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Bi-invariant Randers metrics on Lie groups

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Abstract. In this paper we study the geometry of Lie groups with bi-invariant Randers metric. We first give a necessary and sufficient condition for a left invariant Randers metric to be of Berwald type. We then prove that bi-invariant Randers metrics are of Berwald type. We give an explicit formula for the flag curvature of bi-invariant Randers metrics. Finally a necessary and sufficient condition that left invariant Randers metrics on Lie groups are bi-invariant is given.

1. Introduction

In the past several years, we witness a rapid development in Finsler geometry. This is partially due to the study of some special class of Finsler metrics. Randers spaces are special Finsler spaces, rather close to Riemannian spaces.

A Randers metric on a manifold M is a Finsler metric defined in the following form:

$$F = \alpha + \beta$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M.

Randers metrics were first introduced by physicist G. Randers in 1941 ([13]) from the standpoint of general relativity. Later on, these metrics were applied to the theory of electron microscope by R. S. Ingarden who first named them Randers metrics. Randers metrics also arise naturally from the navigation problem on a Riemannian space (M, h) under the influence of an external force field W [2].

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Not only physicists, but also pure geometers started to show interest in the subject, because Randers manifolds supply one of the most basic examples of Finsler manifolds: by adding a 1-form, their fundamental function perturbs the fundamental function of a Riemannian manifold. A lot of invariants in Finsler geometry were explicitly calculated for the first time for Randers manifolds. For a general survey of results and applications of Randers manifolds, we refer to [1], [3], [8], [11].

The study of invariant structures on Lie groups and homogeneous spaces is an important problem in differential geometry. Lie groups are, in a sense, the nicest examples of manifolds and are good spaces on which to test conjectures. Therefore it is important to study invariant Finsler metrics. S. Deng and Z. Hou studied invariant Finsler metrics on homogeneous spaces and gave some descriptions of these metrics [4], [5]. There are some recent papers on invariant Finsler metrics on two-step nilpotent Lie groups [14] and homogeneous manifolds [6]. Also, in [9], [10] we have studied the homogeneous Finsler spaces and the homogeneous geodesics in homogeneous Finsler spaces.

Among the invariant metrics the bi-invariant ones are the simplest kind. They have nice and simple geometric properties, but still form a large enough class to be of interest. In [5] S. Deng and Z. Hou gave an algebraic description of bi-invariant Finsler metrics and in [10] we prove that the geodesics on a Lie group, relative to a bi-invariant Finsler metric, are the cosets of the one-parameter subgroups.

The purpose of this paper is to study the geometric properties of bi-invariant Randers metrics on Lie groups. A necessary and sufficient condition that left-invariant Randers metrics are of Berwald type is given. We prove that bi-invariant Randers metrics are of Berwald type. We give an explicit formula for the flag curvature of bi-invariant Randers metrics. A necessary and sufficient condition that left invariant Randers metrics on Lie groups are bi-invariant is given. This paper provided a convenient method to construct bi-invariant Randers metrics on Lie groups. Our result provides many new examples of Randers space of Berwald type which is neither Riemannian nor locally Minkowskian.

2. Left invariant Randers metrics on Lie groups

A Finsler metric on a manifold M is a continuous function, $F : TM \longrightarrow [0,\infty)$ differentiable on $TM \setminus \{0\}$ and satisfying three conditions:

(a) $F(y) = 0$ if and only if $y = 0$;

(b) $F(\lambda y) = \lambda F(y)$ for any $y \in T_xM$ and $\lambda > 0$;

(c) For any non-zero $y \in T_xM$, the symmetric bilinear form $g_y : T_xM \times T_xM \longrightarrow R$ given by

$$\mathrm{g}_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)] \mid_{s=t=0}$$

is positive definite.

For each $y \in T_xM - \{0\}$, define

$$C_y(u,v,w) = \frac{1}{4}\frac{\partial^3}{\partial s \partial t \partial r}[F^2(y + su + tv + rw)] \mid_{s=t=r=0} .$$

C is called the Cartan torsion.

Let G be a connected Lie group with Lie algebra $\mathfrak{g} = T_eG$. We may identify the tangent bundle TG with $G \times \mathfrak{g}$ by means of the diffeomorphism that sends (g, X) to $(L_g)_*X \in T_gG$.

Definition 2.1. A Finsler function $F : TG \longrightarrow R_+$ will be called G-invariant if F is constant on all G-orbits in $TG = G \times \mathfrak{g}$; that is, $F(g,X) = F(e,X)$ for all $g \in G$ and $X \in \mathfrak{g}$.

The G-invariant Finsler functions on TG may be identified with the Minkowski norms on $\mathfrak{g}$. If $F : TG \longrightarrow R_+$ is an G-invariant Finsler function, then we may define $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$ by $\widetilde{F}(X) = F(e,X)$, where e denotes the identity in G. Conversely, if we are given a Minkowski norm $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$, then $\widetilde{F}$ arises from an G-invariant Finsler function $F : TG \longrightarrow R_+$ given by $F(g,X) = \widetilde{F}(X)$ for all $(g,X) \in G \times \mathfrak{g}$.

Randers metrics are built from a Riemannian metric $a = a_{ij}dx^i \otimes dx^j$, and a 1-form $b = b_i dx^i$, both living globally on the smooth n-dimensional manifold M. The Finsler function of Randers type has the defining form $F = \alpha + \beta$, where

$$\alpha(x,y) = \sqrt{a_{ij}(x,y)y^iy^j}, \quad \beta(x,y) = b_i(x)y^i.$$

For Randers metrics, strong convexity holds if and only if $\|b\| < 1$; see [1]. The Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ induces the musical bijections between 1-forms and vector fields on M. In this way the 1-form b corresponds to a vector field $b^\sharp$ on M. Thus a Randers metric F with Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and 1-form b can be showed by

$$F(x,y) = \sqrt{a_x(y,y)} + a_x(b^\sharp, y) \qquad x \in M, y \in T_xM,$$

where $a_x(b^\sharp, b^\sharp) < 1 \ \forall x \in M$.

3. Bi-invariant Randers metrics on Lie groups

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $F(x,y) = \sqrt{a_x(y,y)} + a_x(X,y)$ be a left invariant Randers metric on G. It is easy to check that the underlying Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the vector field X are also left invariant.

It is known that the Randers metric defined by Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and 1-form b is of Berwald type if and only if b is parallel with respect to the Levi–Civita connection induced by the Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ (see [1]). It is obvious that b is parallel if and only if the corresponding vector field $b^\sharp$ is parallel with respect to a.

The following proposition is a simple reformation of the well-known characterization of Randers spaces being Berwald.

Proposition 3.1. *Let G be a Lie group with a left-invariant Randers metric F defined by the Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the vector field X. Then the Randers metric F is of Berwald type if and only if ad_X is skew-adjoint with respect to a and $a(X, [\mathfrak{g}, \mathfrak{g}]) = 0$.*

Corollary 3.1. *Let G be a Lie group with a left-invariant Randers metric F defined by the Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the vector field X. If the Randers metric F is of Berwald type then the one-parameter subgroup $t \longrightarrow \exp(tX)$ is a geodesic of F.*

Proof. The corollary is a consequence of Proposition 3.1 and Theorem 3.1 of [10]. □

A geodesic $\gamma(t)$ through the origin o of $M = \frac{G}{H}$ is called homogeneous if it is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tZ)(o), \qquad t \in R,$$

where Z is a nonzero vector in the Lie algebra $\mathfrak{g}$ of G. For results on homogeneous geodesics in homogeneous Finsler manifolds we refer to [10].

Corollary 3.2. *Let G be a Lie group with a left-invariant Randers metric F defined by the Riemannian metric $a = a_{ij}dx_i \otimes dx_j$ and the vector field X. If the Randers metric F is of Berwald type then the Ricci curvature of a in the direction $u = \frac{X}{\sqrt{a(X,X)}}$ is zero.*

Proof. Let G be a Lie group endowed with a left-invariant Riemannian metric a. If $x \in \mathfrak{g}$ is a-orthogonal to the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$, then $\mathrm{Ricci}\,(x) \leq$

0, with equality if and only if ad_x is skew-adjoint with respect to a [12]. So the corollary is a direct consequence of Proposition 3.1. $\square$

Some Lie groups may posses a Randers metric which is invariant not only under left translation but also under right translation. In the following theorem we show that bi-invariant Randers metrics are of Berwald type.

Theorem 3.1. *Let G be a Lie group with a bi-invariant Randers metric F defined by the Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the vector field X. Then the Randers metric F is of Berwald type.*

Proof. Let $F(p,y) = \sqrt{a_p(y,y)} + a_p(X,y)$.
Now for $s,t \in R$

$$F^2(y+su+tv) = a(y+su+tv, y+su+tv) + a^2(X, y+su+tv) + 2\sqrt{a(y+su+tv, y+su+tv)}a(X, y+su+tv).$$

By definition

$$\mathrm{g}_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial r \partial s}F^2(y+ru+sv)\mid_{r=s=0}.$$

So by a direct computation we get

$$\mathrm{g}_y(u,v) = a(u,v) + a(X,u)a(X,v) + \frac{a(u,v)a(X,y)}{\sqrt{a(y,y)}} - \frac{a(v,y)a(u,y)a(X,y)}{a(y,y)\sqrt{a(y,y)}} + \frac{a(X,v)a(u,y)}{\sqrt{a(y,y)}} + \frac{a(X,u)a(v,y)}{\sqrt{a(y,y)}}. \tag{1}$$

So for all $y, z \in \mathfrak{g}$ we have

$$\begin{aligned}\mathrm{g}_y(y,[y,z]) &= a(y,[y,z]) + a(X,y)a(X,[y,z]) \\ &\quad + \frac{a(y,[y,z])a(X,y)}{\sqrt{a(y,y)}} + a(X,[y,z])\sqrt{a(y,y)} \\ &= a(y,[y,z])\left(1 + \frac{a(X,y)}{\sqrt{a(y,y)}}\right) + a(X,[y,z])\left(a(X,y) + \sqrt{a(y,y)}\right).\end{aligned}$$

So we have

$$\mathrm{g}_y(y,[y,z]) = a(y,[y,z])\left(\frac{F(y)}{\sqrt{a(y,y)}}\right) + a(X,[y,z])F(y) \tag{2}$$

Since a is bi-invariant, $a(y,[y,z]) = 0$ and $ad(x)$ is skew-adjoint for every $x \in \mathfrak{g}$. Since F is bi-invariant, $\mathrm{g}_y(y,[y,z]) = 0$ [10]. So From (2) we get $a(X,[y,z]) = 0$ for all $y, z \in \mathfrak{g}$. Therefore, by Proposition 3.1, we see that (G,F) is of Berwald type. $\square$

Corollary 3.3. *Let G be a Lie group with a bi-invariant Randers metric F then*

(a) *The Chern–Rund connection of (G, F) is given by $\nabla_Y Z = \frac{1}{2}[Y, Z]$ for all $Y, Z \in \mathfrak{g}$.*

(b) *The geodesic of (G, F) starting at e are the one-parameter subgroup of G.*

Proof. For Randers spaces of Berwald type the Riemannian connection and the Chern–Rund connection coincide, [1], so (a) is a direct consequence of Theorem 3.1. The direct proof of (b) can be found in our previous paper [10]. □

By a Routine calculation we obtain the following corollary.

Corollary 3.4. *Let G be a Lie group with a bi-invariant Randers metric F defined by the Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the vector field X. Let (P, y) be a flag in $\mathfrak{g}$ such that $\{y, u\}$ is an orthonormal basis of P with respect to $a = a_{ij}dx^i \otimes dx^j$. Then the flag curvature of the flag (P, y) in $\mathfrak{g}$ is given by*

$$K(P, y) = \frac{1}{(1 + a(X, y))^2}\frac{1}{4}\|[u, y]\|^2 + \frac{a(X, u)}{(1 + a(X, y))^3}\frac{1}{4}a([u, y], [X, y]),$$

where $\|[u, y]\|$ denotes the norm of $[u, y]$ with respect to $a = a_{ij}dx^i \otimes dx^j$.

Corollary 3.4 can be found in [6] in a more general setting. For results on Flag curvature of invariant Randers metrics on homogeneous manifolds we refer to [6].

In the following a necessary and sufficient condition that left invariant Randers metrics on Lie groups are bi-invariant is given. This result is well known for Riemannian manifolds.

Theorem 3.2. *Let G be a connected Lie group with a left invariant Randers metric F defined by the Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the the vector field X. Then the Randers metric F is right invariant, hence bi-invariant if and only if $ad(x)$ is skew-adjoint with respect to the $a = a_{ij}dx^i \otimes dx^j$ for every $x \in \mathfrak{g}$ and $a(X, [\mathfrak{g}, \mathfrak{g}]) = 0$.*

Proof. Suppose that $ad(x)$ is skew-adjoint for every $x \in \mathfrak{g}$ and $a(X, [\mathfrak{g}, \mathfrak{g}]) = 0$. We show that the left invariant Randers metric F is also right invariant. We note, by using Proposition 3.1, that (G, F) is of Berwald type. According to the Formula (1) we get

$$\mathrm{g}_y([z, u], v) = a([z, u], v)\left(1 + \frac{a(X, y)}{\sqrt{a(y, y)}}\right) + a([z, u], y)\left(\frac{a(X, v)}{\sqrt{a(y, y)}} - \frac{a(v, y)a(X, y)}{a(y, y)\sqrt{a(y, y)}}\right)$$

$$\mathrm{g}_y(u,[z,v]) = a(u,[z,v])\left(1+\frac{a(X,y)}{\sqrt{a(y,y)}}\right) + a([z,v],y)\left(\frac{a(X,u)}{\sqrt{a(y,y)}} - \frac{a(u,y)a(X,y)}{a(y,y)\sqrt{a(y,y)}}\right).$$

By definition

$$C_y(z,u,v) = \frac{1}{2}\frac{d}{dt}[\mathrm{g}_{y+tv}(z,u)]|_{t=0}.$$

So by a direct computation we get

$$\begin{aligned} C_y([z,y],u,v) = a([z,y],u) &\left(\frac{a(X,v)}{\sqrt{a(y,y)}} - \frac{a(y,v)a(X,y)}{a(y,y)\sqrt{a(y,y)}}\right) \\ &+ a([z,y],v)\left(\frac{a(X,u)}{\sqrt{a(y,y)}} - \frac{a(X,y)a(u,y)}{a(y,y)\sqrt{a(y,y)}}\right). \end{aligned}$$

Therefore

$$\begin{aligned} &\mathrm{g}_y([z,u],v) + \mathrm{g}_y(u,[z,v]) + 2C_y([z,y],u,v) \\ &\quad = (a([z,u],v) + a(u,[z,v]))\left(1 + \frac{a(X,y)}{\sqrt{a(y,y)}}\right) \\ &\qquad + (a([z,u],y) + a([z,y],u))\left(\frac{a(X,v)}{a(y,y)} - \frac{a(v,y)a(X,y)}{a(y,y)\sqrt{a(y,y)}}\right) \\ &\qquad + (a([z,v],y) + a([z,y],v))\left(\frac{a(X,u)}{\sqrt{a(y,y)}} - \frac{a(X,y)a(u,y)}{a(y,y)\sqrt{a(y,y)}}\right). \end{aligned}$$

Since $ad(x)$ is skew-adjoint for every $x \in \mathfrak{g}$ we have

$$\mathrm{g}_y([z,u],v) + \mathrm{g}_y(u,[z,v]) + 2C_y([z,y],u,v) = 0.$$

Now, we consider the function

$$\psi(t) = \mathrm{g}_{Ad(\exp(tz))y}(Ad(\exp(tz))u, Ad(\exp(tz)v)).$$

Taking the derivative with respect to t, we see that $\psi'(t) = 0$ therefor $\psi(t) = \psi(0)$, $\forall t \in R$. Since G is connected we have

$$\mathrm{g}_y(u,v) = \mathrm{g}_{Ad(g)y}(Ad(g)u, Ad(g)v), \quad \forall g \in G.$$

Using the formula $F(y) = \sqrt{\mathrm{g}_y(y,y)}$, we have $F(Ad(g)u) = F(u)$ for all $g \in G$, $u \in \mathfrak{g}$. Therefore

$$F((R_{g^{-1}})_* u) = F(u), \quad \forall g \in G,\ u \in \mathfrak{g}. \qquad \square$$

Remark 3.1. Let $\mathfrak{g}$ be a real Lie algebra, F be a Minkowski norm on $\mathfrak{g}$. Then $\{\mathfrak{g}, F\}$ is called a Minkowski Lie algebra if the following condition is satisfied:

$$\mathrm{g}_y([x,u],v) + \mathrm{g}_y(u,[x,v]) + 2C_y([x,y],u,v) = 0,$$

where $y \in \mathfrak{g} - \{0\}, x, u, v \in \mathfrak{g}$. In the proof of Theorem 3.2 we basically show that $\{\mathfrak{g}, F\}$ is a Minkowski Lie algebra and use the method described in [5].

References

[1] D. Bao, S. S. Chern and Z. Shen, An Introduction to Riemann–Finsler geometry, *Springer-Verlag, New York*, 2000.

[2] D. Bao, C. Robles and Z. Shen, Zermelo navigation on Riemannian manifolds, *J. Differential Geom.* **66** (2004), 377–435.

[3] D. Bao, Randers space forms, *Period. Math. Hungar.* **48** (2004), 3–15.

[4] S. Deng and Z. Hou, Invariant Randers metrics on homogeneous Riemannian manifolds, *J. Phys. A: Math. Gen.* **37** (2004), 4353–4360, Corrigendum, *J. Phy. A: Math. Gen.* **39** (2006) 5249–5250.

[5] S. Deng and Z. Hou, Invariant Finsler metrics on homogeneous manifolds, *J. Phys. A: Math. Gen.* **37** (2004), 8245–8253.

[6] E. Esrafilian and H. R. Salimi Moghaddam, Flag curvature of invariant Randers metrics on homogeneous manifolds, *J. Phys. A: Math. Gen.* **39** (2006), 3319–3324.

[7] S. Helgason, Differential geometry, Lie Groups and Symmetric Space, *Academic Press, New York*, 1978.

[8] L. Kozma, On Randers spaces, *Bull. Soc. Sci. Lett. Ldz Ser. Rech. Deform.* **51** (2006), 91–99.

[9] D. Latifi and A. Razavi, On homogeneous Finsler spaces, *Rep. Math. Phys.* **57** (2006), 357–366, Erratum: *Rep. Math. Phys*, **60** (2007) 347.

[10] D. Latifi, Homogeneous geodesics in homogeneous Finsler spaces, *J. Geom. Phys.* **57** (2007), 1421–1433.

[11] M. Matsumoto and H. Shimada, The corrected fundamental theorem on the Randers spaces of constant curvature, *Tensor (N.S)* **63** (2002), 43–47.

[12] J. Milnor, Curvature of left invariant metrics on Lie groups, *Advances in Math.* **21** (1976), 293–329.

[13] G. Randers, On an asymmetrical metric in the four-space of general relativity, *Phys. Rev.* **59** (1941), 195–199.

[14] A. Tóth and Z. Kovács, On the geometry of two-step nilpotent groups with left invariant Finsler metrics, *Acta Math. Acad. Paed. Nyíregyháziensis* **24** (2008), 155–168.

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