

$$P(r) := r^n + \lambda_{n-1} r^{n-1} + \cdots + \lambda_1 r + \lambda_0 = 0. \tag{4}$$

In order to discuss equation (3) with the general order n, YANG and ZHANG applied the tool of linear difference form (or simply called LΔ form), introduced first in [9], to simplify the problem, finding all continuous solutions in the four hyperbolic cases: (IHe) $1 < r_1 < \ldots < r_n$ (expensive subcase), (IHc) $0 < r_1 < \cdots < r_n < 1$ (contractive subcase), (DHe) $r_1 < \cdots < r_n < -1$ (expansive subcase), and (DHc) $-1 < r_1 < \cdots < r_n < 0$ (contractive subcase), proving non-existence of continuous solutions in the case of no real characteristic roots, and giving the method of lowering order in the case of n-multiple characteristic roots.

So far, as remarked in the last section of [16], no results on the problem of all continuous solutions of (2) are given even for $n = 3$ in the case of n distinct characteristic roots with one of the following conditions:

(ID) P has both positive characteristic roots and negative ones, i.e., increasing factor and decreasing one are both included;

(EC) P has both roots greater than 1 in absolute value and roots less than 1 in absolute value, i.e., expansive factor and contractive one are both included;

(K) P has a root of absolute value being 1, refereed to "critical case" in [8];

($\mathbf{R}_p$) P has exactly p real roots but $2 \leq p < n$.

For $n \geq 3$ we also do not have a complete result when P has multiple roots.

In this paper we employ some properties of linear difference form, shown in Section 2, to discuss equation (2) for general n in the cases ($\mathbf{R}_p$) and ($\mathbf{K}$). In Section 3 we prove in the case ($\mathbf{R}_p$) that all continuous solutions are decided totally by those real characteristic roots, which not only gives a method to lower the order but also partly answers the question raised in Remark 8 in [16]. In Section 4 we give all continuous solutions of equation (2) with $c = 0$ when all characteristic roots are simple and the smallest one is 1. Section 5 is devoted to the case that all characteristic roots are equal to 1, in which equation (2) with even n and $c \neq 0$ is proved to have no continuous solutions.

2. Preliminaries

Suppose that all λ's and c are reals. We discuss equation (2) with $\lambda_0 \neq 0$. Similar to the case of $c = 0$ shown in [9], [16], its solutions have the following basic property.

Lemma 1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of equation* (2). *If $\lambda_0 \neq 0$, then f is a homeomorphism.*

PROOF. It is easy to see that f is one-to-one. Next we prove that the limits $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ are infinite. Suppose that $\lim_{x \to \infty} f(x) = y_0$, where $y_0 \in \mathbb{R}$. Rewrite (2) as

$$f^n(x) + \cdots + \lambda_1 f(x) = -\lambda_0 x + c. \tag{5}$$

By the continuity of f^k on the whole $\mathbb{R}$, $k = 1, \ldots, n$, the left-hand side of (5) is convergent as $x \to \infty$ to a finite limit but the right-hand side of (5) is not. The contradiction shows that $f(\mathbb{R}) = \mathbb{R}$. $\square$

Let $r_1, r_2, \ldots, r_n$ be n complex roots of equation (4). Then equation (2) is equivalent to

$$\sum_{j=0}^{n} (-1)^j \sum_{1 \leq s_1 < \cdots < s_j \leq n} r_{s_1} r_{s_2} \ldots r_{s_j} f^{n-j}(x) = c, \tag{6}$$

where we make a convention for the term of $j = 0$ that

$$\sum_{1 \leq s_1 < \cdots < s_j \leq n} r_{s_1} r_{s_2} \ldots r_{s_j} = 1,$$

and the inverse $g = f^{-1}$ satisfies the *dual equation*

$$\sum_{j=0}^{n} (-1)^j \sum_{1 \leq s_1 < \cdots < s_j \leq n} r_{s_1}^{-1} r_{s_2}^{-1} \ldots r_{s_j}^{-1} g^{n-j}(x) = (-1)^n r_1^{-1}, \ldots, r_n^{-1} c. \qquad (7)$$

As mentioned in our Introduction, the linear difference form, defined in [9], [16], is a useful tool in computation for polynomial-like iterative equations. Suppose that X is a vector space over $\mathbb{C}$ and let $\mathcal{S}(X)$ denote the set of all bilateral sequences $(x_\ell)_{\ell \in \mathbb{Z}} := (\ldots, x_{-1}, x_0, x_1, \ldots)$ in X. Given a positive integer n, complex numbers $\gamma_1, \ldots, \gamma_n$ and an integer k the n-th order linear difference form $\mathcal{F}_k[\gamma_1, \ldots, \gamma_n] : \mathcal{S}(X) \to X$ is defined by

$$\mathcal{F}_k[\gamma_1, \ldots, \gamma_n](x_\ell)_{\ell \in \mathbb{Z}} := \sum_{j=0}^{n} (-1)^j \sum_{1 \leq s_1 < \cdots < s_j \leq n} \gamma_{s_1} \ldots \gamma_{s_j} x_{k+n-j}.$$

We complementarily define $\mathcal{F}_k[\emptyset](x_\ell)_{\ell \in \mathbb{Z}} := x_k$, where $\emptyset$ is the empty set. Sometimes we call this form an n-th order LΔ form or an n-form shortly.

We can apply the concept of LΔ form to the sequences of both $(f^\ell)_{\ell \in \mathbb{Z}}$ and $(f^\ell(x))_{\ell \in \mathbb{Z}}$ with an arbitrary given $x \in \mathbb{R}$. It is clear that $\mathcal{F}_k[\gamma_1, \ldots, \gamma_n](f^\ell)_{\ell \in \mathbb{Z}}(x) = \mathcal{F}_k[\gamma_1, \ldots, \gamma_n](f^\ell(x))_{\ell \in \mathbb{Z}}$. Then equation (6) can be simplified as

$$\mathcal{F}_0[r_1, \ldots, r_n](f^\ell(x))_{\ell \in \mathbb{Z}} = c, \quad \forall x \in \mathbb{R}, \qquad (8)$$

and we can also simplify the dual equation (7) as

$$\mathcal{F}_0[r_1^{-1}, \ldots, r_n^{-1}](f^{-\ell}(x))_{\ell \in \mathbb{Z}} = (-1)^n r_1^{-1}, \ldots, r_n^{-1} c, \quad \forall x \in \mathbb{R}. \qquad (9)$$

The following properties of LΔ form are useful in computation and can be found in [16, Lemmas 1, 2 and 3].

Lemma 2. *Assume $n \geq 1$ and k are integers, and $\gamma_1, \ldots, \gamma_n$ are complex numbers. If $(\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n)$ is a permutation of $(\gamma_1, \ldots, \gamma_n)$, then $\mathcal{F}_k[\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n] = \mathcal{F}_k[\gamma_1, \ldots, \gamma_n]$. Moreover,*

(i) (Lower Order) *for integer $1 \leq q \leq n$,*

$$\mathcal{F}_k[\gamma_1, \ldots, \gamma_n] = \sum_{j=0}^{q} (-1)^j \sum_{n-q+1 \leq s_1 < \cdots < s_j \leq n} \gamma_{s_1} \ldots \gamma_{s_j} \mathcal{F}_{k+q-j}[\gamma_1, \ldots, \gamma_{n-q}];$$

(ii) (Reduce to Dual) *if* $\gamma_j \neq 0$ $(j = 1, \ldots, n)$ *then*

$$\mathcal{F}_k[\gamma_1, \ldots, \gamma_n](f^\ell)_{\ell \in \mathbb{Z}} = (-1)^n \gamma_1 \ldots \gamma_n \, \mathcal{F}_{-(k+n)}[\gamma_1^{-1}, \ldots, \gamma_n^{-1}](f^{-\ell})_{\ell \in \mathbb{Z}};$$

(iii) (Shift on Iteration) *if the continuous function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies that* $\mathcal{F}_0[r_1, \ldots, r_n](f^\ell)_{\ell \in \mathbb{Z}} = 0$, *then*

$$\mathcal{F}_{k+p}[r_1, \ldots, r_{t-1}, r_{t+1}, \ldots, r_n](f^\ell)_{\ell \in \mathbb{Z}} = r_t^p \mathcal{F}_k[r_1, \ldots, r_{t-1}, r_{t+1}, \ldots, r_n](f^\ell)_{\ell \in \mathbb{Z}}$$

for arbitrary positive integers p, t *with* $1 \leq t \leq n$.

Another important thing is the expression of the general iterate of solutions with its first n-th iterates, which is useful to reducing order of equation (8). As in [16], let

$$A := \begin{pmatrix} 1 & -\sum_{j=2}^{n} r_j & \sum_{2 \leq j < k \leq n} r_j r_k & \cdots & (-1)^{n-1} r_2 \ldots r_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -\sum_{j=1}^{n-1} r_j & \sum_{1 \leq j < k \leq n-1} r_j r_k & \cdots & (-1)^{n-1} r_1 \ldots r_{n-1} \end{pmatrix}$$

and let $\Delta(r_1, \ldots, r_n)$ and A_{j1} $(j = 1, \ldots, n)$ denote respectively the determinant and the algebraic complement minors of the matrix A. If the polynomial P, defined in (4), has n distinct roots $r_1, \ldots, r_n$ in $\mathbb{C}$ and $f : \mathbb{R} \to \mathbb{R}$ is a solution of equation (8), by (3.15) in [16, p. 88], we have

$$f^{n+m} = \sum_{j=1}^{n} \frac{A_{j1}}{\Delta(r_1, \ldots, r_n)} r_j^{m+1} \mathcal{F}_0[r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n](f^\ell)_{\ell \in \mathbb{Z}} \qquad (10)$$

for any integer $m \geq 0$.

The following lemma gives the relation between the equation (8) with $c \neq 0$ and the equation (8) with $c = 0$. For $-\infty \leq \alpha < \beta \leq +\infty$, let $|\alpha, \beta|$ denote generally one of the intervals: $[\alpha, \beta]$, $(-\infty, \beta]$, $[\alpha, +\infty)$ and $(-\infty, +\infty)$.

Lemma 3. *Suppose that all numbers* $\gamma_j \neq 0$ $(j = 1, \ldots, n)$ *are real and none of them is equal to 1. Then the equation* $\mathcal{F}_0[\gamma_1, \ldots, \gamma_n](f^\ell(x))_{\ell \in \mathbb{Z}} = c$ *for all* $x \in |\alpha, \beta|$ *can be reduced to the equation* $\mathcal{F}_0[\gamma_1, \ldots, \gamma_n](\tilde{f}^\ell(x))_{\ell \in \mathbb{Z}} = 0$ *for all* $x \in |\alpha - \xi, \beta - \xi|$ *by the substitution* $\tilde{f}(x) = f(x + \xi) - \xi$, *where* $\xi := c / \prod_{\varsigma=1}^{n}(1 - \gamma_\varsigma)$, *and vice versa.*

PROOF. Obviously, $\widetilde{f}^k(x) = f^k(x + \xi) - \xi$ for all $k = 0, 1, \ldots$. For $x \in |\alpha - \xi, \beta - \xi|$,

$$\mathcal{F}_0[\gamma_1, \ldots, \gamma_n](\widetilde{f}^\ell(x))_{\ell \in \mathbb{Z}} = \sum_{j=0}^{n} (-1)^j \sum_{1 \le s_1 < \cdots < s_j \le n} \gamma_{s_1} \gamma_{s_2} \ldots \gamma_{s_j} (f^{n-j}(x + \xi) - \xi)$$

$$= \sum_{j=0}^{n} (-1)^j \sum_{1 \le s_1 < \cdots < s_j \le n} \gamma_{s_1} \gamma_{s_2} \ldots \gamma_{s_j} (f^{n-j}(x + \xi)) - \xi \prod_{\varsigma=1}^{n} (1 - \gamma_\varsigma)$$

$$= \mathcal{F}_0[\gamma_1, \ldots, \gamma_n](f^\ell(x + \xi))_{\ell \in \mathbb{Z}} - c = 0.$$

Noting that the substitution is invertible, we can similarly prove the opposite direction. This completes the proof.   □

3. Case of complex roots

In Remark 8 of [16] there is raised a question: *If equation (2) with $c = 0$ has exactly k real characteristic roots and $k < n$, can we obtain all of its continuous real solutions from the equation $\mathcal{F}_0[r_1, \ldots, r_k](f^\ell)_{\ell \in \mathbb{Z}} = 0$, i.e., the same type of iterative equation of lower order determined exactly by those k real characteristic roots?* The following theorem gives an answer to this question. As usual, for $z \in \mathbb{C}$ by $\overline{z}$ and $|z|$ we denote the *conjugate* and *module* of z respectively..

Theorem 1. *Suppose that the characteristic equation (4) has roots $r_1, \ldots, r_p \in \mathbb{R}$ and $r_{p+1}, \ldots, r_s, \overline{r}_{p+1}, \ldots, \overline{r}_s \in \mathbb{C} \backslash \mathbb{R}$ of multiplicities $k_1, \ldots, k_p$ and $k_{p+1}, \ldots, k_s, k_{p+1}, \ldots, k_s$ respectively, where $p < s \in \mathbb{N}$ and $\sum_{j=1}^{p} k_j + 2 \sum_{j=p+1}^{s} k_j = n$, and that $|r_1| \le \cdots \le |r_p| < |r_{p+1}| < \cdots < |r_s|$. Then equation (2) with $c = 0$ has the same continuous solutions $f : \mathbb{R} \to \mathbb{R}$ as the lower order equation*

$$\sum_{j=0}^{k_1 + \cdots + k_p} (-1)^j \sum_{1 \le s_1 < \cdots < s_j \le k_1 + \cdots + k_p} \gamma_{s_1} \ldots \gamma_{s_j} f^{k_1 + \cdots + k_p - j}(x) = 0, \qquad (11)$$

where $(\gamma_1, \ldots, \gamma_{k_1 + \cdots + k_p}) := (\overbrace{r_1, \ldots, r_1}^{k_1}, \ldots, \overbrace{r_p, \ldots, r_p}^{k_p})$.

This theorem gives a method to lower the order when complex characteristic roots are involved. It shows that an equation having complex characteristic roots can be reduced to an equation of the same type having those real characteristic roots only.

PROOF OF THEOREM 1. Suppose that f is a continuous solution of equation (8) (equivalent to (2)) with $c = 0$. For arbitrarily fixed $x_0 \in \mathbb{R}$, let $x_\ell := f^\ell(x_0)$ for all $\ell \in \mathbb{Z}$. Then the sequence $(x_m)_{m \in \mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, is a solution of the linear difference equation

$$\mathcal{F}_m[\overbrace{r_1, \ldots, r_1}^{k_1}, \ldots, \overbrace{r_s, \ldots, r_s}^{k_s}, \ldots, \overbrace{\overline{r}_s, \ldots, \overline{r}_s}^{k_s}](x_\ell)_{\ell \in \mathbb{Z}} = 0, \quad \forall m \in \mathbb{N}_0, \qquad (12)$$

associated with initial data $x_0, \ldots, x_{n-1}$. Using the formula of general solution of linear difference equations, seen for example in [3, Corollary 2.24, p. 77], we see that all solutions of equation (12) can be presented in the form

$$x_m = \sum_{j=1}^{p}(\mu_{j,0} + \cdots + \mu_{j,k_j-1}m^{k_j-1})r_j^m + \sum_{j=p+1}^{s}\left\{(\mu_{j,0} + \cdots + \mu_{j,k_j-1}m^{k_j-1})r_j^m\right.$$
$$\left. + (\nu_{j,0} + \cdots + \nu_{j,k_j-1}m^{k_j-1})\overline{r}_j^m\right\}, \qquad (13)$$

where all $\mu_{j,k}$'s and $\nu_{j,k}$'s are complex depending on $x_0, \ldots, x_{n-1}$. Comparing the expressions of x_m and $\overline{x}_m$ and noting that $x_m = \overline{x}_m \in \mathbb{R}$, we easily see that $\mu_{j,0}, \ldots, \mu_{j,k_j-1} \in \mathbb{R}$, $j = 1, \ldots, p$ and $\mu_{j,0} = \overline{\nu}_{j,0}, \ldots, \mu_{j,k_j-1} = \overline{\nu}_{j,k_j-1}$, $j = p+1, \ldots, s$.

To simplify the formula (13), we suppose that $\theta, \omega \in [0, 2\pi)$ satisfy

$$r_s = |r_s|(\cos\theta + \mathbf{i}\sin\theta), \quad \mu_{s,k_s-1} = |\mu_{s,k_s-1}|(\cos\omega + \mathbf{i}\sin\omega).$$

It follows from (13) that

$$x_m = A(m) + \mu_{s,k_s-1}m^{k_s-1}r_s^m + \nu_{s,k_s-1}m^{k_s-1}\overline{r}_s^m$$
$$= A(m) + |\mu_{s,k_s-1}|\,m^{k_s-1}|r_s|^m\{(\cos\omega + \mathbf{i}\sin\omega)(\cos m\theta + \mathbf{i}\sin m\theta)$$
$$+ (\cos\omega - \mathbf{i}\sin\omega)(\cos m\theta - \mathbf{i}\sin m\theta)\}$$
$$= A(m) + 2|\mu_{s,k_s-1}|\,m^{k_s-1}|r_s|^m\cos(m\theta + \omega), \qquad (14)$$

where

$$A(m) := \sum_{j=1}^{p}(\mu_{j,0} + \cdots + \mu_{j,k_j-1}m^{k_j-1})r_j^m + \sum_{j=p+1}^{s-1}\left\{(\mu_{j,0} + \cdots + \mu_{j,k_j-1}m^{k_j-1})r_j^m\right.$$
$$\left. + (\nu_{j,0} + \cdots + \nu_{j,k_j-1}m^{k_j-1})\overline{r}_j^m\right\} + (\mu_{s,0} + \cdots + \mu_{s,k_s-2}m^{k_s-2})r_s^m$$
$$+ (\nu_{s,0} + \cdots + \nu_{s,k_s-2}m^{k_s-2})\overline{r}_s^m. \qquad (15)$$

Next we claim that

$$2|\mu_{s,k_s-1}|\, m^{k_s-1}|r_s|^m \cos(m\theta + \omega) = 0, \quad \forall m \in \mathbb{N}_0. \tag{16}$$

Noticing (14), where the terms of r_s and $\overline{r}_s$ of the highest degree are separated from other terms, which are all included in $A(m)$, we obtain that

$$\frac{x_{2(m+1)} - x_{2m}}{(2m)^{k_s-1}|r_s|^{2m}} = \frac{A(2(m+1)) - A(2m)}{(2m)^{k_s-1}|r_s|^{2m}} + 2|\mu_{s,k_s-1}|$$

$$\times \left\{ \frac{(m+1)^{k_s-1}}{m^{k_s-1}}|r_s|^2 \cos(2(m+1)\theta + \omega) - \cos(2m\theta + \omega) \right\} = \frac{A(2(m+1)) - A(2m)}{(2m)^{k_s-1}|r_s|^{2m}}$$

$$+ 2|\mu_{s,k_s-1}|\left\{ \left(\frac{(m+1)^{k_s-1} - m^{k_s-1}}{m^{k_s-1}} + 1 \right)|r_s|^2 \cos(2(m+1)\theta + \omega) - \cos(2m\theta + \omega) \right\}$$

$$= B(m) + 2|\mu_{s,k_s-1}|\{|r_s|^2 \cos(2(m+1)\theta + \omega) - \cos(2m\theta + \omega)\}$$

$$= B(m) + 2|\mu_{s,k_s-1}|\{|r_s|^2 (\cos 2\theta \cos(2m\theta + \omega) - \sin 2\theta \sin(2m\theta + \omega)) - \cos(2m\theta + \omega)\}$$

$$= B(m) + 2|\mu_{s,k_s-1}|\{(|r_s|^2 \cos 2\theta - 1)\cos(2m\theta + \omega) - r_s|^2 \sin 2\theta \sin(2m\theta + \omega)\}$$

$$= B(m) + 2|\mu_{s,k_s-1}|\{(|r_s|^2 \cos 2\theta - 1)^2 + (|r_s|^2 \sin 2\theta)^2\}^{1/2} \cos(2m\theta + \omega + \varphi), \tag{17}$$

where

$$B(m) := \frac{A(2(m+1)) - A(2m)}{(2m)^{k_s-1}|r_s|^{2m}}$$

$$+ 2|\mu_{s,k_s-1}| \left(\frac{(m+1)^{k_s-1} - m^{k_s-1}}{m^{k_s-1}} \right)|r_s|^2 \cos(2(m+1)\theta + \omega),$$

$$\varphi := \begin{cases} \tan^{-1}\{|r_s|^2 \sin 2\theta/(|r_s|^2 \cos 2\theta - 1)\}, & \text{if } |r_s|^2 \cos 2\theta - 1 \neq 0, \\ \pi/2 \text{ or } -\pi/2, & \text{if } |r_s|^2 \cos 2\theta - 1 = 0. \end{cases}$$

It is worth mentioning that the term $(|r_s|^2 \cos 2\theta - 1)^2 + (|r_s|^2 \sin 2\theta)^2$ in (17) cannot be 0. In fact, if $|r_s|^2 \cos 2\theta - 1 = \sin 2\theta = 0$ then $\cos 2\theta = 1$. It implies that $\theta = 0$ or π, which contradicts with the fact that $r_s \in \mathbb{C}\backslash\mathbb{R}$. Observing (15), we conclude that $\lim_{m\to\infty} B(m) = 0$. Furthermore, put

$$y_m := 2|\mu_{s,k_s-1}|\{(|r_s|^2 \cos 2\theta - 1)^2 + (|r_s|^2 \sin 2\theta)^2\}^{1/2} \cos(2m\theta + \omega + \varphi)$$

for all $m \in \mathbb{N}_0$, which appears in the last row of (17). Case (i): $\theta \neq \pi/2, 3\pi/2$. Suppose $|\mu_{s,k_s-1}| \neq 0$. Then the sequence $(y_m : m \geq 0)$ does not have a uniform sign and neither the subsequence of its all positive terms nor the subsequence of its all negative terms tends to 0 since $\theta \neq 0, \pi$. On the other hand, for arbitrary $x_0 \in \mathbb{R}$, if $f^2(x_0) = x_0$ then $f^{2(m+1)}(x_0) - f^{2m}(x_0) = 0$ for all $m \in \mathbb{Z}$; if

$f^2(x_0) \neq x_0$ then $f^{2(m+1)}(x_0) - f^{2m}(x_0) > 0$ (or < 0) for all $m \in \mathbb{Z}$ by the monotonicity of f, given in Lemma 1. It implies that the left hand side of (17) is equal to 0 or has a definite sign for all m, which contradicts the fact that the right hand side of (17) switches its sign for large m. Hence $|\mu_{s,k_s-1}| = 0$. Case (ii): $\theta = \pi/2$ or $3\pi/2$. In this case we have $\sin 2\theta = 0$ and $|r_s|^2 \cos 2\theta - 1 \neq 0$, implying that

$$\varphi = \tan^{-1}\{|r_s|^2 \sin 2\theta / (|r_s|^2 \cos 2\theta - 1)\} = 0.$$

Thus $\cos(2m\theta + \omega + \varphi) = \cos(m\pi + \omega)$ for all $m \in \mathbb{N}_0$. Here, if $\omega \neq \pi/2$ and $3\pi/2$ then, by the discussion in the above case (i), we get $|\mu_{s,k_s-1}| = 0$; if $\omega = \pi/2$ or $3\pi/2$ then

$$\cos(m\pi + \omega) = 0, \quad \forall m \in \mathbb{N}_0.$$

Thus the claimed result (16) is proved, i.e., $x_m = A(m)$ for $m \in \mathbb{N}_0$.

Note that $A(m)$, defined in (15), is of the same form as the formula in (13) but contains neither the term of $\mu_{s,k_s-1}m^{k_s-1}r_s^m$ nor the term of $\nu_{s,k_s-1}m^{k_s-1}\overline{r}_s^m$. We can similarly prove that $\mu_{s,k_s-2} = 0$ (or $\mu_{s-1,k_{s-1}-1} = 0$ when $k_s = 1$). Thus, repeating the same procedure as before, we can prove that $\mu_{ij} = 0$ for all $i = s, s-1, \ldots, p+1$ and all $j = k_{i-1}, \ldots, k_0$ and finally obtain from (13) that

$$x_m = \sum_{j=1}^{p}(\mu_{j,0} + \cdots + \mu_{j,k_j-1}m^{k_j-1})r_j^m,$$

which by the theory of linear difference equations [3, Corollary 2.24, p. 77] is actually a solution of the difference equation

$$\mathcal{F}_m[\overbrace{r_1, \ldots, r_1}^{k_1}, \ldots, \overbrace{r_p, \ldots, r_p}^{k_p}](x_\ell)_{\ell \in \mathbb{Z}} = 0, \quad \forall m \in \mathbb{N}_0.$$

Since $x_m = f^m(x_0)$ for $m \in \mathbb{N}_0$, it implies that

$$\mathcal{F}_0[\overbrace{r_1, \ldots, r_1}^{k_1}, \ldots, \overbrace{r_p, \ldots, r_p}^{k_p}](f^\ell(x_0))_{\ell \in \mathbb{Z}} = 0. \tag{18}$$

This proves (11).

Conversely, if f satisfies (11) then f is a solution of equation (8) with $c = 0$. In fact, by Lemma 2 (i), $\mathcal{F}_0[\overbrace{r_1, \ldots, r_1}^{k_1}, \ldots, \overbrace{r_s, \ldots, r_s}^{k_s}, \ldots, \overbrace{\overline{r}_s, \ldots, \overline{r}_s}^{k_s}](f^\ell)_{\ell \in \mathbb{Z}}$ can be presented as a linear combination of terms $\mathcal{F}_\eta[\overbrace{r_1, \ldots, r_1}^{k_1}, \ldots, \overbrace{r_p, \ldots, r_p}^{k_p}](f^\ell)_{\ell \in \mathbb{Z}}$, $\eta = 0, \ldots, n - \sum_{j=1}^{p}k_j$. This completes the proof. $\qquad\square$

Combining Theorem 1 with Lemma 3 given in Section 2, we can easily give a similar result for equation (2) generally without the assumption that $c = 0$.

Corollary 1. *Suppose that the characteristic equation (4) has roots $r_1, \ldots,$ $r_p \in \mathbb{R}$ and $r_{p+1}, \ldots, r_s, \overline{r}_{p+1}, \ldots, \overline{r}_s \in \mathbb{C} \backslash \mathbb{R}$ of multiplicities $k_1, \ldots, k_p$ and $k_{p+1}, \ldots,$ $k_s, k_{p+1}, \ldots, k_s$ respectively, where $p < s \in \mathbb{N}$ and $\sum_{j=1}^{p} k_j + 2 \sum_{j=p+1}^{s} k_j = n$, and that $r_1, \ldots, r_p \neq 1$ and $|r_1| \leq \cdots \leq |r_p| < |r_{p+1}| < \cdots < |r_s|$. Then equation (2) has the same continuous solutions $f : \mathbb{R} \to \mathbb{R}$ as the lower order equation*

$$\sum_{j=0}^{k_1+\cdots+k_p} (-1)^j \sum_{1 \leq qs_1 < \cdots < s_j \leq k_1 + \cdots + k_p} \gamma_{s_1} \cdots \gamma_{s_j} f^{k_1 + \cdots + k_p - j}(x) = c / \prod_{\varsigma=p+1}^{s} |1 - r_\varsigma|^{2k_\varsigma},$$

where $(\gamma_1, \ldots, \gamma_{k_1+\cdots+k_p}) := (\overbrace{r_1, \ldots, r_1}^{k_1}, \ldots, \overbrace{r_p, \ldots, r_p}^{k_p})$.

4. Case $1 = r_1 < \cdots < r_n$

This is a case in **(K)**. In this case Lemma 3 does not work because $r_1 = 1$. For $n = 2$ both NABEYA [10] and MATKOWSKI and ZHANG [8] considered this case and proved that all solutions of equation (8) with $c = 0$ are of the piecewise linear form

$$f(x) = \begin{cases} r_2(x - a) + a, & x \in (-\infty, a], \\ x, & x \in (a, b), \\ r_2(x - b) + b, & x \in [b, +\infty), \end{cases} \tag{19}$$

where a, b are constants such that $-\infty \leq a \leq b \leq +\infty$. In this section we consider equation (8) of general n with $c = 0$, a case not discussed in [16]. In comparison with the discussion for $n = 2$ in [8], [10], in our case it is difficult to obtain the inequality $r_1 \leq (f(y) - f(x))/(y - x) \leq r_n$ for all $x \neq y$, which is very useful in the proof of (19) in [8], [10] for $n = 2$.

Theorem 2. *Suppose that equation (4) has n distinct roots $1 = r_1 < r_2 < \cdots < r_n$. Then every continuous solution $f : \mathbb{R} \to \mathbb{R}$ of equation (2) with $c = 0$ is of the form*

$$f(x) = \begin{cases} \phi(x), & x \in (-\infty, a], \\ x, & x \in (a, b), \\ \psi(x), & x \in [b, +\infty), \end{cases} \tag{20}$$

where $-\infty \le a \le b \le +\infty$ and the functions ϕ, ψ are solutions of equations

$$\sum_{j=0}^{n-1}(-1)^j \sum_{2\le s_1<\cdots<s_j\le n} r_{s_1}r_{s_2}\ldots r_{s_j}\phi^{n-1-j}(x) = a\prod_{\varsigma=2}^{n}(1-r_\varsigma), \ \forall x \in (-\infty, a], \quad (21)$$

$$\sum_{j=0}^{n-1}(-1)^j \sum_{2\le s_1<\cdots<s_j\le n} r_{s_1}r_{s_2}\ldots r_{s_j}\psi^{n-1-j}(x) = b\prod_{\varsigma=2}^{n}(1-r_\varsigma), \ \forall x \in ![b, +\infty), \quad (22)$$

respectively.

Observe that if $a = -\infty$ (resp. $b = +\infty$) then we have $f(x) = x$ on $(-\infty, b)$ (resp. on $(a, +\infty)$); if $a = b$ then (20) presents solutions of equation

$$\sum_{j=0}^{n-1}(-1)^j \sum_{2\le s_1<\cdots<s_j\le n} r_{s_1}r_{s_2}\ldots r_{s_j}\phi^{n-1-j}(x) = a\prod_{\varsigma=2}^{n}(1-r_\varsigma), \quad \forall x \in \mathbb{R}.$$

Actually, (20) is consistent with (19) when $n = 2$. For general $n \in \mathbb{N}$, by Lemma 3, if both a and b are finite then equations (21) and (22) can be reduced to the equations

$$\sum_{j=0}^{n-1}(-1)^j \sum_{2\le s_1<\cdots<s_j\le n} r_{s_1}r_{s_2}\ldots r_{s_j}\widetilde{\phi}^{n-1-j}(x) = 0, \quad \forall x \in (-\infty, 0],$$

$$\sum_{j=0}^{n-1}(-1)^j \sum_{2\le s_1<\cdots<s_j\le n} r_{s_1}r_{s_2}\ldots r_{s_j}\widetilde{\psi}^{n-1-j}(x) = 0, \quad \forall x \in [0, +\infty),$$

respectively, where $\widetilde{\phi}(x) := \phi(x+a) - a$ and $\widetilde{\psi}(x) := \psi(x+b) - b$. All continuous solutions $\widetilde{\phi} : (-\infty, 0] \to (-\infty, 0]$ and $\widetilde{\psi} : [0, +\infty) \to [0, +\infty)$ of the reduced equations with the restrictions $\widetilde{\phi}(0) = 0$ and $\widetilde{\psi}(0) = 0$ can be constructed as in Theorem 2 in [16]. Therefore, all continuous solutions f of equation (2) with $c = 0$ can be given by (20), where ϕ and ψ are defined by $\widetilde{\phi}$ and $\widetilde{\psi}$ respectively and satisfy $\phi(a) = a$ and $\psi(b) = b$.

PROOF OF THEOREM 2. Suppose that f is a continuous solution of equation (8) with $c = 0$. By (10) and the fact $r_n > r_j$ $(j = 1, \ldots, n-1)$, we get

$$\lim_{m\to+\infty}\frac{f^{n+m}(x)}{r_n^{m+1}} = \frac{A_{n1}}{\Delta(r_1,\ldots,r_n)}\mathcal{F}_0[r_1,\ldots,r_{n-1}](f^\ell(x))_{\ell\in\mathbb{Z}}, \qquad (23)$$

where we can calculate that $A_{n1} = r_n^{n-1}\Delta(r_1,\ldots,r_{n-1}) \neq 0$. Applying (10) again to the dual equation (9) with $c = 0$ and noting that $0 < r_j^{-1} < 1$ for all $j = 2,\ldots,n$, we obtain

$$\lim_{m\to+\infty} f^{-(n+m)}(x) = C_1\,\mathcal{F}_0[r_2^{-1},\ldots,r_n^{-1}](f^{-\ell}(x))_{\ell\in\mathbb{Z}},$$

where C_1 is a constant. Thus, by Lemma 2 (ii) and (iii),

$$\begin{aligned}
\lim_{m\to+\infty} f^{-(n+m)}(x) &= \frac{(-1)^{n-1}C_1}{r_2\ldots r_n}\mathcal{F}_{-(n-1)}[r_2,\ldots,r_n](f^\ell(x))_{\ell\in\mathbb{Z}} \\
&= C_2\,\mathcal{F}_0[r_2,\ldots,r_n](f^\ell(x))_{\ell\in\mathbb{Z}},
\end{aligned} \tag{24}$$

where $C_2 = (-1)^{n-1}C_1/(r_2\ldots r_n)$.

Now we claim that f is strictly increasing on $\mathbb{R}$. Since f is strictly monotone as mentioned in Lemma 1, for a reduction to absurdity we assume that f is strictly decreasing. It implies that the pointwise limit $\lim_{m\to+\infty} f^{n+m}/r_n^{m+1}$ is both nondecreasing (if $n+m$ is even, then f^{n+m} increases) and nonincreasing (if $n+m$ is odd, then f^{n+m} decreases). So by (23) the function $\mathcal{F}_0[r_1,\ldots,r_{n-1}](f^\ell)_{\ell\in\mathbb{Z}}$ is a constant c_1, i.e.,

$$\mathcal{F}_0[r_1,\ldots,r_{n-1}](f^\ell(x))_{\ell\in\mathbb{Z}} = c_1, \quad \forall x \in \mathbb{R}.$$

Clearly, $c_1 = r_n^m c_1$ for all $m \in \mathbb{Z}$ by Lemma 2 (iii), implying that $c_1 = 0$ since $r_n \neq 1$. Repeating the same procedure we can lower the order by 1 while eliminating an r_j $(j = n-1,\ldots,2)$ each time. Finally we obtain that $\mathcal{F}_0[r_1](f^\ell(x))_{\ell\in\mathbb{Z}} = 0$, i.e., $f(x) = x$, for all $x \in \mathbb{R}$, which makes a contradiction. Hence, f is strictly increasing on $\mathbb{R}$. Moreover, the function $\mathcal{F}_0[r_2,\ldots,r_n](f^\ell)_{\ell\in\mathbb{Z}}$ is monotone by (24).

Furthermore, f has at least one fixed point. In fact, let

$$\widetilde{y} := C_2\,\mathcal{F}_0[r_2,\ldots,r_n](f^\ell(\widetilde{x}))_{\ell\in\mathbb{Z}} \in \mathbb{R}$$

for arbitrary given $\widetilde{x} \in \mathbb{R}$. By (24) and the continuity of f we have

$$\widetilde{y} = \lim_{m\to+\infty} f^{-(n+m)}(\widetilde{x}) = f\big(\lim_{m\to+\infty} f^{-(n+m)-1}(\widetilde{x})\big) = f(\widetilde{y}). \tag{25}$$

Therefore, it is reasonable to let a and b, where $-\infty \leq a \leq b \leq +\infty$, denote the infimum and supremum of the set of fixed points of f respectively. First of all, suppose that $-\infty < a < b < +\infty$. It is obvious that both a and b are also fixed points of the continuous function f. Then f is a self-mapping on each of the intervals $(-\infty,a)$, (a,b) and $(b,+\infty)$ since f is proved to be strictly increasing.

In what follows we discuss equation (8) with $c = 0$ on the interval $(-\infty, a]$ and prove (21) with $\phi := f|_{(-\infty, a]}$. For arbitrary fixed $x_0 \in (-\infty, a)$, by Lemma 2 (iii) we see that

$$\mathcal{F}_m[r_2, \ldots, r_n](\phi^\ell(x_0))_{\ell \in \mathbb{Z}} = \mathcal{F}_0[r_2, \ldots, r_n](\phi^\ell(x_0))_{\ell \in \mathbb{Z}} := c_2, \quad \forall\, m \in \mathbb{Z}, \quad (26)$$

where c_2 is dependent on x_0 but independent of m. On the other hand, since there are no fixed points of f in $(-\infty, a)$, the monotonicity of ϕ implies that either

$$\lim_{m \to \infty} \phi^m(x_0) = -\infty, \quad \text{and} \quad \lim_{m \to \infty} \phi^{-m}(x_0) = a,$$

$$\text{or} \quad \lim_{m \to \infty} \phi^m(x_0) = a, \quad \text{and} \quad \lim_{m \to \infty} \phi^{-m}(x_0) = -\infty.$$

Otherwise, the limit of either $\phi^m(x_0)$ or $\phi^{-m}(x_0)$ in $(-\infty, a)$ is a fixed point by (25). It follows that for arbitrary $x \in (-\infty, a)$ there exists $k \in \mathbb{Z}$ such that $\phi^{k-1}(x_0) \leq x \leq \phi^k(x_0)$ (or $\phi^k(x_0) \leq x \leq \phi^{k-1}(x_0)$). So, we can see that $\mathcal{F}_0[r_2, \ldots, r_n](\phi^\ell(x))_{\ell \in \mathbb{Z}}$ lies between two points of $\mathcal{F}_{k-1}[r_2, \ldots, r_n](\phi^\ell(x_0))_{\ell \in \mathbb{Z}}$ and $\mathcal{F}_k[r_2, \ldots, r_n](\phi^\ell(x_0))_{\ell \in \mathbb{Z}}$ since $\mathcal{F}_0[r_2, \ldots, r_n](\phi^\ell)_{\ell \in \mathbb{Z}}$ is monotone by the last sentence of the second paragraph in this proof. Thus, from (26) we get

$$\mathcal{F}_0[r_2, \ldots, r_n](\phi^\ell(x))_{\ell \in \mathbb{Z}} = c_2, \quad \forall\, x \in (-\infty, a).$$

Furthermore, by the continuity of the function $\mathcal{F}_0[r_2, \ldots, r_n](\phi^\ell)_{\ell \in \mathbb{Z}}$,

$$c_2 = \mathcal{F}_0[r_2, \ldots, r_n](\phi^\ell(a))_{\ell \in \mathbb{Z}} = \mathcal{F}_0[r_2, \ldots, r_n](a)_{\ell \in \mathbb{Z}} = a \prod_{\varsigma=2}^{n} (1 - r_\varsigma).$$

This proves (21). Similarly, we can prove (22).

Finally, on the interval (a, b) the limit in (23) is equal to 0 because $f^{n+m}(x)$ is bounded and $r_n > 1$. Hence

$$\mathcal{F}_0[r_1, \ldots, r_{n-1}](f^\ell(x))_{\ell \in \mathbb{Z}} = 0, \quad \forall\, x \in (a, b). \quad (27)$$

Repeating the same procedure of order reduction to the lower order equation (27), we can reduce the order again and finally obtain that $\mathcal{F}_0[r_1](f^\ell(x))_{\ell \in \mathbb{Z}} = 0$, i.e., $f(x) = x$ for all $x \in (a, b)$. This proves (20).

Similar discussions can be applied to the cases of either a or b is infinite, both a and b are infinite and $a = b$. Conversely, if ϕ is a continuous solution of equation (21) then $\phi((-\infty, a]) \subset (-\infty, a]$. Thus, by Lemma 2, the beginning and (i), setting $k = 0$ and $q = 1$, we have

$$\mathcal{F}_0[r_1, \ldots, r_n](\phi^\ell(x))_{\ell \in \mathbb{Z}} = \mathcal{F}_0[r_2, \ldots, r_n, r_1](\phi^\ell(x))_{\ell \in \mathbb{Z}}$$

$$= \mathcal{F}_1[r_2, \ldots, r_n](\phi^\ell(x))_{\ell \in \mathbb{Z}} - \mathcal{F}_0[r_2, \ldots, r_n](\phi^\ell(x))_{\ell \in \mathbb{Z}}$$

$$= \mathcal{F}_0[r_2,\ldots,r_n](\phi^\ell(\phi(x)))_{\ell\in\mathbb{Z}} - \mathcal{F}_0[r_2,\ldots,r_n](\phi^\ell(x))_{\ell\in\mathbb{Z}}$$

$$= a\prod_{\varsigma=2}^{n}(1-r_\varsigma) - a\prod_{\varsigma=2}^{n}(1-r_\varsigma) = 0, \tag{28}$$

i.e., ϕ satisfies equation (8) with $c=0$ on $(-\infty, a]$. Similarly, the solution ψ of equation (22) also satisfies equation (8) with $c=0$ on $[b,+\infty)$. This completes the proof. $\hspace{1cm}\square$

Remark 1. The case that $0 < r_n < \cdots < r_1 = 1$ can be reduced to the case of Theorem 2 by considering the dual equation.

5. Case of all roots being 1

Theorem 3. *Suppose that all characteristic roots r_j are equal to 1. Then equation (2) with $c \neq 0$ and even n has no continuous solutions.*

PROOF. Consider equation (8) with $c \neq 0$ and even n and let f be its a continuous solution. We first present the general iterate f^m of f in terms of m, which cannot be given by Theorem 1 in [16] because $c \neq 0$ in our case. By Lemma 2 (i), setting $k = m-1$ and $q = 1$, we have

$$c = \mathcal{F}_{m-1}[\overbrace{1,\ldots,1}^{n}](f^\ell)_{\ell\in\mathbb{Z}} = \mathcal{F}_m[\overbrace{1,\ldots,1}^{n-1}](f^\ell)_{\ell\in\mathbb{Z}} - \mathcal{F}_{m-1}[\overbrace{1,\ldots,1}^{n-1}](f^\ell)_{\ell\in\mathbb{Z}}, \quad \forall m\in\mathbb{N},$$

which enables us to prove by induction on m easily that

$$\mathcal{F}_m[\overbrace{1,\ldots,1}^{n-1}](f^\ell)_{\ell\in\mathbb{Z}} = \mathcal{F}_0[\overbrace{1,\ldots,1}^{n-1}](f^\ell)_{\ell\in\mathbb{Z}} + mc, \quad \forall m \in \mathbb{N}. \tag{29}$$

We further claim that

$$\mathcal{F}_m[\overbrace{1,\ldots,1}^{n-j}](f^\ell)_{\ell\in\mathbb{Z}} = \mu_{j,0} + \cdots + \mu_{j,j-1}m^{j-1} + \frac{c}{j!}m^j \tag{30}$$

for all $j = 1,\ldots,n$ and all $m = j, j+1,\ldots$, where all $\mu_{j,i}$'s are functions of x but independent of m. This assertion is true for $j = 1$ by (29). Assume that (30) is true for an integer $j \in \{1,\ldots,n-1\}$ and $m \geq j$. Then, for an arbitrarily fixed integer $m \geq j+1$ and $s = m-1, m-2, \ldots, j$, we have

$$\mathcal{F}_{s+1}[\overbrace{1,\ldots,1}^{n-j-1}](f^\ell)_{\ell\in\mathbb{Z}} - \mathcal{F}_s[\overbrace{1,\ldots,1}^{n-j-1}](f^\ell)_{\ell\in\mathbb{Z}}$$

$$= \mathcal{F}_s[\overbrace{1,\ldots,1}^{n-j}](f^\ell)_{\ell\in\mathbb{Z}} = \mu_{j,0} + \cdots + \mu_{j,j-1}s^{j-1} + \frac{c}{j!}s^j.$$

It follows that for $m \geq j+1$

$$\mathcal{F}_m[\overbrace{1,\ldots,1}^{n-j-1}](f^\ell)_{\ell\in\mathbb{Z}} = \mathcal{F}_j[\overbrace{1,\ldots,1}^{n-j-1}](f^\ell)_{\ell\in\mathbb{Z}} + \sum_{s=j}^{m-1} \mathcal{F}_s[\overbrace{1,\ldots,1}^{n-j}](f^\ell)_{\ell\in\mathbb{Z}}$$

$$= \mathcal{F}_j[\overbrace{1,\ldots,1}^{n-j-1}](f^\ell)_{\ell\in\mathbb{Z}} + \mu_{j,0}(m-j) + \cdots$$

$$+ \mu_{j,j-1}\{(m-1)^{j-1} + (m-2)^{j-1} + \cdots + j^{j-1}\}$$

$$+ \frac{c}{j!}\{(m-1)^j + (m-2)^j + \cdots + j^j\}$$

$$= \mu_{j+1,0} + \cdots + \mu_{j+1,j}m^j + \frac{c}{(j+1)!}m^{j+1},$$

where the known formula of sum of powers ([11])

$$(m-1)^k + (m-2)^k + \cdots + j^k = \frac{1}{k+1}m^{k+1} + a_{k,k}m^k + \cdots + a_{k,0}, \quad \forall k \in \mathbb{N},$$

for some constants $a_{k,i} \in \mathbb{R}$ independent of m is applied and therefore all $\mu_{j+1,i}$'s $(i = 0, \ldots, j)$, determined by those $\mu_{j,i}$'s $(i = 0, \ldots, j-1)$ and constant coefficients $a_{j,i}$'s, are also functions of x but independent of m. Thus the claim (30) is proved by induction. Putting $j = n$ in (30) and considering the dual equation (7), we finally obtain

$$\begin{cases} f^m = \mu_{n,0} + \cdots + \mu_{n,n-1}m^{n-1} + \dfrac{c}{n!}m^n, \\ f^{-m} = \mu_{n,0}^* + \cdots + \mu_{n,n-1}^*m^{n-1} + (-1)^n\dfrac{c}{n!}m^n, \end{cases} \tag{31}$$

where all $\mu_{n,i}^*$'s are functions of x but independent of m.

Since n is even, the right-hand sides of the formulae in (31) both tend to $+\infty$ (resp. $-\infty$) as $m \to +\infty$ for each fixed $x \in \mathbb{R}$ when $c > 0$ (resp. $c < 0$). It follows that

$$\lim_{m\to+\infty} f^m(x) = \lim_{m\to+\infty} f^{-m}(x) = +\infty \quad (\text{resp. } -\infty) \tag{32}$$

for $x \in \mathbb{R}$ when $c > 0$ (resp. $c < 0$). The monotonicity of f implies that f^2 is increasing and either

$$\cdots < f^{-2m}(x_0) < \cdots < f^{-2}(x_0) < x_0 < f^2(x_0) < \cdots < f^{2m}(x_0) < \ldots \tag{33}$$

or

$$\cdots > f^{-2m}(x_0) > \cdots > f^{-2}(x_0) > x_0 > f^2(x_0) > \cdots > f^{2m}(x_0) > \ldots \tag{34}$$

for a fixed $x_0 \in \mathbb{R}$ and $m \in \mathbb{N}$. If $f^{2m}(x_0) \to +\infty$ as $m \to +\infty$, then (32) contradicts to (33), implying that $f^{-2m}(x_0)$ does not tend to $+\infty$ as $m \to +\infty$; if $f^{2m}(x_0) \to -\infty$ as $m \to +\infty$, then (32) contradicts to (34) for the same reason. Hence, equation (8) with $c \neq 0$ has no continuous solutions. □

The case that all characteristic roots are the same was considered early by NABEYA [10] for $n = 2$. Our Theorem 3 obviously generalizes his result to general even n. In contrast to the case of $c = 0$, which was discussed in [8] for $n = 2$ and in [16] for general n, the proof under the case of $c \neq 0$ is quite different because the term $(c/n!)m^n$ in (31), from which we get the contradiction, does not appear in the formula of the general iterate of f when $c = 0$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of equation (2) with $c = 0$ and all characteristic roots being 1. Then, from [16] we can see that $f(x) = x$ if f has fixed points; otherwise, $\mathcal{F}_0[1, \ldots, 1](f^\ell(x))_{\ell \in \mathbb{Z}} = \tau$, where τ is a real constant which is equal to 0 for odd n. Thus, when all characteristic roots are equal to 1, only the case that n is odd and $c \neq 0$ is unsolved.

6. Further discussion

In addition to the general form of (2), which includes the form (3) as considered in [16], this paper is also motivated by the fact that some cases of (3) unsolved in [16] can be reduced to solving a lower order iterative equation of the form (2). For example, as shown in Theorem 2, equation (3) can be reduced to the equation

$$\mathcal{F}_0[r_2, \ldots, r_n](\phi^\ell(x))_{\ell \in \mathbb{Z}} = a \prod_{\varsigma=2}^{n}(1 - r_\varsigma)$$

on the interval $(-\infty, a]$ for a constant $a \in \mathbb{R}$.

As shown in previous sections, both the case (**ID**) and the case (**EC**) are still difficult for $n \geq 3$. Ignoring the difficult cases, we still have some unsolved problems in those discussed cases:

In the case (**$\mathbf{R}_p$**) the order of equation (2) can be lowered as in Section 3 when those characteristic roots $r_1, \ldots, r_p \in \mathbb{R}$ and $r_{p+1}, \ldots, r_s \in \mathbb{C} \backslash \mathbb{R}$ satisfy that

$$|r_1| \leq \cdots \leq |r_p| < |r_{p+1}| < \cdots < |r_s|.$$

The discussion is still difficult when there are two non-conjugate complex roots with the same norm or when a complex root lies between two real ones in norm. In this sense the question raised in Remark 8 in [16] is only answered partly in

Section 3. Additionally, except for Corollary 1, we have no results for equation (8) with $c \neq 0$ when a real characteristic root is equal to 1.

In the case **(K)**, when the smallest characteristic root is equal to 1, although equation (2) with $c = 0$ can be simplified by lowering its order in Section 4, can we do the same for (2) with $c \neq 0$? We also want to know if we can lower the order in the opposite case, i.e., when the largest root is equal to -1. As mentioned in Section 5, when all characteristic roots are equal to the same 1, equation (2) with $c \neq 0$ is unsolved yet for odd n and we want to know if we can lower the order when the n-multiple root is not equal to 1 and n is odd, which was also indicated in Remark 9 of [16].

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