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Quasicompact endomorphisms of Lipschitz algebras of analytic functions

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Abstract. We study power compact and quasicompact endomorphisms of certain subalgebras of analytic Lipschitz algebras. We first give a sufficient condition for a quasicompact endomorphism of these algebras to be power compact. Then in certain cases, we show that this condition is also necessary. Using this, by constructing an example, we show that there exists a quasicompact endomorphism of these algebras which is not power compact. As a final result, we give a necessary condition for the quasicompactness of an endomorphism of these algebras.

1. Introduction

Let A be a Banach function algebra on a compact Hausdorff space X with maximal ideal space M_A. For every $x \in X$, let δ_x be the evaluation map on A defined by $\delta_x(f) = f(x)$ for every $f \in A$. Then the map $\delta : x \to \delta_x$ is a homeomorphism from X onto a (weak*) compact subset $\delta(X)$ of M_A. We will frequently identify X with its image $\delta(X)$ in M_A and use the identification $x \simeq \delta_x$ for every $x \in X$. A Banach function algebra A is called natural when the map δ is onto M_A, i.e., $X = M_A$.

Let A be a natural Banach function algebra on a compact Hausdorff space X. Then every unital endomorphism T of A has the form $Tf = f \circ \varphi$, for some selfmap $\varphi : X \to X$. In this case we say that T is induced by φ. Note that if T is an endomorphism of A induced by the selfmap φ, then T^n is an endomorphism of A induced by the selfmap $\varphi_n : X \to X$ for each $n \in \mathbb{N}$, where φ_n denotes

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the n-th iterate of φ. In this note we consider endomorphisms which are also quasicompact or power compact. For convenience we give the definition of these endomorphisms.

Definition 1.1. Let E be an infinite-dimensional Banach space. We denote by $\mathcal{B}(E)$ and $\mathcal{K}(E)$ the Banach algebra of all bounded operators and compact operators on E, respectively. The essential spectrum $\sigma_e(T)$ of $T \in \mathcal{B}(E)$ is the spectrum of $T+\mathcal{K}(E)$ in the Calkin algebra $\mathcal{B}(E)/\mathcal{K}(E)$. The essential spectral radius $r_e(T)$ of $T \in \mathcal{B}(E)$ is given by the formula $r_e(T) = \lim_{n\to\infty}(\mathrm{dist}(T^n, \mathcal{K}(E)))^{1/n}$. The operator $T \in \mathcal{B}(E)$ is called *Riesz* or *quasicompact* if $r_e(T) = 0$ or $r_e(T) < 1$, respectively. If T^N is compact for some positive integer N, then T is called *power compact.*

Clearly every power compact operator is Riesz. Also every Riesz operator is quasicompact. A question one may ask is when a quasicompact (or Riesz) endomorphism is power compact. Feinstein and Kamowitz in [4, Theorem 2.2] showed that every quasicompact endomorphism of Dales–Davie algebras $D(X, M)$ induced by an analytic selfmap of X is power compact when $D(X, M)$ is a natural Banach function algebra on a connected compact plane set X. They also considered this problem for endomorphisms of certain uniform algebras and the Banach algebra $C^1[0, 1]$ of continuously differentiable functions on $[0, 1]$. In this paper we investigate this question for certain subalgebras of Lipschitz algebras defined as follows.

Definition 1.2. Let (X, d) be a compact metric space and $0 < \alpha \leq 1$. The Lipschitz algebra of order α, $\mathrm{Lip}(X, \alpha)$, is the algebra of all complex-valued functions f on X for which

$$p_\alpha(f) = \sup\left\{\frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in X \text{ and } x \neq y\right\} < \infty.$$

The subalgebra of those functions $f \in \mathrm{Lip}(X, \alpha)$ for which $\frac{|f(x)-f(y)|}{d^\alpha(x,y)} \to 0$ as $d(x, y) \to 0$ is denoted by $\mathrm{lip}(X, \alpha)$.

These Lipschitz algebras were first studied by Sherbert [13], [14]. The algebras $\mathrm{Lip}(X, \alpha)$ for $0 < \alpha \leq 1$ and $\mathrm{lip}(X, \alpha)$ for $0 < \alpha < 1$ are natural Banach function algebras on X under the norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$, where $\|f\|_X = \sup_{x\in X} |f(x)|$.

It was shown in [7] that if φ induces a compact endomorphism of a unital commutative semi-simple Banach algebra A with connected maximal ideal space M_A, then $\bigcap \varphi_n(M_A) = \{a\}$ for some $a \in M_A$. Then in [4, Theorem 1.2], Feinstein and Kamowitz gave a short proof showing that a stronger result holds even for quasicompact endomorphisms, as Theorem 1.1 below. First we give some notations.

Let A be a unital commutative semi-simple Banach algebra. For each a, b in M_A we let $\|a-b\| = \sup\{|\widehat{f}(a) - \widehat{f}(b)| : f \in A,\ \|f\| \leq 1\}$ (the norm of $a-b$ in the dual space A^* of A). Furthermore, we let $B^*(a, \varepsilon) = \{b \in M_A : \|a-b\| < \varepsilon\}$ for $\varepsilon > 0$ and $a \in M_A$.

Theorem 1.1. [4, Theorem 1.2] *Let A be a unital commutative semi-simple Banach algebra with connected maximal ideal space M_A. Suppose that T is a unital quasicompact endomorphism of A induced by the selfmap φ of M_A. Then the following hold:*

(i) *The operators T^n converge in operator norm to a rank-one unital endomorphism S of A, and there exists $a \in M_A$ such that $Sf = \widehat{f}(a)1$ for all $f \in A$. This point a is the unique fixed point of φ.*

(ii) *For each $\varepsilon > 0$, there exists a positive integer N such that $\varphi_N(M_A) \subseteq B^*(a, \varepsilon)$.*

(iii) $\bigcap \varphi_n(M_A) = \{a\}$.

2. Sufficient conditions

Let X be a compact plane set with nonempty interior and $A(X)$ denote the uniform algebra of all continuous complex-valued functions on X which are analytic on $\mathrm{int}X$. Let $0 < \alpha \leq 1$ and $\mathrm{Lip}_A(X, \alpha) = \mathrm{Lip}(X, \alpha) \cap A(X)$. Then $(\mathrm{Lip}_A(X, \alpha), \|.\|_\alpha)$ is a Banach function algebra on X, and its maximal ideal space is X [6]. Therefore every unital endomorphism T of $\mathrm{Lip}_A(X, \alpha)$ has the form $Tf = f \circ \varphi$ for some selfmap $\varphi : X \to X$. Note that in this case $\varphi \in \mathrm{Lip}_A(X, \alpha)$, since $\mathrm{Lip}_A(X, \alpha)$ contains the coordinate function Z. The compact endomorphisms of $\mathrm{Lip}_A(X, \alpha)$ have been studied in [1]. Here we investigate the quasicompact endomorphisms of these algebras.

Let T be a quasicompact endomorphism of $\mathrm{Lip}_A(X, \alpha)$ induced by the selfmap $\varphi : X \to X$. When X is connected, by Theorem 1.1(i), φ has a unique fixed point $z_0 \in X$. In the following theorem we impose a condition on z_0 which implies that T is power compact.

Theorem 2.1. *Let X be a connected compact plane set with nonempty interior and $0 < \alpha \leq 1$. Suppose that T is a quasicompact endomorphism of $\mathrm{Lip}_A(X, \alpha)$ induced by the selfmap $\varphi : X \to X$ with the fixed point z_0. If $z_0 \in \mathrm{int}X$, then T is power compact.*

Proof. Let $B(z_0, \delta) \subseteq \operatorname{int}X$ for some positive number δ. By Theorem 1.1(ii), there exists a positive integer N such that

$$\varphi_n(X) \subseteq B^*\left(z_0, \frac{\delta}{\|Z\|_\alpha}\right), \tag{1}$$

for all $n \geq N$. Since the algebra $\mathrm{Lip}_A(X, \alpha)$ contains the coordinate function Z, one has $|z - w| \leq \|Z\|_\alpha \|z - w\|$ and therefore $B^*(z, r) \subseteq B(z, r\|Z\|_\alpha)$ for each $z \in X$ and $r > 0$. Employing (1), we have $\varphi_n(X) \subseteq B(z_0, \delta)$ for all $n \geq N$. Since T^N is an endomorphism of $\mathrm{Lip}_A(X, \alpha)$ induced by φ_N and $\varphi_N(X) \subseteq \operatorname{int}X$, by [1, Theorem 2.1], it follows that T^N is compact and therefore T is power compact. □

Now we show that the result of Theorem 2.1 also holds true for certain subalgebras of $\mathrm{Lip}_A(X, \alpha)$, namely $\mathrm{lip}_A(X, \alpha) = \mathrm{lip}(X, \alpha) \cap A(X)$ for $0 < \alpha < 1$ and $\mathrm{Lip}_R(X, \alpha)$ for $0 < \alpha \leq 1$, where $\mathrm{Lip}_R(X, \alpha)$ is the closed subalgebra of $\mathrm{Lip}(X, \alpha)$ generated by $R_0(X)$, the algebra of rational functions with poles off X. These are Banach function algebras on X. The maximal ideal space of $\mathrm{lip}_A(X, \alpha)$ and $\mathrm{Lip}_R(X, \alpha)$ coincides with X (for more details about these algebras see [5], [8], [9], [10], [11], and [12]). To prove Theorem 2.1 for these algebras, we need the following result.

Proposition 2.1. *Let A be a natural Banach function algebra on a compact Hausdorff space X. Suppose that the selfmap $\varphi : X \to X$ induces a compact endomorphism of A. If B is a closed subalgebra of A which is closed under the composition of φ, then φ induces a compact endomorphism of B.*

Proof. Let T be a compact endomorphism of A induced by φ. The closedness of B under the composition of φ implies that B is invariant under T. Hence, the compactness of $T : A \to A$ and the completeness of B imply that T is a compact endomorphism of B. □

Corollary 2.1. *Let X be a compact plane set with nonempty interior and T be an endomorphism of $\mathrm{lip}_A(X, \alpha)$ $(0 < \alpha < 1)$ or $\mathrm{Lip}_R(X, \alpha)$ $(0 < \alpha \leq 1)$ induced by the selfmap φ of X. If $\varphi(X) \subseteq \operatorname{int}X$, then T is compact.*

Proof. Let B be any of the algebras $\mathrm{lip}_A(X, \alpha)$ or $\mathrm{Lip}_R(X, \alpha)$ and T be the endomorphism of B induced by φ. Since $Z \in B$, we have $\varphi \in B \subseteq \mathrm{Lip}_A(X, \alpha)$ and since $\varphi(X) \subseteq \operatorname{int}X$, by the Functional Calculus Theorem, $f \circ \varphi \in \mathrm{Lip}_A(X, \alpha)$ for all $f \in \mathrm{Lip}_A(X, \alpha)$. Therefore, $f \mapsto f \circ \varphi$ is an endomorphism of $\mathrm{Lip}_A(X, \alpha)$ which is also compact by [1, Theorem 2.1]. Now Proposition 2.1 implies that T is a compact endomorphism of B. □

Using Corollary 2.1 and the same argument of the proof of Theorem 2.1, we have the following result.

Corollary 2.2. *Let X be a connected compact plane set with nonempty interior. Suppose that T is a quasicompact endomorphism of* $\mathrm{lip}_A(X,\alpha)$ $(0 < \alpha < 1)$ *or* $\mathrm{Lip}_R(X,\alpha)$ $(0 < \alpha \leq 1)$ *induced by the selfmap $\varphi : X \to X$ with the fixed point z_0. If $z_0 \in intX$, then T is power compact.*

We now give a discussion on isolated points of M_A, when A is a closed function subalgebra of $\mathrm{Lip}(X,\alpha)$ $(0 < \alpha \leq 1)$. First, in general, let A be a natural Banach function algebra on a connected compact Hausdorff space X and T be a quasicompact endomorphism of A induced by the selfmap $\varphi : X \to X$ with the fixed point z_0. If z_0 is isolated in the norm topology of $X = M_A$, then by Theorem 1.1(ii) we have $\varphi_N(X) = \{z_0\}$ for some positive integer N. This implies that φ_N is constant on X and therefore T is power compact. So, as [4, Corollary 1.3], we have the following sufficient condition for the power compactness of T.

Proposition 2.2. *Let A be a natural Banach function algebra on a connected compact Hausdorff space X. Let T be a quasicompact endomorphism of A induced by the selfmap φ on X with the fixed point z_0. Then T is power compact if any of the following conditions is satisfied.*

(i) *z_0 is an isolated point of $X = M_A$ in the norm topology.*

(ii) *A has no nonzero point derivation at z_0.*

However, as the following theorem shows when A is a natural closed function subalgebra of $\mathrm{Lip}(X,\alpha)$ $(0 < \alpha \leq 1)$ on a connected compact metric space (X,d), $X = M_A$ cannot have an isolated point in its norm topology. So Proposition 2.2 cannot be useful for this kind of algebra.

Theorem 2.2. *Let (X,d) be a compact metric space and A be a closed function subalgebra of* $\mathrm{Lip}(X,\alpha)$ $(0 < \alpha \leq 1)$. *Then the metric topology, the weak* topology, and the norm topology coincide on X $(\simeq \delta(X))$.*

Proof. Since $\delta(X) \subseteq M_A$ is the homeomorphic image of X under the map $\delta : x \to \delta_x$, the metric topology d and the weak* topology coincide on $\delta(X)$. On the other hand, as we know the weak* topology is weaker than the norm topology on A^* and hence on $\delta(X)$. Therefore, all we must show is that the norm topology is weaker than the metric topology d on $\delta(X)$.

By the definition of p_α for every $x, y \in X$ and $f \in A$, we have

$$|f(x) - f(y)| \leq p_\alpha(f)\, d^\alpha(x,y) \leq \|f\|_\alpha\, d^\alpha(x,y).$$

Consequently, $\|x-y\| = \sup\{|f(x)-f(y)| : f \in A,\ \|f\|_\alpha \le 1\} \le d^\alpha(x,y)$. This implies that the norm topology is weaker than the metric topology d on $\delta(X)$, which completes the proof of the theorem. □

As another application of Theorem 2.2, we have the following corollary about the point derivations of $\mathrm{lip}(X,\alpha)$ $(0<\alpha<1)$. Before stating this corollary we recall that by [2, Theorem 1.6.2] if there exists no nonzero point derivation on a Banach algebra A at a point $a \in M_A$, then a is an isolated point of M_A in the norm topology. Consequently, if (X,d) is a compact metric space and A is a natural closed function subalgebra of $\mathrm{Lip}(X,\alpha)$ $(0<\alpha\le 1)$, then by Theorem 2.2 there exists a nonzero point derivation at each nonisolated point x of (X,d). Sherbert in [14] showed that there exists no nonzero bounded point derivation on $\mathrm{lip}(X,\alpha)$ $(0<\alpha<1)$ at each point $x \in X$. Combining this with the above arguments, we get the following corollary which has already been proved in [3, Theorem 4.4.30(iv)] in a different way.

Corollary 2.3. *Let (X,d) be a compact metric space and $0<\alpha<1$. If x is not an isolated point of (X,d), then there exists an unbounded point derivation on $\mathrm{lip}(X,\alpha)$ at x.*

3. Necessary conditions

In this section we first consider the converse of Theorem 2.1, i.e., for a power compact endomorphism T of $\mathrm{Lip}_A(X,\alpha)$ induced by φ, does the fixed point of φ belong to $\mathrm{int}X$? For the case $\alpha=1$ and certain plane sets X we show that the answer of this question is positive. The type of plane sets which we shall consider are introduced as follows.

Definition 3.1. A plane set X has an internal circular tangent at $c \in \partial X$ if there exists an open disc Δ contained in X with $\overline{\Delta} \cap X = \{c\}$. A plane set X is *strongly accessible from the interior* if it has an internal circular tangent at each point of its boundary.

We say that a compact plane set X has *peak boundary with respect to* $B \subseteq C(X)$ if for each $c \in \partial X$ there exists a nonconstant function $h \in B$ such that $\|h\|_X = h(c) = 1$.

For example, if X is a compact plane set such that $\mathbb{C}\setminus X$ is strongly accessible from the interior, then X has peak boundary with respect to $R_0(X)$ and hence with respect to every subset of $C(X)$ which contains $R_0(X)$.

In the next theorem we consider the converse problem for $\mathrm{Lip}_A(X,1)$ and those compact plane sets X which are strongly accessible from the interior and have peak boundary with respect to $\mathrm{Lip}_A(X,1)$. Such sets include the closed unit disc $\overline{\mathbb{D}}$ and$\overline{B}(z_0,r)\setminus\cup_{k=1}^{n}B(z_k,r_k)$ where closed discs $\overline{B}(z_k,r_k)$ are mutually disjoint in $B(z_0,r)=\{z\in\mathbb{C}:|z-z_0|<r\}$.

Theorem 3.1. *Let Ω be a bounded domain in the plane and $X=\overline{\Omega}$ be strongly accessible from the interior and have peak boundary with respect to* $\mathrm{Lip}_A(X,1)$. *Let T be a nonzero power compact endomorphism of* $\mathrm{Lip}_A(X,1)$ *induced by the nonconstant selfmap $\varphi: X\to X$. If z_0 is the fixed point of φ, then $z_0\in \mathrm{int}X$.*

Proof. Let T be power compact, that is T^N be compact for a positive integer N. By the hypothesis, φ is nonconstant and Ω is a domain, therefore φ_N is nonconstant. Then $\varphi_N(X)\subseteq \mathrm{int}X$, by [1, Theorem 2.5]. Since z_0 is the fixed point of φ, we have $\varphi_N(z_0)=z_0$ and therefore $z_0\in\mathrm{int}X$. □

Note that a slight modification of [1, Theorem 2.5] shows that Theorem 3.1 also holds true for $\mathrm{Lip}_R(X,1)$.

In the following example for $0<\alpha\le 1$ we construct a quasicompact endomorphism of $\mathrm{Lip}_A(\overline{\mathbb{D}},\alpha)$ induced by a selfmap φ with the boundary fixed point $z_0=1$. Therefore in the case of $\alpha=1$, using Theorem 3.1, we have a quasicompact endomorphism of $\mathrm{Lip}_A(\overline{\mathbb{D}},1)$ which is not power compact.

Example 3.1. Let $c>1$. Consider the selfmap $\varphi:\overline{\mathbb{D}}\longrightarrow\overline{\mathbb{D}}$ by $\varphi(z)=\frac{z+(c-1)}{c}$ for every $z\in\overline{\mathbb{D}}$. In fact, φ takes the closed unit disc $\overline{\mathbb{D}}$ onto the closed disc with radius $\frac{1}{c}$ centered at $1-\frac{1}{c}$ and $\varphi(1)=1$. Let $n\in\mathbb{N}$, then by a simple calculation one can show that $\varphi_n(z)=\frac{z+(c^n-1)}{c^n}$, taking the closed unit disc onto the closed disc with radius $\frac{1}{c^n}$ centered at $1-\frac{1}{c^n}$.

For each $0<\alpha\le 1$ and $c>1$, let T_c be the endomorphism of $\mathrm{Lip}_A(\overline{\mathbb{D}},\alpha)$ induced by φ. We show that $r_e(T_c)\le\frac{1}{c^\alpha}$ and therefore T_c is a quasicompact endomorphism of $\mathrm{Lip}_A(\overline{\mathbb{D}},\alpha)$. To this end, let $Lf=f(1)\cdot 1$ for every $f\in\mathrm{Lip}_A(\overline{\mathbb{D}},\alpha)$, then L is a compact (rank-one) endomorphism of $\mathrm{Lip}_A(\overline{\mathbb{D}},\alpha)$ and for each $f\in\mathrm{Lip}_A(\overline{\mathbb{D}},\alpha)$ we have

$$|f(\varphi_n(z))-f(1)|\le p_\alpha(f)|\varphi_n(z)-1|^\alpha\le\|f\|_\alpha\left(\frac{2}{c^n}\right)^\alpha,$$

for all $z\in\overline{\mathbb{D}}$. This yields

$$\|T_c^n f-Lf\|_{\overline{\mathbb{D}}}\le\|f\|_\alpha\left(\frac{2}{c^n}\right)^\alpha. \tag{2}$$

On the other hand, for all $z, w \in \overline{\mathbb{D}}$ with $z \neq w$ we have

$$\frac{|(f(\varphi_n(z)) - f(1)) - (f(\varphi_n(w)) - f(1))|}{|z-w|^\alpha} \leq p_\alpha(f)\left(\frac{1}{c^n}\right)^\alpha \leq \|f\|_\alpha \left(\frac{1}{c^n}\right)^\alpha.$$

Therefore,

$$p_\alpha(T_c^n f - Lf) \leq \|f\|_\alpha \left(\frac{1}{c^n}\right)^\alpha. \tag{3}$$

Adding (2) and (3), one has

$$\|T_c^n f - Lf\|_\alpha \leq \|f\|_\alpha \left(\left(\frac{2}{c^n}\right)^\alpha + \left(\frac{1}{c^n}\right)^\alpha\right) \leq \|f\|_\alpha \frac{3}{(c^n)^\alpha}.$$

Therefore $\|T_c^n - L\| \leq \frac{3}{(c^n)^\alpha}$ which implies that $\operatorname{dist}(T_c^n, \mathcal{K})^{\frac{1}{n}} \leq \frac{3^{\frac{1}{n}}}{c^\alpha}$ where $\mathcal{K} = \mathcal{K}(\mathrm{Lip}_A(\overline{\mathbb{D}}, \alpha))$. Consequently we have $r_e(T_c) \leq \frac{1}{c^\alpha}$.

Note that the unique fixed point of φ, $z_0 = 1$, does not belong to $\mathbb{D}$. Therefore, Theorem 3.1 implies that the quasicompact endomorphisms T_c of $\mathrm{Lip}_A(\overline{\mathbb{D}}, 1)$ are not power compact.

Example 3.1 shows that in the implications

compact $\Rightarrow$ power compact $\Rightarrow$ quasicompact,

the converse of the second implication is not true for the endomorphisms of $\mathrm{Lip}_A(\overline{\mathbb{D}}, 1)$. By giving the following example, we show that neither the converse of the first implication is true.

Example 3.2. Consider the selfmap $\varphi : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ by $\varphi(z) = \frac{1-z^2}{2}$ for every $z \in \overline{\mathbb{D}}$. Let $0 < \alpha \leq 1$ and $f \in \mathrm{Lip}_A(\overline{\mathbb{D}}, \alpha)$, then

$$\begin{aligned}\frac{|f\circ\varphi(z) - f\circ\varphi(w)|}{|z-w|^\alpha} &= \frac{|f(\varphi(z)) - f(\varphi(w))|}{|\varphi(z)-\varphi(w)|^\alpha} \cdot \left|\frac{\varphi(z)-\varphi(w)}{z-w}\right|^\alpha \\ &\leq p_\alpha(f)\left(\frac{|z+w|}{2}\right)^\alpha \leq p_\alpha(f).\end{aligned}$$

Consequently, $f \circ \varphi \in \mathrm{Lip}_A(\overline{\mathbb{D}}, \alpha)$ and hence φ induces an endomorphism T of $\mathrm{Lip}_A(\overline{\mathbb{D}}, \alpha)$. For every $z \in \overline{\mathbb{D}}$ we have

$$|\varphi_2(z)| = \left|\frac{3 - z^4 + 2z^2}{8}\right| \leq \frac{3}{4},$$

so $\varphi_2(\overline{\mathbb{D}}) \subseteq \mathbb{D}$. Therefore, by [1, Theorem 2.1], T_2 is a compact endomorphism of $\mathrm{Lip}_A(\overline{\mathbb{D}}, \alpha)$ $(0 < \alpha \leq 1)$. On the other hand, since $\varphi(i) = 1$, it follows from [1, Theorem 2.5] that T is not compact when $\alpha = 1$.

We conclude by giving a necessary condition for an endomorphism T of $\mathrm{Lip}_A(X,1)$ or $\mathrm{Lip}_R(X,1)$ to be quasicompact, when the inducing selfmap φ is differentiable at its fixed point.

Definition 3.2. Let X be a perfect compact plane set. A complex-valued function f on X is called differentiable at a point $z_0 \in X$ if

$$f'(z_0) = \lim_{\substack{z\to z_0\\ z\in X}} \frac{f(z)-f(z_0)}{z-z_0}$$

exists.

Theorem 3.2. *Let X be a connected compact plane set. Let A be a natural closed function subalgebra of $\mathrm{Lip}(X,1)$ containing the coordinate function Z. Suppose that T is a quasicompact endomorphism of A induced by the selfmap $\varphi : X \to X$ with the fixed point z_0. If φ is differentiable at z_0, then $|\varphi'(z_0)| < 1$.*

Proof. By the continuity of φ, differentiability of φ at its fixed point z_0 implies that φ_n is also differentiable at z_0 and $\varphi_n'(z_0) = (\varphi'(z_0))^n$ for all $n \in \mathbb{N}$. Since T is quasicompact, Theorem 1.1(i) yields $\|T^n f - f(z_0)\cdot 1\|_1 \to 0$ as $n \to \infty$ for every $f \in A$. Considering the coordinate function Z, we get

$$\|\varphi_n - z_0\cdot 1\|_1 \to 0 \quad \text{as } n\to\infty. \tag{4}$$

On the other hand,

$$\sup_{\substack{z,w\in X\\ z\neq w}} \left|\frac{\varphi_n(z)-\varphi_n(w)}{z-w}\right| = p_1(\varphi_n) = p_1(\varphi_n - z_0\cdot 1) \le \|\varphi_n - z_0\cdot 1\|_1.$$

So by (4) we get

$$\sup_{\substack{z,w\in X\\ z\neq w}} \left|\frac{\varphi_n(z)-\varphi_n(w)}{z-w}\right| \to 0 \quad \text{as } n\to\infty.$$

In particular,

$$\sup_{\substack{z\in X\\ z\neq z_0}} \left|\frac{\varphi_n(z)-\varphi_n(z_0)}{z-z_0}\right| < 1,$$

for some n. Hence

$$|\varphi'(z_0)|^n = |\varphi_n'(z_0)| = \lim_{\substack{z\to z_0\\ z\in X}} \left|\frac{\varphi_n(z)-\varphi_n(z_0)}{z-z_0}\right| < 1.$$

Therefore, $|\varphi'(z_0)| < 1$. $\square$

It follows from the above proof that we can state the result of Theorem 3.2 without the assumption of differentiability of φ as follows. *If T is a quasicompact* *endomorphism of A induced by the selfmap φ of X then $p_1(\varphi_n) \to 0$ as $n \to \infty$ and in particular, $p_1(\varphi_N) < 1$ for some positive integer N.*

Corollary 3.1. *Let X be a connected compact plane set. Suppose that T is a quasicompact endomorphism of $\mathrm{Lip}_A(X, 1)$ or $\mathrm{Lip}_R(X, 1)$ induced by the selfmap $\varphi : X \to X$ with the fixed point z_0. If φ is differentiable at z_0, then $|\varphi'(z_0)| < 1$.*

Corollary 3.2. *Let X be a connected compact plane set with nonempty interior. Suppose that T is a quasicompact endomorphism of $\mathrm{Lip}_A(X, 1)$ or $\mathrm{Lip}_R(X, 1)$ induced by the selfmap $\varphi : X \to X$. If φ has an interior fixed point z_0, then $|\varphi'(z_0)| < 1$.*

The following is an immediate consequence of Theorem 3.1 and Corollary 3.2.

Corollary 3.3. *Let Ω be a bounded domain in the plane and $X = \overline{\Omega}$ be strongly accessible from the interior and have peak boundary with respect to $\mathrm{Lip}_A(X, 1)$ $(\mathrm{Lip}_R(X, 1))$. Let T be a nonzero power compact endomorphism of $\mathrm{Lip}_A(X, 1)$ $(\mathrm{Lip}_R(X, 1))$ induced by the nonconstant selfmap $\varphi : X \to X$ with the fixed point z_0. Then $|\varphi'(z_0)| < 1$.*

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References

[1] F. Behrouzi and H. Mahyar, Compact endomorphisms of certain analytic Lipschitz algebras, *Bull. Belg. Math. Soc. Simon Stevin* **12**, no. 2 (2005), 301–312.

[2] A. Browder, Introduction to Function Algebras, *W. A. Benjamin, Inc.*, *New York*, 1969.

[3] H. G. Dales, Banach Algebras and Automatic Continuity, London Mathematical Society Monographs, New Series, Volume 24, *The Clarendon Press*, *Oxford*, 2000.

[4] J. F. Feinstein and H. Kamowitz, Quasicompact and Riesz endomorphisms of Banach algebras, *J. Funct. Anal.* **225**, no. 2 (2005), 427–438.

[5] T. G. Honary and H. Mahyar, Approximation in Lipschitz algebras, *Quaest. Math.* **23**, no. 1 (2000), 13–19.

[6] K. Jarosz, $\mathrm{Lip}_{Hol}(X, \alpha)$, *Proc. Amer. Math. Soc.* **125**, no. 10 (1997), 3129–3130.

[7] H. Kamowitz, Compact endomorphisms of Banach algebras, *Pacific J. Math.* **89**, no. 2 (1980), 313–325.

[8] H. Mahyar, Approximation in Lipschitz algebras and their maximal ideal spaces, PhD Thesis, *Tarbiat Moallem University*, *Tehran, Iran*, 1994.

[9] H. Mahyar, The maximal ideal space of $\mathrm{lip}_A(X, \alpha)$, *Proc. Amer. Math. Soc.* **122**, no. 1 (1994), 175–181.

[10] A. G. O'Farrell, Annihilators of rational modules, *J. Funct. Anal.* **19** (1975), 373–389.

[11] A. G. O'Farrell, Hausdorff content and rational approximation in fractional Lipschitz norms, *Trans. Amer. Math. Soc.* **228** (1977), 187–206.

[12] A. G. O'Farrell, Rational approximation in Lipschitz norms-I, *Proc. Roy. Irish Acad.* **77 A** (1977), 113–115.

[13] D. R. Sherbert, Banach algebras of Lipschitz functions, *Pacific J. Math.* **13** (1963), 1387–1399.

[14] D. R. Sherbert, The structure of ideals and point derivations in Banach algebras of Lipschitz functions, *Trans. Amer. Math. Soc.* **111** (1964), 240–272.

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