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A new Hilbert-type operator and applications

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Abstract. By using the way of weight coefficient and the theory of operators, we define a new Hilbert-type operator with the non-decreasing homogeneous kernel and obtain its norm. As applications, an extended theorem on Hilbert-type inequalities with the homogeneous kernel of $-\lambda-$degree is established, and some particular cases are considered.

1. Introduction

In 1908, H. Weyl published the well known Hilbert's inequality as: If $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are real sequences, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then (cf. [1])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible.

In 1925, G. H. Hardy gave a best extension of (1) by introducing one pair of conjugate exponents (p,q) $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ as (cf. [2]): If $p>1$, $a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{2}$$

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We named of (2) Hardy–Hilbert's inequality. In 1934, Hardy et al. [3] gave a basic theorem with the general kernel as follows (see [3], Theorem 318):

Theorem A. *Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1, k(x, y)$ is a homogeneous function of $-1-$degree, and $k = \int_0^\infty k(u, 1)u^{-1/p}du$ is a positive number. If both $k(u, 1)u^{-1/p}$ and $k(1, u)u^{-1/q}$ are strictly decreasing functions for $u > 0$, $a_n, b_n \geq 0$, $0 < \|a\|_p = (\sum_{n=1}^{\infty} a_n^p)^{\frac{1}{p}} < \infty, 0 < \|b\|_q = (\sum_{n=1}^{\infty} b_n^q)^{\frac{1}{q}} < \infty$, then we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m, n)a_m b_n < k\|a\|_p\|b\|_q; \tag{3}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} k(m, n)a_m\right)^p < k^p\|a\|_p^p, \tag{4}$$

where the constant factors k and k^p are the best possible.

Note. In particular, we find some classical Hilbert-type inequalities as:

(i) For $k(x, y) = \frac{1}{x+y}$, since $k = \pi/\sin(\frac{\pi}{p})$, (3) reduces (2);

(ii) for $k(x, y) = \frac{1}{\max\{x,y\}}, \frac{\ln(x/y)}{x-y}$, (3) reduces to (see [3], Theorem 341, Theorem 342)

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m, n\}} < pq\left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}; \tag{5}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m b_n}{m-n} < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^2\left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{6}$$

Hardy also gave a multiple extension of (3) (see [3], Theorem 322).

In 2001, Yang [4] gave an extension of (1) as: For $0 < \lambda \leq 4$,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2\sum_{n=1}^{\infty} n^{1-\lambda}b_n^2\right)^{\frac{1}{2}}, \tag{7}$$

where the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ is the best possible ($B(u, v)$ is the Beta function). And Yang [5] also gave an extension of (2) as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{\pi}{\lambda\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)}b_n^q\right\}^{\frac{1}{q}}, \tag{8}$$

where the constant factor $\frac{\pi}{\lambda \sin(\frac{\pi}{p})}$ $(0 < \lambda \le 2)$ is the best possible.

In 2004, YANG [6] published the dual form of (2) as follows

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{n=1}^{\infty} n^{p-2}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{q-2}b_n^q\right)^{\frac{1}{q}}, \tag{9}$$

where the constant factor $\pi/\sin\left(\frac{\pi}{p}\right)$ is the best possible. For $p = q = 2$, both (9) and (2) reduce to (1). It means that there are more than two different best extensions of (1). For united expressing (2) and (9), in 2005, YANG [7] gave an extension of (7)–(9) with two pairs of conjugate exponents (p, q), (r, s) $(p, r > 1)$ and two parameters $\alpha, \lambda > 0$ $(\alpha\lambda \le \min\{r, s\})$ as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha + n^\alpha)^\lambda} < k_{\alpha\lambda}(r)\left\{\sum_{n=1}^{\infty} n^{p(1-\frac{\alpha\lambda}{r})-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q(1-\frac{\alpha\lambda}{s})-1}b_n^q\right\}^{\frac{1}{q}}, \tag{10}$$

where the constant factor $k_{\alpha\lambda}(r) = \frac{1}{\alpha}B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)$ is the best possible. Setting some particular parameters in (10), it reduces to (2) and (7)–(9). T. K. POGANY [8] also considered a best extension of (2) with the general kernel as $\frac{1}{(\lambda_m+\rho_n)^\mu}$ $(\mu, \lambda_m, \rho_n > 0)$.

In 2006–2008, some authors also considered the operator expressing of (3)–(4). Suppose that $k(x, y)(\ge 0)$ is a symmetric function with $k(y, x) = k(x, y)$, and $k_0(p) := \int_0^\infty k(x, y)\left(\frac{x}{y}\right)^{\frac{1}{r}} dy (r = p, q; x > 0)$ is a positive number independent of x. Define an operator $T : l^r \to l^r$ $(r = p, q)$ as: for $a_m \ge 0, a = \{a_m\}_{m=1}^\infty \in l^p$, there exists only $Ta = c = \{c_n\}_{n=1}^\infty \in l^p$, satisfying

$$(Ta)(n) = c_n := \sum_{m=1}^{\infty} k(m, n)a_m \quad (n \in \mathbb{N}). \tag{11}$$

Then we may define the formal inner product of Ta and b as

$$(Ta, b) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m, n)a_m b_n. \tag{12}$$

In 2007, YANG [9] proved that if for $\varepsilon \ge 0$ small enough, $k(x, y)\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{r}}$ is strictly decreasing for $y > 0$, the integral $\int_0^\infty k(x, y)\left(\frac{x}{y}\right)^{\frac{1+\varepsilon}{r}} dy = k_\varepsilon(p)$ is also a positive number independent of $x > 0$, $k_\varepsilon(p) = k_0(p) + o(1)(\varepsilon \to 0^+)$, and

$$\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\int_0^1 k(m, t)\left(\frac{m}{t}\right)^{\frac{1+\varepsilon}{r}} dt = O(1)(\varepsilon \to 0^+; r = p, q), \tag{13}$$

then $\|T\|_p = k_0(p)$; in this case, if $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p$, $\|b\|_q > 0$, then we have two equivalent inequalities as:

$$(Ta, b) < \|T\|_p \|a\|_p \|b\|_q; \|Ta\|_p < \|T\|_p \|a\|_p, \tag{14}$$

where the constant $\|T\|_p$ is the best possible. In particular, for $k(x, y)$ being $-1-$ degree homogeneous, inequalities (14) reduce to (3)–(4) (in the symmetric kernel, cf. [10]). Yang et al. [11] also considered (14) in l^2. On the integral analogues of (14), Árpád Bényi et al. [12], [13], [14], [15] and [16] gave some new results.

In this paper, by using the way of weight coefficient and the theory of operators as [9], we define a new Hilbert-type operator with the non-decreasing homogeneous kernel and obtain its norm. As applications, an extended theorem on Hilbert-type inequality with the non-decreasing homogeneous kernel of $-\lambda-$degree is established, and some particular cases are considered.

2. A new Hilbert-type operator and its norm

If $k_\lambda(x, y)$ is a measurable function, satisfying for $\lambda, u, x, y > 0$, $k_\lambda(ux, uy) = u^{-\lambda} k_\lambda(x, y)$, then we call $k_\lambda(x, y)$ the homogeneous function of -λ-degree.

Lemma 1. *If $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\lambda > 0, k_\lambda(x, y)(\geq 0)$ is a homogeneous function of $-\lambda$-degree, and $k_\lambda(r) := \int_0^\infty k(u, 1) u^{\frac{\lambda}{r}-1} du$ is a positive number, then,* (i) $\int_0^\infty k(1, u) u^{\frac{\lambda}{s}-1} du = k_\lambda(r)$; (ii) *for $x, y \in (0, \infty)$, setting*

$$\omega_\lambda(r, y) := \int_0^\infty k_\lambda(x, y) \frac{y^{\frac{\lambda}{s}}}{x^{1-\frac{\lambda}{r}}} dx, \quad \varpi_\lambda(s, x) := \int_0^\infty k_\lambda(x, y) \frac{x^{\frac{\lambda}{r}}}{y^{1-\frac{\lambda}{s}}} dy, \tag{15}$$

then we have $\omega_\lambda(r, y) = \varpi_\lambda(s, x) = k_\lambda(r)$.

Proof. (i) Setting $v = \frac{1}{u}$, by the assumption, we obtain $\int_0^\infty k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du = \int_0^\infty k_\lambda(v, 1) v^{\frac{\lambda}{r}-1} dv = k_\lambda(r)$.

(ii) Setting $x = yu$ in the integrals $\omega_\lambda(r, y)$ and $\varpi_\lambda(s, x)$, in view of (i), we still find that $\omega_\lambda(r, y) = \varpi_\lambda(s, x) = k_\lambda(r)$. The lemma is proved.

For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we set $\phi(x) = x^{p(1-\frac{\lambda}{r})-1}$, $\psi(x) = x^{q(1-\frac{\lambda}{s})-1}$ and $\psi^{p-1}(x) = x^{\frac{p\lambda}{s}-1}$, $x \in (0, \infty)$. Define the space of real sequences as: $l_\phi^p := \{a = \{a_n\}_{n=1}^\infty; \|a\|_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\}$. We may still define l_ψ^q and $l_{\psi^{1-p}}^p$. Define the weight coefficient $W_\lambda(r, n)$ and $\widetilde{W}_\lambda(s, m)$ $(m, n \in \mathbb{N})$ as

$$W_\lambda(r, n) := \sum_{m=1}^\infty k_\lambda(m, n) \frac{n^{\frac{\lambda}{s}}}{m^{1-\frac{\lambda}{r}}}, \widetilde{W}_\lambda(s, m) := \sum_{n=1}^\infty k_\lambda(m, n) \frac{m^{\frac{\lambda}{r}}}{n^{1-\frac{\lambda}{s}}}. \tag{16}$$

□

Lemma 2. *As the assumption of Lemma 1, for* $a_m \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l_{\phi}^p$, *setting* $c_n = \sum_{m=1}^{\infty} k_\lambda(m,n)a_m$, *if*

$$W_\lambda(r,n) < k_\lambda(r), \widetilde{W}_\lambda(s,m) < k_\lambda(r) \quad (m,n \in \mathbb{N}), \tag{17}$$

then $c = \{c_n\}_{n=1}^{\infty} \in l_{\psi^{1-p}}^p$.

Proof. By Hölder's inequality [17] and (16)–(17), we obtain

$$\begin{aligned}
c_n^p &= \left\{ \sum_{m=1}^{\infty} k_\lambda(m,n) \left[\frac{m^{(1-\frac{\lambda}{r})/q}}{n^{(1-\frac{\lambda}{s})/p}} a_m \right] \left[\frac{n^{(1-\frac{\lambda}{s})/p}}{m^{(1-\frac{\lambda}{r})/q}} \right] \right\}^p \\
&\leq \left[\sum_{m=1}^{\infty} k_\lambda(m,n) \frac{m^{(1-\frac{\lambda}{r})p/q}}{n^{1-\frac{\lambda}{s}}} a_m^p \right] \left[\sum_{m=1}^{\infty} k_\lambda(m,n) \frac{n^{(1-\frac{\lambda}{s})q/p}}{m^{1-\frac{\lambda}{r}}} \right]^{p-1} \\
&= \left[\sum_{m=1}^{\infty} k_\lambda(m,n) \frac{m^{(1-\frac{\lambda}{r})p/q}}{n^{1-\frac{\lambda}{s}}} a_m^p \right] \left[n^{q(1-\frac{\lambda}{s})-1} W_\lambda(r,n) \right]^{p-1} \\
&\leq k_\lambda^{p-1}(r) n^{1-\frac{p\lambda}{s}} \sum_{m=1}^{\infty} k_\lambda(m,n) \frac{m^{(1-\frac{\lambda}{r})p/q}}{n^{1-\frac{\lambda}{s}}} a_m^p;
\end{aligned}$$

$$\begin{aligned}
\|c\|_{p,\psi^{1-p}} &= \left\{ \sum_{n=1}^{\infty} n^{1-\frac{p\lambda}{s}} c_n^p \right\}^{\frac{1}{p}} = \left\{ \sum_{n=1}^{\infty} n^{\frac{p\lambda}{s}-1} \left[\sum_{m=1}^{\infty} k_\lambda(m,n) a_m \right]^p \right\}^{\frac{1}{p}} \\
&\leq k_\lambda^{\frac{1}{q}}(r) \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k_\lambda(m,n) \frac{m^{(1-\frac{\lambda}{r})p/q}}{n^{1-\frac{\lambda}{s}}} a_m^p \right\}^{\frac{1}{p}} \\
&= k_\lambda^{\frac{1}{q}}(r) \left\{ \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} k_\lambda(m,n) \frac{m^{\frac{\lambda}{r}}}{n^{1-\frac{\lambda}{s}}} \right] m^{p(1-\frac{\lambda}{r})-1} a_m^p \right\}^{\frac{1}{p}} \\
&= k_\lambda^{\frac{1}{q}}(r) \left\{ \sum_{m=1}^{\infty} \widetilde{W}_\lambda(s,m) m^{p(1-\frac{\lambda}{r})-1} a_m^p \right\}^{\frac{1}{p}} < k_\lambda(r) \|a\|_{p,\phi}.
\end{aligned} \tag{18}$$

Therefore $c = \{c_n\}_{n=1}^{\infty} \in l_{\psi^{1-p}}^p$. The lemma is proved.

For $a_m \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l_\phi^p$, define a Hilbert-type operator $T : l_\phi^p \to l_{\psi^{1-p}}^p$ as: $Ta = c$, satisfying $c = \{c_n\}_{n=1}^{\infty}$,

$$(Ta)(n) := c_n = \sum_{m=1}^{\infty} k_\lambda(m,n) a_m \quad (n \in \mathbb{N}). \tag{19}$$

In view of Lemma 2, $c \in l_{\psi^{1-p}}^p$ and the operator T exists. If there exists $M > 0$, such that for any $a \in l_\phi^p$, $\|Ta\|_{p,\psi^{1-p}} \leq M\|a\|_{p,\phi}$, then
$\|T\| = \sup_{\|a\|_{p,\phi}=1} \|Ta\|_{p,\psi^{1-p}} \leq M$. Hence by (18), we find $\|T\| \leq k_\lambda(r)$ and T is bounded.

Theorem 1. *As the assumption of Lemma 2, if for any $r > 1$,*

$$k_\lambda(r) - O\left(\frac{1}{m^{\lambda(r)}}\right) \le \widetilde{W}_\lambda(s,m)\,(\lambda(r) > 0; \quad m \in \mathbb{N}), \tag{20}$$

then we have $\|T\| = k_\lambda(r)$.

Proof. For $a_m, b_n \ge 0$, $a = \{a_m\}_{m=1}^\infty \in l_\phi^p$, $b = \{b_n\}_{n=1}^\infty \in l_\psi^q$, $\|a\|_{p,\phi} > 0$, $\|b\|_{q,\psi} > 0$, by Hölder's inequality [**?**], we find

$$\begin{aligned}(Ta,b) &= \sum_{n=1}^\infty \left[n^{\frac{\lambda}{s}-\frac{1}{p}} \sum_{m=1}^\infty k_\lambda(m,n) a_m\right]\left[n^{\frac{-\lambda}{s}+\frac{1}{p}} b_n\right] \\ &\le \left\{\sum_{n=1}^\infty n^{\frac{p\lambda}{s}-1}\Big[\sum_{m=1}^\infty k_\lambda(m,n)a_m\Big]^p\right\}^{\frac{1}{p}} \|b\|_{q,\psi}.\end{aligned} \tag{21}$$

Then by (18), we have

$$(Ta,b) < k_\lambda(r)\|a\|_{p,\phi}\|b\|_{q,\psi}. \tag{22}$$

For $0 < \varepsilon < \min\left\{\frac{p\lambda}{r}, \frac{q\lambda}{s}\right\}$, setting $\widetilde{a} = \{\widetilde{a}_n\}_{n=1}^\infty$, $\widetilde{b} = \{\widetilde{b}_n\}_{n=1}^\infty$ as $\widetilde{a}_n = n^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}$, $\widetilde{b}_n = n^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}$, for $n \in \mathbb{N}$, if there exists a constant $0 < k \le k_\lambda(r)$, such that (22) is still valid when we replace $k_\lambda(r)$ by k, then,

$$(T\widetilde{a},\widetilde{b}) = \sum_{n=1}^\infty \sum_{m=1}^\infty k(m,n)\widetilde{a}_m\widetilde{b}_n < k\|\widetilde{a}\|_{p,\phi}\|\widetilde{b}\|_{q,\psi} = k\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}}; \tag{23}$$

$$(T\widetilde{a},\widetilde{b}) = \sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}} \sum_{n=1}^\infty k_\lambda(m,n) m^{\lambda(\frac{1}{r}+\frac{\varepsilon}{q\lambda})} n^{\lambda(\frac{1}{s}-\frac{\varepsilon}{q\lambda})-1}. \tag{24}$$

Setting (r', s') as $\frac{1}{r'} = \frac{1}{r} + \frac{\varepsilon}{q\lambda} > 0$, $\frac{1}{s'} = \frac{1}{s} - \frac{\varepsilon}{q\lambda} > 0$, then $\frac{1}{r'} + \frac{1}{s'} = 1$ with $r' > 1$. Hence by (20),

$$\sum_{n=1}^\infty k_\lambda(m,n) m^{\lambda(\frac{1}{r}+\frac{\varepsilon}{q\lambda})} n^{\lambda(\frac{1}{s}-\frac{\varepsilon}{q\lambda})-1} = \widetilde{W}_\lambda(s',m) \ge k_\lambda(r') - O\left(\frac{1}{m^{\lambda(r')}}\right),$$

and then by (23) and (24), it follows

$$\begin{aligned}k\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} &> (T\widetilde{a},\widetilde{b}) = \sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}}\widetilde{W}_\lambda(s',m) \ge \sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}}\left[k_\lambda(r') - O\left(\frac{1}{m^{\lambda(r')}}\right)\right] \\ &= \sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}} k_\lambda(r') - \sum_{m=1}^\infty \frac{1}{m^{1+\varepsilon}} O\left(\frac{1}{m^{\lambda(r')}}\right)\end{aligned}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \left[k_\lambda(r') - \left(\sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \right)^{-1} \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} O\left(\frac{1}{m^{\lambda(r')}} \right) \right];$$

$$k > k_\lambda(r') - \left(\sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \right)^{-1} \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} O\left(\frac{1}{m^{\lambda(r')}} \right) \; (\lambda(r') > 0).$$

Since by Watou's Lemma,

$$\underline{\lim}_{\varepsilon \to 0^+} k_\lambda(r') \geq \int_0^\infty \lim_{\varepsilon \to 0^+} k(u,1) u^{\frac{\lambda}{r'}-1} du = k_\lambda(r),$$

then $k \geq k_\lambda(r)(\varepsilon \to 0^+)$, and $k = k_\lambda(r)$ is the best value of (22). We conform that $k_\lambda(r)$ is the best value of (18). Otherwise we can get a contradiction by (21) that the constant factor in (22) is not the best possible. It follows that $\|T\| = k_\lambda(r)$. The theorem is proved. □

3. Some applications

Lemma 3. *As the assumption of Lemma 1, if $k_\lambda(u,1)u^{\frac{\lambda}{r}-1}$ and $k_\lambda(1,u)u^{\frac{\lambda}{s}-1}$ are decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$, and $k_\lambda(1,u) = o\big(\frac{1}{u^\alpha}\big)\big(u \to 0^+;\ 0 < \alpha < \frac{\lambda}{s}\big)$, then both (17) and (20) are valid.*

Proof. By the assumption and Lemma 1, we find

$$\begin{aligned} k_\lambda(r) - \int_0^{\frac{1}{m}} k_\lambda(1,u) u^{\frac{\lambda}{s}-1} du &= \int_1^\infty k_\lambda\left(1, \frac{y}{m}\right) \left(\frac{y}{m}\right)^{\frac{\lambda}{s}-1} d\left(\frac{y}{m}\right) \leq \widetilde{W}_\lambda(s,m) \\ &= \frac{1}{m} \sum_{n=1}^{\infty} k_\lambda\left(1, \frac{n}{m}\right) \left(\frac{n}{m}\right)^{\frac{\lambda}{s}-1} < \int_0^\infty k_\lambda\left(1, \frac{y}{m}\right) \left(\frac{y}{m}\right)^{\frac{\lambda}{s}-1} d\left(\frac{y}{m}\right) \\ &= \varpi_\lambda\left(s, \frac{y}{m}\right) = k_\lambda(r). \end{aligned}$$

Since $u^\alpha k_\lambda(1,u) \to 0$ $(u \to 0^+)$, there exists a constant $\delta \in (0,1)$, such that for $u \in (0,\delta)$, $u^\alpha k_\lambda(1,u) \leq 1$. Since in $[\delta,1]$, $k_\lambda(1,u) \leq k_\lambda(1,\delta) \leq \frac{L}{u^\alpha}$ $(L > 1)$, then $k_\lambda(1,u) \leq \frac{L}{u^\alpha}$, $u \in (0,1]$. Hence, setting $\lambda(r) = \frac{\lambda}{s} - \alpha > 0$, it follows

$$0 \leq \int_0^{\frac{1}{m}} k_\lambda(1,u) u^{\frac{\lambda}{s}-1} du \leq L \int_0^{\frac{1}{m}} u^{\lambda(r)-1} du = \frac{L}{\lambda(r)} \left(\frac{1}{m^{\lambda(r)}} \right),$$

and (20) is valid. By the same way, it follows $W_\lambda(r,n) < k_\lambda(r)$, and then we have (17). The lemma is proved. □

Theorem 2. *Suppose that* $p, r > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\lambda > 0$, $k_\lambda(x, y)$ (≥ 0) *is a homogeneous function of* $-\lambda$*-degree,* $k_\lambda(r) = \int_0^\infty k_\lambda(u, 1)u^{\frac{\lambda}{r}-1}du$ *is a positive number, and the weight coefficients* $W_\lambda(r, n)$ *and* $\widetilde{W}_\lambda(s, m)$ *satisfy inequalities* (17) *and* (20). *If* $a_n, b_n \geq 0$, $a = \{a_n\}_{n=1}^\infty \in l_\phi^p$, $b = \{b_n\}_{n=1}^\infty \in l_\psi^q$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, *then we have the following equivalent inequalities:*

$$(Ta, b) = \sum_{n=1}^\infty \sum_{m=1}^\infty k_\lambda(m, n)a_m b_n < k_\lambda(r)\|a\|_{p,\phi}\|b\|_{q,\psi}; \tag{25}$$

$$\|Ta\|_{p,\psi^{1-p}}^p = \sum_{n=1}^\infty n^{\frac{p\lambda}{s}-1}\left(\sum_{m=1}^\infty k_\lambda(m, n)a_m\right)^p < k_\lambda^p(r)\|a\|_{p,\phi}^p, \tag{26}$$

where the constant factors $k_\lambda(r)$ *and* $k_\lambda^p(r)$ *are the best possible.*

Replacing the conditions (17) *and* (20) *by* (a) $k_\lambda(u, 1)u^{\frac{\lambda}{r}-1}$ *and* $k_\lambda(1, u)u^{\frac{\lambda}{s}-1}$ *are decreasing in* $(0, \infty)$ *and strictly decreasing in a subinterval of* $(0, \infty)$; (b) $k_\lambda(1, u)u^{\frac{\lambda}{s}-1} = o\big(\frac{1}{u^\alpha}\big)$ $\big(u \to 0^+; 0 < \alpha < \frac{\lambda}{s}\big)$, *we still have* (25) *and* (26).

Proof. In view of (22) and (18), we have (25) and (26). Base on the proof of Theorem 1, it follows that the both the constant factors in (25) and (26) are the best possible.

If (26) is valid, then by (21), we have (25). Suppose that (25) is valid. By (18), $\|Ta\|_{p,\psi^{1-p}}^p < \infty$. If $\|Ta\|_{p,\psi^{1-p}}^p = 0$, then (26) is naturally valid; if $\|Ta\|_{p,\psi^{1-p}}^p > 0$, setting $b_n = n^{\frac{p\lambda}{s}-1}(\sum_{m=1}^\infty k_\lambda(m, n)a_m)^{p-1}$, then we find $0 < \|b\|_{q,\psi}^q = \|Ta\|_{p,\psi^{1-p}}^p < \infty$. By (25), we obtain

$$\|b\|_{q,\psi}^q = \|Ta\|_{p,\psi^{1-p}}^p = (Ta, b) < k_\lambda(r)\|a\|_{p,\phi}\|b\|_{q,\psi};$$

$$\|b\|_{q,\psi}^{q-1} = \|Ta\|_{p,\psi^{1-p}} < k_\lambda(r)\|a\|_{p,\phi},$$

and we have (26). Hence (25) and (26) are equivalent.

By (a), (b) and Lemma 3, we still have (25)–(26). The theorem is proved. □

Remark. (i) For $\lambda = 1$, $s = p$, $r = q$, (25) and (26) reduce respectively to (6) and (7). It is obvious that Theorem 2 is an extension of Theorem 1.

(ii) If we reserve (b) and replace the condition (a) by (a)' for $0 < \lambda \leq \min\{r, s\}$, $k_\lambda(u, 1)$ and $k_\lambda(1, u)$ are decreasing in $(0, \infty)$ and strictly decreasing in a subinterval of $(0, \infty)$, then (25)–(26) are still valid. Hence in particular, for $k_{\alpha\lambda}(x, y) = \frac{1}{(x^\alpha + y^\alpha)^\lambda}$ $(\alpha, \lambda > 0, \alpha\lambda \leq \min\{r, s\})$ in (25), we find

$$k_{\alpha\lambda}(r) = \int_0^\infty \frac{u^{\frac{\alpha\lambda}{r}-1}}{(u^\alpha + 1)^\lambda}du = \frac{1}{\alpha}\int_0^\infty \frac{v^{\frac{\lambda}{r}-1}}{(v + 1)^\lambda}dv = \frac{1}{\alpha}B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right),$$

$0 \le k_{\alpha\lambda}(1,u) \le 1$, and then it deduces to (10); for $k_\lambda(x,y) = \frac{1}{(\max\{x,y\})^\lambda}$ $(0 < \lambda \le \min\{r,s\})$ in (25), we find

$$k_\lambda(r) = \int_0^\infty \frac{1}{(\max\{u,1\})^\lambda} u^{\frac{\lambda}{r}-1} du = \frac{rs}{\lambda},$$

$0 \le k_\lambda(1,u) \le 1 (u \in (0,1])$ and it deduces to the best extension of (5) as

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(\max\{m,n\})^\lambda} < \frac{rs}{\lambda} \|a\|_{p,\phi} \|b\|_{q,\psi}; \tag{27}$$

for $k_\lambda(x,y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$ $(0 < \lambda \le \min\{r,s\})$ in (25), we find (cf. [3])

$$k_\lambda(r) = \int_0^\infty \frac{\ln u}{u^\lambda - 1} u^{\frac{\lambda}{r}-1} du = \left[\frac{\pi}{\lambda \sin(\frac{\pi}{r})}\right]^2,$$

$k_\lambda(1,u) = o\big(\frac{1}{u^\alpha}\big)$ $\big(u \to 0^+; 0 < \alpha < \frac{\lambda}{s}\big)$ and $(\frac{\ln u}{u^\lambda - 1})' < 0$, then it deduces to the best extension of (6) as

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\ln(m/n) a_m b_n}{m^\lambda - n^\lambda} < \left[\frac{\pi}{\lambda \sin(\frac{\pi}{r})}\right]^2 \|a\|_{p,\phi} \|b\|_{q,\psi}. \tag{28}$$

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