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Uniqueness of meromorphic functions that share three sets II

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Abstract. We employ the notion of weighted sharing of sets to prove three uniqueness theorems which will improve and supplement two recent results of Lü and Xu [11] related to a well known question of Gross.

1. Introduction, definitions and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \quad (r \longrightarrow \infty, r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM. Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S)=\bigcup_{a\in S}\{z : f(z)-a=0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S}\{z : f(z)-a=0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S)=E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S)=\overline{E}_g(S)$, we say

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that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values.

Inspired by the Nevanlinna's five and four value theorem, in 1970s F. Gross and C. C. Yang started to study the similar but more general questions of two functions that share sets of distinct elements instead of values. For instance, they proved that if f and g are two non-constant entire functions and S_1, S_2 and S_3 are three distinct finite sets such that $f^{-1}(S_i) = g^{-1}(S_i)$ for $i = 1, 2, 3$, then $f \equiv g$. In [6] Gross posed the following question:

Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

Fang and Xu [5] considered the case of meromorphic functions and proved the following result.

Theorem A. *Let $S_1 = \{0\}$, $S_2 = \{z : z^3 - z^2 - 1 = 0\}$ and $S_3 = \{\infty\}$. Suppose that f and g are two non-constant meromorphic functions satisfying $\Theta(\infty; f) > \frac{1}{2}$ and $\Theta(\infty; g) > \frac{1}{2}$. If $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ then $f \equiv g$.*

Dealing with the question of Gross in 2002 Qui and Fang [12] and in 2004 Yi and Lin [15] also obtained some different types of results corresponding to Theorem A. In [16] Zhang generalised Theorem A. In 2001 the idea of gradation of sharing of values and sets known as weighted sharing has been introduced in [9], [10] which measures how close a shared value is to being shared IM or to being shared CM. We now give the definition.

Definition 1.1 ([9], [10]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2 ([9]). Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\cup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

The notion of weighted sharing of set has rendered an useful tool in order to deal with the problem of Gross. Recently in [1]–[3], Theorem A has been improved resorting to the above notion.

It is to be noted that to prove the uniqueness of meromorphic functions in Theorem A and all its subsequent improvements some additional suppositions have been taken under consideration. Also in all the stated results no investigation has been carried out on further relaxation of the nature of sharing the set $\{\infty\}$.

In 1999 Fang considered the uniqueness problem of admissible meromorphic function sharing three sets on unit disc without the aid of any extra condition at the expense of allowing the range set S_1 to be slightly modified. Fang [4] proved the following result.

Theorem B. *Let* $S_1 = \{0, \frac{2}{3}\}$, $S_2 = \{z : z^3 - z^2 - 1 = 0\}$ *and* $S_3 = \{\infty\}$. *If* $E_f(S_j, \infty) = E_g(S_j, \infty)$ *for* $j = 1, 2, 3$ *then* $f \equiv g$.

Lü and Xu [11] improved Theorem B for non-constant meromorphic functions in the complex plane by relaxing the nature of sharing of the range sets. They [11] proved the following two theorems.

Theorem C. *Let* S_i *be given as in Theorem B. If* $E_f(S_1, 0) = E_g(S_1, 0)$, $E_f(S_2, 3) = E_g(S_2, 3)$, $E_f(S_3, 1) = E_g(S_3, 1)$, *then* $f \equiv g$.

Theorem D. *Let* S_i *be given as in Theorem B. If* $E_f(S_1, \infty) = E_g(S_1, \infty)$, $E_f(S_2, 2) = E_g(S_2, 2)$, $E_f(S_3, 1) = E_g(S_3, 1)$, *then* $f \equiv g$.

The purpose of the paper is to investigate the possibility of further relaxing the sharing nature of the range sets. The following three theorems are the main results of the paper.

Theorem 1.1. *Let* S_i *be given as in Theorem B. If* $E_f(S_1, 0) = E_g(S_1, 0)$, $E_f(S_2, 3) = E_g(S_2, 3)$, $E_f(S_3, 0) = E_g(S_3, 0)$, *then* $f \equiv g$.

Theorem 1.2. *Let* S_i *be given as in Theorem B. If* $E_f(S_1, 1) = E_g(S_1, 1)$, $E_f(S_2, 2) = E_g(S_2, 2)$, $E_f(S_3, 1) = E_g(S_3, 1)$, *then* $f \equiv g$.

Theorem 1.3. *Let* S_i *be given as in Theorem B. If* $E_f(S_1, 0) = E_g(S_1, 0)$, $E_f(S_2, 2) = E_g(S_2, 2)$, $E_f(S_3, 2) = E_g(S_3, 2)$, *then* $f \equiv g$.

Remark 1.1. Theorem 1.1 and Theorem 1.2 improves Theorem C and Theorem D respectively. Also Theorem 1.3 improves Theorem B.

Though for the standard definitions and notations of the value distribution theory we refer to [7], we now explain some notations which are used in the paper.

Definition 1.3 ([8]). For $a \in \mathbb{C} \cup \{\infty\}$we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a points of f

whose multiplicities are not greater(less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r,a;f \mid \le m)$ ($\overline{N}(r,a;f \mid \ge m)$) are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f \mid < m)$, $N(r,a;f \mid > m)$, $\overline{N}(r,a;f \mid < m)$ and $\overline{N}(r,a;f \mid > m)$ are defined analogously.

Definition 1.4 ([10]). We denote by $N_2(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid \ge 2)$.

Definition 1.5 ([9], [10]). Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and in particular if f and g share (a,p) then $\overline{N}_*(r,a;f,g) \le \overline{N}(r,a;f \mid \ge p+1) = \overline{N}(r,a;g \mid \ge p+1)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$ as follows

$$F = f^3 - f^2, \quad G = g^3 - g^2. \tag{2.1}$$

Henceforth we shall denote by H and Φ the following two functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{2.2}$$

and

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1}. \tag{2.3}$$

Lemma 2.1 ([10], Lemma 1)**.** *Let F, G be two non-constant meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$. Then*

$$N(r,1;F \mid = 1) = N(r,1;G \mid = 1) \le N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *Let S_1, S_2 and S_3 be defined as in Theorem B and F, G be given by* (2.1)*. If for two non-constant meromorphic functions f and g $E_f(S_1,p) = E_g(S_1,p)$, $E_f(S_2,0) = E_g(S_2,0)$, $E_f(S_3,0) = E_g(S_3,0)$, where $0 \le p < \infty$ and $H \not\equiv 0$ then*

$$N(r,H) \le \overline{N}(r,0;f \mid \ge p+1) + \overline{N}\left(r,f;\frac{2}{3} \mid \ge p+1\right) + \overline{N}_*(r,1;F,G)$$
$$+\overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'),$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f\big(f-\frac{2}{3}\big)(F-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. First we note that

$$H=\frac{f'}{f-\frac{2}{3}}-\frac{g'}{g-\frac{2}{3}}+\frac{f'}{f}-\frac{g'}{g}+\frac{f''}{f'}-\frac{g''}{g'}-\left(\frac{2F'}{F-1}-\frac{2G'}{G-1}\right).$$

Since $E_f(S_1,0)=E_g(S_1,0)$ it follows that if z_0 be 0-point of f (g) then either $g(z_0)=0$ $(f(z_0)=0)$ or $g(z_0)=\frac{2}{3}$ $\big(f(z_0)=\frac{2}{3}\big)$. Clearly F and G share $(1,0)$. Since H has only simple poles, the lemma can easily be proved by simple calculation. □

Lemma 2.3. *Let f and g be two meromorphic meromorphic functions sharing $(1,m)$, where $1\le m<\infty$. Then*

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)$$
$$\le\frac{1}{2}\left[N(r,1;f)+N(r,1;g)\right]$$

Proof. Let z_0 be a 1- point of f of multiplicity p and a 1-point of g of multiplicity q.

Since f, g share $(1,m)$, we note that the 1-points of f and g upto multiplicity m are same and as a result when $p=q\le m$, z_0 is counted $\min(2,p)$ times in the left hand side of the above inequality whereas it is counted p times in the right hand side of the same. If $p=m+1$ then the possible values of q are as follows. (i) $q=m+1$, (ii) $q\ge m+2$. When $p=m+2$ then q can take the following possible values (i) $q=m+1$, (ii) $q=m+2$, (iii) $q\ge m+3$. Similar explanations hold if we interchange p and q. Clearly when $p=q\ge m+1$, z_0 is counted 2 times in the left hand side and $p\ge m+1$ times in the right hand side of the above inequality. When $p>q\ge m+1$, in view of Definition 1.5 we know z_0 is counted $m+\frac{3}{2}$ times in the left hand side and $\frac{p+q}{2}\ge m+\frac{3}{2}$ times in the right hand side of the above inequality. When $q>p$ we can explain similarly. Hence the lemma follows. □

Lemma 2.4 ([13])**.** *Let f be a nonconstant meromorphic function and $P(f)=a_0+a_1f+a_2f^2+\ldots+a_nf^n$, where $a_0,a_1,a_2\ldots,a_n$ are constants and $a_n\ne 0$. Then $T(r,P(f))=nT(r,f)+O(1)$.*

Lemma 2.5. *Let S_1, S_2 and S_3 be defined as in Theorem B and F, G be given by* (2.1)*. If for two nonconstant meromorphic functions f and g $E_f(S_1,p)=$*

$E_g(S_1,p)$, $E_f(S_2,m)=E_g(S_2,m)$, $E_f(S_3,k)=E_g(S_3,k)$ *and* $\Phi\not\equiv 0$ *then*

$$\begin{aligned}&(2p+1)\left\{\overline{N}\left(r,0;f\mid\geq p+1\right)+\overline{N}\left(r,\frac{2}{3};f\mid\geq p+1\right)\right\}\\&\quad\leq \overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;f,g)+S(r,f)+S(r,g).\end{aligned}$$

Proof. By the given condition clearly F and G share $(1,m)$. Also we see that

$$\Phi=\frac{3f\big(f-\frac{2}{3}\big)f'}{F-1}-\frac{3g\big(g-\frac{2}{3}\big)g'}{G-1}.$$

Let z_0 be a zero or a $\frac{2}{3}$- point of f with multiplicity r. Since $E_f(S_1,p)=E_g(S_1,p)$ then that would be a zero of Φ of multiplicity $2r-1$ if $r\leq p$ and a zero of at least $2(p+1)-1=2p+1$ if $r>p$. So using Lemma 2.4 by a simple calculation we can write

$$\begin{aligned}&(2p+1)\left\{\overline{N}(r,0;f\mid\geq p+1)+\overline{N}\left(r,\frac{2}{3};f\mid\geq p+1\right)\right\}\\&\quad\leq N(r,0;\Phi)\leq T(r,\Phi)\leq N(r,\infty;\Phi)+S(r,F)+S(r,G)\\&\quad\leq \overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;f,g)+S(r,f)+S(r,g).\end{aligned}$$

□

Lemma 2.6 ([14], Lemma 6)**.** *If* $H\equiv 0$, *and* F, G *share* $(\infty,0)$ *then* F, G *share* $(1,\infty)$ *and* (∞,∞).

Lemma 2.7. *Let* S_1, S_2 *and* S_3 *be defined as in Theorem B and* F, G *be given by* (2.1). *If for two nonconstant meromorphic functions* f *and* g $E_f(S_1,p)=E_g(S_1,p)$, $E_f(S_2,m)=E_g(S_2,m)$, $E_f(S_3,k)=E_g(S_3,k)$, *where* $0\leq p<\infty$, $2\leq m<\infty$ *and* $H\not\equiv 0$. *Then*

$$\begin{aligned}&4\{T(r,f)+T(r,g)\}\\&\quad\leq 2\left\{\overline{N}(r,0;f)+\overline{N}\left(r,\frac{2}{3};f\right)\right\}+\overline{N}(r,0;f\mid\geq p+1)+\overline{N}\left(r,\frac{2}{3};f\mid\geq p+1\right)\\&\qquad+2\overline{N}(r,\infty;f)+\overline{N}_*(r,\infty;f,g)+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]\\&\qquad-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Proof. By the second fundamental theorem we get

$$4\{T(r,f)+T(r,g)\}\leq \overline{N}(r,1;F)+\overline{N}(r,0;f)+\overline{N}\left(r,\frac{2}{3};f\right)+\overline{N}(r,\infty;f)$$

$$+\overline{N}(r,1;G)+\overline{N}(r,0;g)+\overline{N}\left(r,\frac{2}{3};g\right)+\overline{N}(r,\infty;g)$$
$$-N_0(r,0;f')-N_0(r,0;g')+S(r,f)+S(r,g). \tag{2.4}$$

Using Lemmas 2.1, 2.2, 2.3 and 2.4 we see that

$$\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\le \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+N(r,1;F\mid =1)\\
&-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\le \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+\overline{N}(r,0;f\mid \ge p+1)\\
&+\overline{N}\left(r,\frac{2}{3};f\mid \ge p+1\right)+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\
&+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r,f)+S(r,g).
\end{aligned} \tag{2.5}$$

Using (2.5) in (2.4) and noting that $\overline{N}(r,0;f)+\overline{N}\left(r,\frac{2}{3};f\right)=\overline{N}(r,0;g)+\overline{N}\left(r,\frac{2}{3};g\right)$ and $\overline{N}(r,\infty;f)=\overline{N}(r,\infty;g)$ the lemma follows. □

Lemma 2.8 ([11]). *Let S_1, S_2 and S_3 be defined as in Theorem B. If for two nonconstant meromorphic functions f and g $E_f(S_1,0)=E_g(S_1,0)$, $E_f(S_3,0)=E_g(S_3,0)$, then $S(r,f)=S(r,g)$.*

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then F and G share $(1,3)$ and f, g share $(\infty,1)$. We consider the following cases.

Case 1. Suppose that $\Phi\not\equiv 0$.

Subcase 1.1. Let $H\not\equiv 0$. Then using Lemma 2.4 and Lemma 2.7 with $m=3$, $p=0$ and $k=1$ and Lemma 2.5 with $p=0$ in view of Definition 1.5 we obtain

$$\begin{aligned}
&4\{T(r,f)+T(r,g)\}\le 3\left\{\overline{N}(r,0;f)+\overline{N}\left(r,\frac{2}{3};f\right)\right\}+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
&+\overline{N}(r,\infty;f\mid\ge 2)+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]-\frac{3}{2}\overline{N}_*(r,1;F,G)+S(r,f)\\
&+S(r,g)\le \frac{3}{2}\overline{N}(r,1;F\mid\ge 4)+N_2(r,\infty;f)+N_2(r,\infty;g)\\
&+2\overline{N}(r,\infty;f\mid\ge 2)+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+S(r,f)+S(r,g)\\
&\le \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+\frac{3}{4}\overline{N}(r,1;F\mid\ge 4)+\frac{3}{4}\overline{N}(r,1;G\mid\ge 4)
\end{aligned}$$

$$+ N_2(r,\infty;f) + N_2(r,\infty;g) + \overline{N}(r,\infty;f\mid\geq 2) + \overline{N}(r,\infty;g\mid\geq 2)$$
$$+ S(r,f) + S(r,g) \leq \frac{33}{16}\{N(r,1;f) + N(r,1;g)\}$$
$$+ \frac{3}{2}\{N(r,\infty;f) + N(r,\infty;g)\} + S(r,f) + S(r,g)$$
$$\leq \frac{57}{16}\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \tag{3.1}$$

(3.1) gives a contradiction.
Subcase 1.2. Let $H \equiv 0$. From (2.2) we have

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.2}$$

where ($A \neq 0$), B are constants. Then by Lemma 2.6 F, G share $(1,\infty)$ and f, g share (∞,∞). Hence by Lemma 2.5 we get $\overline{N}(r,0;f) + \overline{N}\big(r,\frac{2}{3};f\big) = S(r,f) + S(r,g)$. So when ∞ is a Picard exceptional value, by the second fundamental theorem and Lemma 2.8 we see that

$$T(r,f) \leq \overline{N}(r,0;f) + \overline{N}\left(r,\frac{2}{3};f\right) + \overline{N}(r,\infty;f) + S(r,f) \leq S(r,f),$$

a contradiction. Now suppose ∞ is not a Picard exceptional value. We now follow the same procedure as adopted in [11]. For the sake of convenience we are giving the outline of the proof. To this end let us suppose there is a point $z_0 \in \mathbb{C}$ such that

$$f(z_0) = \infty.$$

Substituting z_0 in (3.2) we get $B = 0$ and so

$$A(f^3 - f^2 - 1) = (g^3 - g^2 - 1). \tag{3.3}$$

Clearly from above and the fact F and G share $(1,\infty)$ we get $E_f(S_j,\infty) = E_g(S_j,\infty)$ for $j = 1,2,3$ and so from Theorem A we have $f \equiv g$.

Case 2. Suppose that $\Phi \equiv 0$. Now from (2.3) we get (3.3). Now proceeding the same way as done in Subcase 1.2 we can prove $f \equiv g$. □

Proof of Theorem 1.2. Let F, G be given by (2.1). Then F and G share $(1,2)$, and f, g share $(\infty,1)$. We consider the following cases.

Case 1. Suppose that $\Phi \not\equiv 0$.
Subcase 1.1. Let $H \not\equiv 0$. Then using Lemma 2.4 and Lemma 2.7 with $m = 2$,

$p = 1$ and $k = 1$ and Lemma 2.5 with $p = 0$, $p = 1$ and in view of Definition 1.5 we obtain

$$
\begin{aligned}
4\{T(r,f)+T(r,g)\} &\le 2\left\{\overline{N}(r,0;f)+\overline{N}\left(r,\frac{2}{3};f\right)\right\}+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
&\quad+\overline{N}(r,\infty;f\mid\ge 2)+\overline{N}(r,0;f\mid\ge 2)+\overline{N}\left(r,\frac{2}{3};f\mid\ge 2\right)\\
&\quad+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]-\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\le N_2(r,\infty;f)+N_2(r,\infty;g)+\frac{4}{3}\overline{N}(r,\infty;f\mid\ge 2)+\frac{11}{6}\overline{N}(r,1;F\mid\ge 3)\\
&\quad+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+S(r,f)+S(r,g)\\
&\le \frac{29}{12}\{N(r,1;f)+N(r,1;g)\}+\frac{4}{3}\{N(r,\infty;f)+N(r,\infty;g)\}+S(r,f)+S(r,g)\\
&\le \frac{45}{12}\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \qquad (3.4)
\end{aligned}
$$

(3.4) gives a contradiction.

Subcase 1.2. Let $H \equiv 0$. We now omit the proof since the rest of the proof is similar to that of Theorem 1.1. □

Proof of Theorem 1.3. Let F, G be given by (2.1). Then F and G share $(1,2)$, and f, g share $(\infty,4)$. We consider the following cases.

Case 1. Suppose that $\Phi \not\equiv 0$.

Subcase 1.1. Let $H \not\equiv 0$. Then using Lemma 2.4 and Lemma 2.7 with $m = 2$, $p = 0$ and $k = 2$ and Lemma 2.5 with $p = 0$, in view of Definition 1.5 we obtain

$$
\begin{aligned}
4\{T(r,f)+T(r,g)\} &\le 3\left\{\overline{N}(r,0;f)+\overline{N}\left(r,\frac{2}{3};f\right)\right\}+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
&\quad+\overline{N}(r,\infty;f\mid\ge 3)+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]-\frac{1}{2}\overline{N}_*(r,1;F,G)\\
&\quad+S(r,f)+S(r,g)\le\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+2\overline{N}(r,\infty;f\mid\ge 3)\\
&\quad+2\overline{N}(r,\infty;g\mid\ge 3)+\frac{5}{2}\overline{N}(r,1;F\mid\ge 3)+\frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]\\
&\quad+S(r,f)+S(r,g)\le\frac{33}{12}\{N(r,1;f)+N(r,1;g)\}+\{N(r,\infty;f)+N(r,\infty;g)\}\\
&\quad+S(r,f)+S(r,g)\le\frac{45}{12}\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \qquad (3.5)
\end{aligned}
$$

(3.5) gives a contradiction.

Subcase 1.2. Let $H \equiv 0$. We now omit the proof since the rest of the proof is similar to that of Theorem 1.1. □

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