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Polynomial bases of split simple Lie algebras

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Abstract. We show that every simple Lie algebra $\mathfrak{g}$ of real rank at least two is isomorphic to a space of polynomials defined on the group $N = \exp \mathfrak{n}$, where $\mathfrak{n}$ is the nilpotent component of the Iwasawa decomposition of $\mathfrak{g}$. Using suitable coordinates on N, we then write a basis of this space of polynomials when $\mathfrak{g}$ is split.

1. Introduction

When $n \geq 3$, the action of the conformal group $O(1,4)$ on $\mathbb{R}^3 \cup \{\infty\}$ may be characterized in differential geometric terms: Liouville proved in 1850 that a C^4 map between domains $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{R}^3$ whose differential is a multiple of an isometry at each point of $\mathcal{U}$ is the restriction to $\mathcal{U}$ of the action of some $g \in O(1,4)$. This type of result has been extended to $\mathbb{R}^n$ with weaker smoothness assumptions and to more general spaces, see for instance [3]–[5], [7]–[10].

In [4], the authors consider the problem of characterizing the action of a semisimple Lie group G on the homogeneous spaces G/P, where P is a minimal parabolic subgroup. More precisely, they prove a Liouville type theorem for every semisimple Lie group G with rank at least two. The proof of this theorem passes through a polynomial representation of simple real Lie algebras, that we intend to make explicit. In particular, it is possible to define an isomorphism I between the Lie algebra of G and a space of polynomials on N, the nilpotent component of the Iwasawa decomposition of G. The isomorphism induces a Lie algebra structure on this space of polynomials. We are interested in investigating the polynomial representation of the simple Lie algebras given by I.

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The paper is organized as follows. In Section 2 we fix the notations and recall a result of [4] that we are going to need. In particular, we give the definition of multicontact map and vector field, and recall (Theorem 1) that the space of multicontact vector fields on N is isomorphic to the simple Lie algebra $\mathfrak{g}$ whose nilradical is $\mathfrak{n} = \mathrm{Lie}(N)$. In Section 3 we discuss the isomorphism I in some details. First we introduce the notion of homogeneous function and vector field and observe that the space of multicontact vector fields is generated as a vector space by its homogeneous parts. In fact, there is a one to one correspondence between suitable bases of $\mathfrak{g}$ and homogeneous generators of the multicontact vector fields. This correspondence allows us to define I. The idea is to fix a basis of each root space and therefore a basis of $\mathfrak{g}$. Hence I is the linear map that assigns to each such basis element a suitable vector of polynomials. In Section 4 we restrict to the case of split simple Lie algebras $\mathfrak{g}$. In this case the image $I(X)$ is exactly one polynomial. In Lemma 2 we give a formula for computing $I(X)$, whenever X lies in a root space or in the Cartan subspace. We then use this in Proposition 3 to find an explicit basis of the space of the polynomials in canonical coordinates. In the last section we consider the case where $\mathfrak{g}$ is $\mathfrak{sl}(3,\mathbb{R})$ and therefore N is the Heisenberg group and apply Proposition 3 in order to write the polynomial basis of $\mathfrak{sl}(3,\mathbb{R})$.

2. Notations and preliminaries

We introduce some tools which come from the classical theory of semisimple Lie groups [1], [6], as well as some further properties proved in [4]. Let $\mathfrak{g}$ be a simple Lie algebra with Killing form B and Cartan involution θ. Then $B_\theta(X,Y) = -B(X,\theta Y)$ is an inner product on $\mathfrak{g}$. Let $\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k} = \{X \in \mathfrak{g} : \theta X = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} : \theta X = -X\}$. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and denote by Σ the set of restricted roots, Σ is a subset of the dual $\mathfrak{a}'$ of $\mathfrak{a}$, which is endowed with an inner product $(\,\cdot\,,\cdot\,)$ induced by B_θ. Choose an ordering $\succeq$ on $\mathfrak{a}'$. Call Σ_+ and $\Delta = \{\delta_1,\dots,\delta_r\}$ the subsets for positive and simple positive restricted roots. We call rank of $\mathfrak{g}$ the cardinality of Δ. Every positive root α can be written as $\alpha = \sum_{i=1}^r n_i\delta_i$ for uniquely defined non-negative integers $n_1,\dots,n_r$. The positive integer $\mathrm{ht}(\alpha) = \sum_{i=1}^r n_i$ is called the height of α. It is well-known that there is exactly one root ω, called the highest root, that satisfies $\omega \succ \alpha$ (strictly) for every other root α. The root space decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{m}\oplus\mathfrak{a}\oplus\bigoplus_{\alpha\in\Sigma}\mathfrak{g}_\alpha$, where $\mathfrak{m} = \{X \in \mathfrak{k} : [X,H] = 0,\ H \in \mathfrak{a}\}$. The Iwasawa decomposition is $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$, where $\mathfrak{n} = \oplus_{\gamma\in\Sigma_+}\mathfrak{g}_\gamma$. We write $\mathfrak{n}_i = \oplus_{\mathrm{ht}(\gamma)=i}\mathfrak{g}_\gamma$

for every $i = 1, \dots, \mathrm{ht}(\omega)$. Then $\mathfrak{n}$ is a stratified nilpotent Lie algebra, that is $[\mathfrak{n}_i, \mathfrak{n}_1] = \mathfrak{n}_{i+1}$. Finally, we denote by G a Lie group whose Lie algebra is $\mathfrak{g}$.

Consider a diffeomorphism f between open subsets $\mathcal{U}$ and $\mathcal{V}$ of N. For every positive root α, the space $\mathfrak{g}_\alpha$ defines a subspace of the tangent space of N at the identity and by left translation it defines a sub-bundle of the tangent bundle, for which we abuse the notation $\mathfrak{g}_\alpha$. We say that f is a multicontact mapping if its differential f_* preserves $\mathfrak{g}_\delta$, for every simple root δ. This is a generalized notion of contact mapping in the usual sense, because $\mathfrak{n}_1 = \oplus_{\delta \in \Delta} \mathfrak{g}_\delta$ and a basis of left invariant vector fields of $\mathfrak{n}_1$ generates via Lie bracket the whole algebra of left invariant vector fields. If $\mathcal{U} = \mathcal{V}$ we can compose two multicontact mappings, obtaining another multicontact map. We define a multicontact vector field as a vector field V on $\mathcal{U}$ whose local flow $\{\phi_t^V\}$ consists of multicontact maps. Such a vector field satisfies

$$[V, \mathfrak{g}_\delta] \subset \mathfrak{g}_\delta,$$

for every simple root δ. The group G acts on G/P. By means of the Bruhat decomposition, the action can be restricted to N. Let $\mathfrak{X}(N)$ denote the Lie algebra of vector fields on N. We define a representation of $\mathfrak{g}$ as vector fields on N

$$\tau : \mathfrak{g} \to \mathfrak{X}(N)$$

as

$$(\tau(X)f)(n) = \frac{d}{dt} f([\exp(tX)n])\Big|_{t=0}.$$

Hence $[\exp(tX)n]$ is the N-component of the product $\exp(tX) \cdot n$ in the Bruhat decomposition of G/P (see [4] for more details). The following theorem is proved in [4] and its proof contains the results we need.

Theorem 1 ([4])**.** *Suppose that $\mathfrak{g}$ has real rank at least two. Then every C^1 multicontact vector field is in fact smooth, and the Lie algebra of multicontact vector fields on $\mathcal{U}$ consists of the restrictions of $\tau(\mathfrak{g})$ to $\mathcal{U}$.*

3. The polynomial algebra $\mathcal{P}$

From now on we assume that $\mathfrak{g}$ has real rank at least two. For every $\alpha \in \Sigma_+$, denote by m_α the dimension of $\mathfrak{g}_\alpha$ and fix a basis $\{X_{\alpha,i} : \alpha \in \Sigma_+,\ i = 1, \dots, m_\alpha\}$ of $\mathfrak{n}$ consisting of left-invariant vector fields on N. A smooth vector field V on $\mathcal{U}$ is

$$V = \sum_{\alpha \in \Sigma_+} \sum_{i=1}^{m_\alpha} v_{\alpha,i} X_{\alpha,i}, \tag{1}$$

with smooth functions $v_{\alpha,i}$. The proof of Theorem 1 points out that a multicontact vector field is determined by its component along the directions corresponding to the highest root, namely $\{v_{\omega,i} : i = 1, \ldots, m_\omega\}$, the remaining components being obtained differentiating those. Further, the functions $v_{\omega,i}$ are in fact polynomials in canonical coordinates.

We select an element H_0 in the Cartan subspace $\mathfrak{a}$ such that $\delta(H_0) = -1$ for all simple roots δ. We say that a function v on N is homogeneous of degree r if it does not vanish identically and satisfies $\tau(H_0)v = rv$. A vector field V is said to be homogeneous of degree s if it does not vanish identically and satisfies $[\tau(H_0), V] = sV$. Hence

$$\begin{aligned} \deg(vV) &= \deg(V) + \deg(v), \\ \deg(V(v)) &= \deg(v) + \deg(V) \quad \text{(except when } V(v) = 0), \\ \deg([V, W]) &= \deg(V) + \deg(V) \quad \text{(except when } V \text{ and } W \text{ commute).} \end{aligned}$$

The Lie algebra of multicontact vector fields is then generated by its homogeneous parts. More precisely, the set

$$\{\tau(X_{\alpha,i}), \alpha \in \Sigma \cup \{0\},\ i = 1, \ldots, m_\alpha\}$$

defines a basis. Since τ is a representation, we have

$$[\tau(H_0), \tau(X_{\alpha,i})] = \tau([H_0, X_{\alpha,i}]) = \alpha(H_0)\tau(X_{\alpha,i}) = -\mathrm{ht}(\alpha)\tau(X_{\alpha,i}).$$

Let p be a ω-component of $\tau(X_{\alpha,i})$. Then the height of α and the degree of p are related:

$$-\mathrm{ht}(\alpha) = \deg(\tau(X_{\alpha,i})) = \deg(pX_{\omega,j}) = \deg(p) + \deg(X_{\omega,j}) = \deg(p) - \mathrm{ht}(\omega),$$

whence

$$\deg(p) = \mathrm{ht}(\omega) - \mathrm{ht}(\alpha).$$

Define

$$I : \mathfrak{g} \longrightarrow \mathcal{P}, \tag{2}$$

by extending linearly the assignment $I(X_{\alpha,i}) = (v_{\omega,1}, \ldots, v_{\omega,m_\omega})$, the vector of polynomials that corresponds to the coefficients of $\tau(X_{\alpha,i})$ along ω. Here $\mathcal{P}$ is a vector space of polynomial vectors, namely the image of the above mapping inside the m_ω-fold cartesian product of the algebra of polynomials in $\dim(\mathfrak{n})$ indeterminates over the reals. The map I is an isomorphism, that induces a Lie algebra structure on $\mathcal{P}$.

Since the homogeneity degree of the polynomials $I(X_{\alpha,i})$ depends only on the root space $\mathfrak{g}_\alpha$, all the components of a single basis vector have the same degree. Let α be a positive root, $X \in \mathfrak{g}_{\pm\alpha}$ or $X \in \mathfrak{m} \oplus \mathfrak{a}$, and let p be a ω-component of $\tau(X)$. The following diagram clarifies the various notions of degree:

root space	$\deg(\tau(X))$	$\deg p$
$\mathfrak{g}_\alpha$	$-\mathrm{ht}(\alpha)$	$\mathrm{ht}(\omega) - \mathrm{ht}(\alpha)$
$\mathfrak{m} \oplus \mathfrak{a}$	0	$\mathrm{ht}(\omega)$
$\mathfrak{g}_{-\alpha}$	$\mathrm{ht}(\alpha)$	$\mathrm{ht}(\omega) + \mathrm{ht}(\alpha)$.

In particular each polynomial has degree between 0 and $2h$, where $h = \mathrm{ht}(\omega)$.

4. The split case

We compute explicit formulas for a basis of $\mathcal{P}$. We restrict our discussion to the case of the split real form $\mathfrak{g}$ of a simple complex Lie algebra. The most relevant consequences of this assumption for our considerations are that $\mathfrak{m} = \{0\}$ and that each restricted root space has real dimension one. In particular, this implies that $I(\mathfrak{g}_\alpha)$ consists for all α of the real multiples of a single polynomial.

Our decomposition formulas are relative to a suitable decomposition of the restricted root system (see e.g. [2]), namely $\Sigma_+ = \Sigma_0 \oplus \Sigma_{1/2} \oplus \Sigma_1$, where

$$\Sigma_0 = \{\beta \in \Sigma_+ : (\omega, \beta) = 0\},$$
$$\Sigma_{1/2} = \left\{\beta \in \Sigma_+ : (\omega, \beta) = \frac{1}{2}(\omega, \omega)\right\},$$
$$\Sigma_1 = \{\beta \in \Sigma_+ : (\omega, \beta) = (\omega, \omega)\} = \{\omega\}.$$

We shall write $\Delta_{1/2} = \Sigma_{1/2} \cap \Delta$ and $\Delta_0 = \Sigma_0 \cap \Delta$. According to the decomposition of Σ_+, we put

$$\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_{1/2} \oplus \mathfrak{n}_1,$$

with obvious notations. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, and $(\alpha + \beta, \omega) = (\alpha, \omega) + (\beta, \omega)$, it follows that $\mathfrak{n}_0$ is a subalgebra and $\mathfrak{n}_{1/2} \oplus \mathfrak{n}_1$ is an ideal in $\mathfrak{n}$. The Cartan involution θ maps each root space $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$, so that $\overline{\mathfrak{n}} = \theta\mathfrak{n} = \oplus_{\gamma \in \Sigma_-} \mathfrak{g}_\gamma$, where $\Sigma_- = -\Sigma_+$. We write

$$\overline{\mathfrak{n}} = \overline{\mathfrak{n}}_0 \oplus \overline{\mathfrak{n}}_{1/2} \oplus \overline{\mathfrak{n}}_1,$$

so that

$$\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{n}_{1/2} \oplus \mathfrak{n}_0 \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}}_0 \oplus \overline{\mathfrak{n}}_{1/2} \oplus \overline{\mathfrak{n}}_1. \tag{3}$$

By linearity of the scalar product, the following commutation rules hold

$$\begin{array}{lll} [\mathfrak{a}, \mathfrak{n}_0] \subset \mathfrak{n}_0 & [\mathfrak{a}, \mathfrak{n}_{1/2}] \subset \mathfrak{n}_{1/2} & [\mathfrak{a}, \mathfrak{n}_1] \subset \mathfrak{n}_1 \\ [\mathfrak{n}_0, \overline{\mathfrak{n}}_0] \subset \mathfrak{n}_0 \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}}_0 & [\mathfrak{n}_0, \overline{\mathfrak{n}}_{1/2}] \subset \overline{\mathfrak{n}}_{1/2} & [\mathfrak{n}_0, \overline{\mathfrak{n}}_1] = \{0\} \\ [\mathfrak{n}_{1/2}, \overline{\mathfrak{n}}_0] \subset \mathfrak{n}_{1/2} & [\mathfrak{n}_{1/2}, \overline{\mathfrak{n}}_{1/2}] \subset \mathfrak{n}_0 \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}}_0 & [\mathfrak{n}_{1/2}, \overline{\mathfrak{n}}_1] \subset \overline{\mathfrak{n}}_{1/2} \\ [\mathfrak{n}_1, \overline{\mathfrak{n}}_0] = \{0\} & [\mathfrak{n}_1, \overline{\mathfrak{n}}_{1/2}] \subset \mathfrak{n}_{1/2} & [\mathfrak{n}_1, \overline{\mathfrak{n}}_1] \subset \mathfrak{a} \end{array} \tag{4}$$

We fix the following canonical coordinates on N:

$$n = n_1 n_{\frac{1}{2}} n_0 = \exp(zZ) \exp\Big(\sum_{\alpha \in \Sigma_{1/2}} y_\alpha Y_\alpha \Big) \exp\Big(\sum_{\beta \in \Sigma_0} x_\beta X_\beta \Big), \tag{5}$$

where $\{X_\beta, \beta \in \Sigma_0\}$, $\{Y_\alpha, \alpha \in \Sigma_{1/2}\}$ and Z are a basis of $\mathfrak{n}_0$, $\mathfrak{n}_{1/2}$ and $\mathfrak{n}_1$ respectively.

Set $X \in \mathfrak{g}_\alpha$, $\alpha \in \Sigma \cup \{0\}$ and n in N. By the Bruhat decomposition, for t small enough there exists $b(t) \in P$ such that $\exp(tX)nb(t) \in N$. Then consider the decomposition of $n^{-1}\exp(tX)nb(t)$ with respect to the chosen coordinates, namely

$$n^{-1}\exp(tX)nb(t) = n_1^X(t) n_{1/2}^X(t) n_0^X(t).$$

Lemma 2. *With the notations as above, writing $n = n_1 n_{1/2} n_0$, we have*

(i) *there exists $A \in \mathfrak{n}_1$ and $B \in \mathfrak{n}_{1/2} \oplus \mathfrak{n}_0 \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}}$ such that*

$$n_{1/2}^{-1} n_1^{-1} \exp(tX) n_1 n_{1/2} = \exp(tA) \exp(tB) \exp(o(t));$$

(ii)

$$\frac{d}{dt} \left(n_1^X(t) \right) \Big|_{t=0} = A;$$

(iii) *If I is the isomorphism defined in* (2), *then $I(X) = A$.*

Proof. Write

$$n^{-1}\exp(tX)n = n_0^{-1} n_{1/2}^{-1} n_1^{-1} \exp(tX) n_1 n_{1/2} n_0.$$

Observe first that since $n_1 = \exp(zZ)$,

$$n_1^{-1} \exp(tX) n_1 = \exp(e^{-\operatorname{ad}(zZ)} tX).$$

Now, by (4)

$$[Z,X] \in \begin{cases} \mathfrak{n}_1 & \text{if } X \in \mathfrak{a} \\ \mathfrak{n}_{1/2} & \text{if } X \in \overline{\mathfrak{n}}_{1/2} \\ \mathfrak{a} & \text{if } X \in \overline{\mathfrak{n}}_1, \end{cases}$$

and if X belongs to some other summand in the decomposition (3), then $[Z,X]=0$. Therefore by the Baker–Campbell–Hausdorff formula

$$\begin{aligned} n_1^{-1} \exp(tX) n_1 &= \exp(tX + t(H_1 + A_{1/2} + A_1) + o(t)) \\ &= \exp(tA_1 + o(t)) \exp(tX + t(H_1 + A_{1/2}) + o(t)), \end{aligned} \tag{6}$$

where $H_1 \in \mathfrak{a}$, $A_{1/2} \in \mathfrak{n}_{1/2}$ and $A_1 \in \mathfrak{n}_1$.

Secondly, since $\mathfrak{n}_1$ commutes with $\mathfrak{n}$, we consider

$$n_{1/2}^{-1} \exp(tX + t(H_1 + A_{1/2})) n_{1/2} = \exp(e^{-\sum_\alpha y_\alpha \operatorname{ad} Y_\alpha}(tX + tH_1 + tA_{1/2})).$$

Since $n_{1/2}$ is the exponential of some element in $\mathfrak{n}_{1/2}$, in the above formula $\alpha \in \Sigma_{1/2}$. Therefore, if the commutator $[Y_\alpha, X] \neq 0$, then by (4)

$$[Y_\alpha, X] \in \begin{cases} \overline{\mathfrak{n}}_{1/2} & \text{if } X \in \overline{\mathfrak{n}}_1 \\ \mathfrak{a} & \text{if } X \in \overline{\mathfrak{n}}_{1/2} \\ \mathfrak{n}_{1/2} & \text{if } X \in \overline{\mathfrak{n}}_0 \oplus \mathfrak{n}_0 \oplus \mathfrak{a} \\ \mathfrak{n}_1 & \text{if } X \in \mathfrak{n}_{1/2}. \end{cases}$$

Moreover,

$$[Y_\alpha, H_1] \in \mathfrak{a}, \quad [Y_\alpha, A_{1/2}] \in \mathfrak{n}_1.$$

Hence

$$\begin{aligned} n_{1/2}^{-1} \exp(tX + t(H_1 + A_{1/2})) n_{1/2} &= \exp(tX + t(B_{1/2}^- + H_2 + B_{1/2} + B_1) + o(t)), \\ &= \exp(tB_1 + o(t)) \exp(tX + t(B_{1/2}^- + H_2 + B_{1/2}) + o(t)), \end{aligned} \tag{7}$$

for some $B_{1/2}^- \in \overline{\mathfrak{n}}_{1/2}$, $H_2 \in \mathfrak{a}$, $B_{1/2} \in \mathfrak{n}_{1/2}$ and $B_1 \in \mathfrak{n}_1$. Also, observe that by the Baker–Campbell–Hausdorff formula

$$\exp(tL + o(t)) = \exp(tL) \exp(o(t)) \tag{8}$$

for any $L \in \mathfrak{g}$. Thus, by (6) and (4) we obtain that

$$n_{1/2}^{-1} n_1^{-1} \exp(tX) n_1 n_{1/2} = \exp(tA) \exp(tB) \exp(o(t)),$$

with

$$A = \begin{cases} A_1 + B_1 & \text{if } X \notin \mathfrak{n}_1 \\ A_1 + B_1 + X & \text{if } X \in \mathfrak{n}_1 \end{cases}$$

and

$$B = \begin{cases} B^-_{1/2} + H_2 + B_{1/2} & \text{if } X \in \mathfrak{n}_1 \\ B^-_{1/2} + H_2 + B_{1/2} + X & \text{if } X \notin \mathfrak{n}_1. \end{cases}$$

This proves (i).

Next, consider

$$n_0^{-1} \exp(tX + t(B^-_{1/2} + H_2 + B_{1/2}))n_0 = \exp(e^{-\operatorname{ad}(\sum_\beta x_\beta X_\beta)}(tX + tB^-_{1/2} + tH_2 + tB_{1/2})).$$

If $[X_\beta, X] \neq 0$, then by (4)

$$[X_\beta, X] \in \begin{cases} \overline{\mathfrak{n}}_{1/2} & \text{if } X \in \overline{\mathfrak{n}}_{1/2} \\ \mathfrak{n}_0 \oplus \overline{\mathfrak{n}}_0 \oplus \mathfrak{a} & \text{if } X \in \mathfrak{n}_0 \oplus \overline{\mathfrak{n}}_0 \oplus \mathfrak{a} \\ \mathfrak{n}_{1/2} & \text{if } X \in \mathfrak{n}_{1/2}. \end{cases}$$

Furthermore,

$$[X_\beta, B^-_{1/2}] \in \overline{\mathfrak{n}}_{1/2}, \quad [X_\beta, H_2] \in \mathfrak{n}_0, \qquad [X_\beta, B_{1/2}] \in \mathfrak{n}_{1/2}.$$

Hence

$$n_0^{-1} \exp(tX + t(B^-_{1/2} + H_2 + B_{1/2}))n_0 = \exp(tX + t(C^-_{1/2} + C^-_0 + H_3 + C^+_0 + C_{1/2}) + o(t)), \quad (9)$$

for some $C^-_{1/2} \in \overline{\mathfrak{n}}_{1/2}$, $C^-_0 \in \overline{\mathfrak{n}}_0$, $H_3 \in \mathfrak{a}$, $C^+_0 \in \mathfrak{n}_0$ and $C_{1/2} \in \mathfrak{n}_{1/2}$.

Since $\mathfrak{n}_1$ commutes with $\mathfrak{n}$, using (6), (4), (8) and (9) we obtain

$$\begin{aligned} n^{-1} \exp(tX)n &= \exp(tA_1 + tB_1 + o(t)) \\ &\quad \times \exp(tX + t(C^-_{1/2} + C^-_0 + H_3 + C^+_0 + C_{1/2}) + o(t)) \\ &= \exp(tA_1 + tB_1 + o(t)) \exp(tX + tC^+_0 tC_{1/2} + o(t)) \\ &\quad \times \exp(t(C^-_{1/2} + C^-_0 + H_3) + o(t)) \\ &= \exp(tA_1 + tB_1 + tk_1(X)) \exp(tC_{1/2} + tk_{1/2}(X)) \\ &\quad \times \exp(tC_0 + tk_0(X)) \exp(tC^-_{1/2} + tC^-_0 + tH_3 + tk(X)) \exp(o(t)) \\ &= \exp(tA) \exp(tC) \exp(tD) \exp(tE) \exp(o(t)), \end{aligned} \quad (10)$$

where

$$k(X) = \begin{cases} X & \text{if } X \in \mathfrak{a} \oplus \overline{\mathfrak{n}} \\ 0 & \text{otherwise,} \end{cases}$$

$$k_i(X) = \begin{cases} X & \text{if } X \in \mathfrak{n}_{(i)},\ i = 0, 1/2, 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$C = C_{1/2} + k_{1/2}(X), \quad D = C_0 + k_0(X), \quad E = C^-_{1/2} + C^-_0 + H_3 + k(X).$$

On the other hand, by hypothesis

$$n^{-1} \exp(tX) n = n^X_1(t) n^X_{1/2}(t) n^X_0(t) b(t)^{-1}. \tag{11}$$

Observe that since $n^{-1}\exp(tX)n$ is the identity for $t = 0$, then necessarily $n^X_r(0) = e$ for every $r = 1, 1/2, 0$, and $b(0) = e$. Therefore, comparing (10) and (11),

$$\frac{d}{dt}\left(\exp(tA)\exp(tC)\exp(tD)\exp(tE)\exp(o(t))\right)\Big|_{t=0} = \frac{d}{dt}\left(n^X_1(t) n^X_{1/2}(t) n^X_0(t) b(t)^{-1}\right)\Big|_{t=0},$$

whence

$$\begin{aligned} A + C + D + E = {} & \frac{d}{dt}\left(n^X_1(t)\right)\Big|_{t=0} n^X_{1/2}(0) n^X_0(0) b(0)^{-1} \\ & + n^X_1(0) \frac{d}{dt}\left(n^X_{1/2}(t)\right)\Big|_{t=0} n^X_0(0) b(0)^{-1} \\ & + n^X_1(0) n^X_{1/2}(0) \frac{d}{dt}\left(n^X_0(t)\right)\Big|_{t=0} b(0)^{-1} \\ & + n^X_1(0) n^X_{1/2}(0) n^X_0(0) \frac{d}{dt}\left(b(t)^{-1}\right)\Big|_{t=0}. \end{aligned}$$

This implies

$$A = \frac{d}{dt}\left(n^X_1(t)\right)\Big|_{t=0},$$

because A and $\frac{d}{dt}\left(n^X_1(t)\right)\big|_{t=0}$ are the only two terms in the above sum that lie along Z. Thus also (ii) is proved.

In order to prove (iii), consider the multicontact vector field associated to X:

$$\tau(X) f(n) = \frac{d}{dt} f([\exp(tX)n])\Big|_{t=0},$$

where $[\exp(tX)n]$ is the N- component of $\exp(tX)n$ in the Bruhat decomposition.

This is equivalent to saying that for t small enough there exists $b(t) \in P$ such that $[\exp(tX)n] = \exp(tX)nb(t) \in N$. Hence

$$\begin{aligned}\tau(X)f(n) &= \frac{d}{dt}f(\exp(tX)nb(t))\Big|_{t=0}\\ &= \frac{d}{dt}f(nn^{-1}\exp(tX)nb(t))\Big|_{t=0}\\ &= \frac{d}{dt}f(n_1n_{1/2}n_0n_1^X(t)n_{1/2}^X(t)n_0^X(t))\Big|_{t=0}.\end{aligned}$$

Consider the left-invariant vector fields corresponding to the basis of $\mathfrak{n}$ chosen in (5) and write $\tau(X)$ accordingly. Then the image of X via the isomorphism I defined in (2) is p, the coefficient along Z of $\tau(X)$. We observed that $n_r^X(0) = e$, for every $r = 0, 1/2, 1$. Therefore,

$$p = \frac{d}{dt}(n_1^X(t))\Big|_{t=0}, \tag{12}$$

and so $p = A$. □

We showed above that $p(n)$ is obtained in two steps: first we compute the conjugation $n_{1/2}^{-1}n_1^{-1}\exp(tX)n_1n_{1/2}$ and then write it in the form $\exp(tA)\exp(tB + o(t))$, where $A \in \mathfrak{n}_1$ and B has no components along $\mathfrak{n}_1$, according to the decomposition (3).

We shall obtain explicit formulas for the homogeneous polynomials corresponding to $\mathfrak{g}$ using (12). We consider separately the cases with α in Σ_0, $\Sigma_{1/2}$, Σ_1, $\{0\}$, $-\Sigma_1$, $-\Sigma_{1/2}$, $-\Sigma_0$. The resulting polynomials are a basis of the space $\mathcal{P}$ and we collect them in the next proposition. We define on $\Sigma_{1/2}$ the equivalence relation $\sim$ given by

$$\alpha \sim \beta \Leftrightarrow \alpha + \beta = \omega,$$

and we choose one representative for each element of the quotient $(\Sigma_{1/2}/\sim)$. Denote the set of such representatives by $\tilde{\Sigma}_{1/2}$.

Proposition 3. *Denote $p^\alpha = I(X_\alpha)$ for every $X_\alpha \in \mathfrak{g}_\alpha$ and every non zero root α and $p^H = I(H)$ for every $H \in \mathfrak{a}$. We write $c_{\alpha,\beta}$ for the structure constants of $[X_\alpha, X_\beta]$ and H_γ for the unique element in $\mathfrak{a}$ for which $\gamma(H_\gamma) = 1$. Then the following formulas hold.*

(i) *If $\gamma \in \Sigma_{1/2}$, then $p^\gamma(n) = c_{\gamma,\omega-\gamma}y_{\omega-\gamma}$.*

(ii) *If $H \in \mathfrak{a}$, then*

$$p^H(n) = \omega(H)z - \frac{1}{2}\sum_{\alpha\in\tilde{\Sigma}_{1/2}} y_\alpha y_{\omega-\alpha}\left((\omega-\alpha)(H) - \alpha(H)\right)c_{\alpha,\omega-\alpha}.$$

(iii) $p^{\omega}(n) = 1$.

(iv) *If* $\nu \in \Sigma_0 \cup -\Sigma_0$, *then*

$$p^{\nu}(n) = \frac{1}{2} \sum_{\nu+\alpha_1+\alpha_2=\omega} c_{\alpha_1,\nu} c_{\alpha_2,\nu+\alpha_1} y_{\alpha_1} y_{\alpha_2},$$

where α_1 *and* α_2 *vary in* $\Sigma_{1/2}$.

(v) *If* $\gamma \in \Sigma_{1/2}$, *then*

$$\begin{aligned} p^{-\gamma}(n) = -\omega(H_\gamma) y_\gamma z \frac{1}{6} \sum_{\alpha \in \Sigma_{1/2}} \alpha(H_\gamma) c_{\omega-\alpha,\alpha} y_\gamma y_\alpha y_{\omega-\alpha} \\ - \frac{1}{6} \sum_{-\gamma+\alpha_1+\alpha_2+\alpha_3=\omega} c_{\alpha_1,-\gamma} c_{\alpha_2,-\gamma+\alpha_1} c_{\alpha_3,-\gamma+\alpha_1+\alpha_2} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3}, \end{aligned}$$

with α_1,α_2,$\alpha_3 \in \Sigma_{1/2}$.

(vi) *Finally,*

$$p^{-\omega}(n) = -\frac{1}{2} z^2 \omega(H_\omega) + \frac{1}{2} z \sum_{\alpha \in \Sigma_{1/2}} c_{\omega-\alpha,\alpha} \alpha(H_\omega) y_\alpha y_{\omega-\alpha}$$

$$+ \frac{t}{24} \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \Sigma_{1/2}} c_{\alpha_1,-\omega} c_{\alpha_2,-\omega+\alpha_1} c_{\alpha_3,-\omega+\alpha_1+\alpha_2} c_{\alpha_4,-\omega+\alpha_1+\alpha_2+\alpha_3} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} y_{\alpha_4}.$$

Proof. (i) We will repeatedly use the following simple observation: if $\alpha, \gamma \in \Sigma_{1/2}$ and $\alpha + \gamma \in \Sigma$, then $\gamma = \omega - \alpha$. Indeed $(\alpha + \gamma, \omega) = (\alpha, \omega) + (\gamma, \omega) = (\omega, \omega)$. This implies that $\alpha = \omega - \gamma$. Since $[Z, \mathfrak{n}] = 0$,

$$\begin{aligned} n_{1/2}^{-1} n_1^{-1} \exp(tY_\gamma) n_1 n_{1/2} &= n_{1/2}^{-1} \exp\left(\sum_{n=0}^{+\infty} (-1)^n \frac{(\operatorname{ad} zZ)^n}{n!} tY_\gamma \right) n_{1/2} \\ &= n_{1/2}^{-1} \exp(tY_\gamma) n_{1/2} \\ &= \exp\left(\sum_{n=0}^{+\infty} (-1)^n \frac{(\operatorname{ad}(\sum_{\alpha \in \Sigma_{1/2}} y_\alpha Y_\alpha))^n}{n!} tY_\gamma \right) \\ &= \exp(tY_\gamma - t y_{\omega-\gamma} [Y_{\omega-\gamma}, Y_\gamma]) \\ &= \exp(t c_{\gamma,\omega-\gamma} y_{\omega-\gamma} Z) \exp\big(tY_\gamma\big). \end{aligned}$$

By (12) and the remark thereafter, we have $p^{\gamma}(n) = c_{\gamma,\omega-\gamma} y_{\omega-\gamma}$.

(ii) Since $[\mathfrak{n}_{1/2}, \mathfrak{n}_{1/2}] \subseteq \mathfrak{n}_1$, every bracket involving three or more vectors in $\mathfrak{n}_{1/2}$ is zero. If $H \in \mathfrak{a}$, then

$$\begin{aligned} n_{1/2}^{-1} n_1^{-1} \exp(tH) n_1 n_{1/2} &= n_{1/2}^{-1} \exp(tH - tz[Z,H]) n_{1/2} \\ &= \exp(t\omega(H)zZ) \exp\Big(tH - t \sum_{\alpha \in \Sigma_{1/2}} y_\alpha [Y_\alpha, H] + t/2 \sum_{\alpha+\beta=\omega} y_\alpha y_\beta \big[Y_\beta, [Y_\alpha, H]\big]\Big) \\ &= \exp\Big(\omega(H) z + \frac{1}{2} \sum_{\alpha+\beta=\omega} \alpha(H) c_{\alpha,\beta} y_\alpha y_\beta\Big) tZ \ldots \end{aligned}$$

where the only relevant component is the linear term in t along Z. Therefore

$$p^H(n) = \omega(H) z - \frac{1}{2} \sum_{\alpha \in \tilde{\Sigma}_{1/2}} y_\alpha y_{\omega-\alpha} \left((\omega - \alpha)(H) - \alpha(H)\right) c_{\alpha,\omega-\alpha},$$

as required.

(iii) Since $[Z, \mathfrak{n}] = 0$, the conclusion is obvious.

(iv) If $\alpha \in \Sigma_{1/2}$, then $(\nu + \alpha, \omega) = (\nu, \omega) + (\alpha, \omega) = \frac{1}{2}(\omega, \omega)$, whence $\nu + \alpha \in \Sigma_{1/2}$, provided it is a root. Moreover by definition $\omega + \nu$ is not a root. Therefore

$$\begin{aligned} n_{1/2}^{-1} n_1^{-1} \exp(tX_\nu) n_1 n_{1/2} &= n_{1/2}^{-1} \exp(tX_\nu) n_{1/2} \\ &= \exp\Big(tX_\nu - t \sum_{\alpha \in \Sigma_{1/2}} y_\alpha [Y_\alpha, X_\nu] + \frac{t}{2} \sum_{\alpha_1, \alpha_2 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} \big[Y_{\alpha_2}, [Y_{\alpha_1}, X_\nu]\big]\Big) \\ &= \exp\Big(\frac{t}{2} \sum_{\nu+\alpha_1+\alpha_2=\omega} c_{\alpha_1,\nu} c_{\alpha_2,\nu+\alpha_1} y_{\alpha_1} y_{\alpha_2} Z\big) \exp\big(tX_\nu - t \sum_{\alpha \in A} c_{\alpha,\nu} y_\alpha Y_{\alpha+\nu}\Big) \end{aligned}$$

So (12) gives $p^\nu(n) = \frac{1}{2} \sum_{\nu+\alpha_1+\alpha_2=\omega} c_{\alpha_1,\nu} c_{\alpha_2,\nu+\alpha_1} y_{\alpha_1} y_{\alpha_2}$, where α_1 and α_2 are in $\Sigma_{1/2}$.

(v) Take $\gamma \in \Sigma_{1/2}$. Then

$$\begin{aligned} n_{1/2}^{-1} n_1^{-1} \exp(tY_{-\gamma}) n_1 n_{1/2} &= n_{1/2}^{-1} \exp(tY_{-\gamma} - tc_{\omega,-\gamma} z Y_{\omega-\gamma}) n_{1/2} \\ &= \exp\Big(tY_{-\gamma} - tc_{\omega,-\gamma} z Y_{\omega-\gamma} - t \sum_{\alpha \in \Sigma_{1/2}} y_\alpha [Y_\alpha, Y_{-\gamma}] \\ &\quad + tz \sum_{\alpha \in \Sigma_{1/2}} c_{\omega,-\gamma} y_\alpha [Y_\alpha, Y_{\omega-\gamma}] + \frac{t}{2} \sum_{\alpha_1, \alpha_2 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} [Y_{\alpha_2}, [Y_{\alpha_1}, Y_{-\gamma}] \\ &\quad - \frac{t}{6} \sum_{\alpha_1, \alpha_2, \alpha_3 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} \big[Y_{\alpha_3}, [Y_{\alpha_2}, [Y_{\alpha_1}, Y_{-\gamma}]]\big]\Big). \end{aligned}$$

Since $(-\gamma+\alpha,\omega) = -(\gamma,\omega)+(\alpha,\omega) = -\frac{1}{2}(\omega,\omega)+\frac{1}{2}(\omega,\omega) = 0$ for every $\alpha \in \Sigma_{1/2}$, it follows that $-\gamma+\alpha$ is either in $\pm\Sigma_0$ or 0, or not a root. This implies that the bracket $[Y_{\alpha_1}, Y_{-\gamma}]$ is respectively in $\mathfrak{n}_0$, $\mathfrak{a}$ or zero. Then (12) yields the desired expression for $p^{-\gamma}(n)$, since the Jacobi identity implies that $c_{\omega,-\gamma}c_{\gamma,\omega-\gamma} = -\omega(H_\gamma)$.

(vi) Notice that in order to obtain ω we must add to $-\omega$ exactly four roots in $\Sigma_{1/2}$. We have

$$
\begin{aligned}
&n_{1/2}^{-1}n_1^{-1}\exp tX_{-\omega}n_1n_{1/2} = n_{1/2}^{-1}\exp\left(tX_{-\omega} - tzH_\omega - \frac{t}{2}z^2\omega(H_\omega)Z\right)n_{1/2}\\
&= \exp\left(-\frac{t}{2}z^2\omega(H_\omega)Z\right)\exp\left(tX_{-\omega} - tzH_\omega - t\sum_{\alpha\in\Sigma_{1/2}} c_{\alpha,-\omega}y_\alpha Y_{-\omega+\alpha}\right.\\
&\quad - tz\sum_{\alpha\in\Sigma_{1/2}}\alpha(H_\omega)y_\alpha Y_\alpha + \frac{t}{2}\sum_{\alpha_1,\alpha_2\in\Sigma_{1/2}} y_{\alpha_1}y_{\alpha_2}\big[Y_{\alpha_2},[Y_{\alpha_1},X_{-\omega}]\big]\\
&\quad + \frac{t}{2}z\sum_{\alpha\in\Sigma_{1/2}} c_{\omega-\alpha,\alpha}\alpha(H_\omega)y_\alpha y_{\omega-\alpha}Z - \frac{t}{6}\sum_{\alpha_1,\alpha_2,\alpha_3\in\Sigma_{1/2}} y_{\alpha_1}y_{\alpha_2}y_{\alpha_3}\big[Y_{\alpha_3},[Y_{\alpha_2},[Y_{\alpha_1},X_{-\omega}]]\big]\\
&\quad \left.+ \frac{t}{24}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\Sigma_{1/2}} y_{\alpha_1}y_{\alpha_2}y_{\alpha_3}y_{\alpha_4}\big[Y_{\alpha_4},[Y_{\alpha_3},[Y_{\alpha_2},[Y_{\alpha_1},X_{-\omega}]]]\big]\right)\\
&= \exp\Bigg(\bigg(-\frac{t}{2}z^2\omega(H_\omega) + \frac{t}{2}z\sum_{\alpha\in\Sigma_{1/2}} c_{\omega-\alpha,\alpha}\alpha(H_\omega)y_\alpha y_{\omega-\alpha}\\
&\quad + \frac{t}{24}\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\Sigma_{1/2}} c_{\alpha_1,-\omega}c_{\alpha_2,-\omega+\alpha_1}\\
&\qquad \times c_{\alpha_3,-\omega+\alpha_1+\alpha_2}c_{\alpha_4,-\omega+\alpha_1+\alpha_2+\alpha_3}y_{\alpha_1}y_{\alpha_2}y_{\alpha_3}y_{\alpha_4}\bigg)Z\Bigg)\cdots.
\end{aligned}
$$

Therefore (vi) follows. □

5. Example

We consider $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$, the simple Lie algebra of real 3×3 matrices with zero trace. Its Iwasawa nilpotent Lie algebra $\mathfrak{n}$ is given by the matrices

$$
\nu(x,y,z) = \begin{bmatrix} 0 & x & z\\ 0 & 0 & y\\ 0 & 0 & 0\end{bmatrix},
$$

for x, y and z in $\mathbb{R}$. Notice that this is the Lie algebra of the three dimensional Heisenberg group. Take α and β to be the simple roots relative to the standard Cartan subspace $\mathfrak{a}$ of $\mathfrak{sl}(3,\mathbb{R})$ of diagonal matrices: $\alpha(\operatorname{diag}(a,b,c)) = (a-b)$ and $\beta((\operatorname{diag}(a,b,c)) = (b-c)$. Then

$$\mathfrak{g}_\alpha = \{\nu(x,0,0) : x \in \mathbb{R}\},$$
$$\mathfrak{g}_\beta = \{\nu(0,y,0) : y \in \mathbb{R}\},$$
$$\mathfrak{g}_{\alpha+\beta} = \{\nu(0,0,z) : z \in \mathbb{R}\},$$

where $\alpha+\beta$ is the highest root also denoted ω. The Lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{a} \oplus \theta(\mathfrak{g}_\alpha) \oplus \theta(\mathfrak{g}_\beta) \oplus \theta(\mathfrak{g}_{\alpha+\beta}),$$

where θ is the Cartan involution. We choose the basis of $\mathfrak{n}$ given by $X = \nu(1,0,0)$, $Y = \nu(0,1,0)$ and $Z = \nu(0,0,1)$ and the basis of $\mathfrak{a}$

$$H_\alpha = \operatorname{diag}\left(\frac{1}{2}, -\frac{1}{2}, 0\right)$$
$$H_\beta = \operatorname{diag}\left(0, \frac{1}{2}, -\frac{1}{2}\right).$$

We can complete $\{X, Y, Z, H_\alpha, H_\beta\}$ to a basis of $\mathfrak{sl}(3,\mathbb{R})$ adding $\theta(X) = -X^{tr}$, $\theta(Y) = -Y^{tr}$ and $\theta(Z) = -Z^{tr}$, which are a basis of $\mathfrak{g}_{-\alpha}$, $\mathfrak{g}_{-\beta}$ and $\mathfrak{g}_{-\alpha-\beta}$ respectively. In order to apply the formulas of Proposition 3 to the chosen basis of $\mathfrak{g}$ we need the structure constants, that can be easily computed, and the vector $H_\omega = H_\alpha + H_\beta = \operatorname{diag}(1/2, 0, -1/2)$. The indeterminates of the polynomials are the canonical coordinates $n = (x,y,z) = \exp(zZ)\exp(xX + yY)$. Hence, a straightforward calculation yields the following polynomials.

$$p^\alpha(n) = y, \qquad p^{H_\alpha}(n) = \frac{1}{2}z + \frac{3}{4}xy, \quad p^{-\alpha}(n) = -\frac{1}{2}xz + \frac{1}{12}x^2y,$$
$$p^\beta(n) = -x, \quad p^{H_\beta}(n) = \frac{1}{2}z - \frac{1}{4}xy, \quad p^{-\beta}(n) = -\frac{1}{2}yz + \frac{5}{4}xy^2,$$
$$p^{\alpha+\beta}(n) = 1, \qquad\qquad p^{-\alpha-\beta}(n) = -\frac{1}{2}z^2 - \frac{1}{6}x^2y^2.$$

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