

On the supremum of iterated local time

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Dedicated to the 100th anniversary of the birthday of Béla Györfi

Abstract. We obtain upper and lower class integral tests for the space-wise supremum of the iterated local time of two independent Wiener processes. We then establish a strong invariance principle between this iterated local time and the local time process of the simple symmetric random walk on the two-dimensional comb lattice. The latter, in turn, enables us to conclude upper and lower class tests for the local time of simple symmetric random walk on the two-dimensional comb lattice as well.

1. Introduction and main results

Let $\{W(t); t \geq 0\}$ be a standard Wiener process (Brownian motion), i.e., a Gaussian process with

$$E(W(t)) = 0, \quad E(W(t_1)W(t_2)) = \min(t_1, t_2), \quad t, t_1, t_2 \geq 0.$$

The local time process $\{\eta(x, t); x \in \mathbb{R}, t \geq 0\}$ is defined via

$$\int_A \eta(x, t) dx = \lambda \{s : 0 \leq s \leq t, W(s) \in A\} \quad (1.1)$$

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for any $t \geq 0$ and Borel set $A \subset \mathbb{R}$, where $\lambda(\cdot)$ is the Lebesgue measure, and $\eta(\cdot, \cdot)$ is frequently referred to as Wiener or Brownian local time.

Let $\eta_1(x, t)$ and $\eta_2(x, t)$ be two independent Brownian local times. The iterated local time is defined by

$$\Upsilon(x, t) := \eta_1(x, \eta_2(0, t)).$$

Denote

$$\Upsilon^*(t) := \sup_{x \in \mathbb{R}} \Upsilon(x, t). \quad (1.2)$$

First we give asymptotic values for the upper and lower tails of the distribution of $\Upsilon^*(t)$.

Theorem 1.1. *As $z \rightarrow \infty$*

$$P(\Upsilon^*(t) > zt^{1/4}) \sim \frac{2^{11/3} z^{2/3}}{(3\pi)^{1/2}} \exp\left(-\frac{3z^{4/3}}{2^{5/3}}\right) \quad (1.3)$$

and as $z \rightarrow 0$,

$$P(\Upsilon^*(t) < zt^{1/4}) \sim \frac{4z^2}{(2\pi)^{1/2}} \int_0^\infty \frac{G(s)}{s^3} ds, \quad (1.4)$$

for all $t > 0$, where

$$G(s) := P\left(\sup_{x \in \mathbb{R}} \eta(x, 1) < s\right).$$

Note that an explicit formula for $G(s)$ in terms of Bessel functions is given in CSÁKI and FÖLDES [9].

The following integral tests are obtained.

Theorem 1.2. *Let $f(t) > 0$ be a non-decreasing function and put*

$$I(f) := \int_1^\infty \frac{f^2(t)}{t} \exp\left(-\frac{3}{2^{5/3}} f^{4/3}(t)\right) dt.$$

Then

$$P(\Upsilon^*(t) > t^{1/4} f(t) \text{ i.o. as } t \rightarrow \infty) = 0 \text{ or } 1$$

according as $I(f)$ converges or diverges.

Theorem 1.3. *Let $g(t) > 0$ be a non-increasing function and put*

$$J(g) := \int_1^\infty \frac{g^2(t)}{t} dt.$$

Then

$$P(\Upsilon^*(t) < t^{1/4} g(t) \text{ i.o. as } t \rightarrow \infty) = 0 \text{ or } 1$$

according as $J(g)$ converges or diverges.

In particular, we have the following law of the iterated logarithm:

$$\limsup_{t \rightarrow \infty} \frac{\Upsilon^*(t)}{t^{1/4}(\log \log t)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.}$$

To compare the above results with similar integral tests for $\Upsilon(0, t)$, note that $\{\eta(0, t); t \geq 0\}$ has the same distribution as $\{\sup_{0 \leq s \leq t} W(s); t \geq 0\}$. Consequently, $\{\Upsilon(0, t); t \geq 0\}$ has the same distribution as $\{\sup_{0 \leq s \leq t} W_1(\eta_2(0, s)); t \geq 0\}$, or, as easily seen, the same distribution as $\{\sup_{0 \leq s \leq t} \bar{W}_1(W_2(s) \vee 0); t \geq 0\}$. From BERTOIN [2] we obtain the following integral tests.

Theorem A. *Put*

$$\begin{aligned} \hat{I}(f) &:= \int_1^\infty \frac{f^{2/3}(t)}{t} \exp\left(-\frac{3}{2^{5/3}} f^{4/3}(t)\right) dt, \\ \hat{J}(g) &:= \int_1^\infty \frac{g(t)}{t} dt. \end{aligned}$$

Then

$$P(\Upsilon(0, t) > t^{1/4} f(t) \text{ i.o. as } t \rightarrow \infty) = 0 \text{ or } 1$$

according as $\hat{I}(f)$ converges or diverges. Moreover,

$$P(\Upsilon(0, t) < t^{1/4} g(t) \text{ i.o. as } t \rightarrow \infty) = 0 \text{ or } 1$$

according as $\hat{J}(g)$ converges or diverges.

In particular, we have the same law of the iterated logarithm as for $\Upsilon^*(t)$:

$$\limsup_{t \rightarrow \infty} \frac{\Upsilon(0, t)}{t^{1/4}(\log \log t)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.}$$

In the subsequent sections the proofs of Theorem 1.1, 1.2 and 1.3 will be given. In Section 5 we apply the results for the local time of the simple random walk on the 2-dimensional comb.

In the proofs unimportant constants of possibly different positive values will be denoted by c, c_0, c_1, c_2 .

2. Proof of Theorem 1.1

Since

$$\frac{\Upsilon^*(t)}{t^{1/4}} = \frac{\eta_1^*(\eta_2(0, t))}{(\eta_2(0, t))^{1/2}} \sqrt{\frac{\eta_2(0, t)}{t^{1/2}}},$$

it has the same distribution as $\eta_1^*(1)\sqrt{|N|}$, where $\eta_1^*(s) = \sup_{x \in \mathbb{R}} \eta_1(x, s)$ and N is a standard normal random variable independent of $\eta_1^*(1)$. Hence, denoting by φ the standard normal density,

$$P(\Upsilon^*(t) > zt^{1/4}) = 2 \int_0^\infty \left(1 - G\left(\frac{z}{\sqrt{u}}\right)\right) \varphi(u) du. \quad (2.1)$$

For the upper tail of G we have (see CsÁKI [5])

$$1 - G(z) \sim 4\sqrt{\frac{2}{\pi}} z \exp\left(-\frac{z^2}{2}\right), \quad z \rightarrow \infty. \quad (2.2)$$

Now split the integral in (2.1) into three parts:

$$\int_0^\infty = \int_0^{z^{2/3}/2} + \int_{z^{2/3}/2}^{2z^{2/3}} + \int_{2z^{2/3}}^\infty = I_1 + I_2 + I_3.$$

Using (2.2), it is easy to see that

$$I_1 \leq c(1 - G(2^{1/2}z^{2/3})) \leq cz^{2/3} \exp(-z^{4/3}),$$

$$I_3 \leq c \int_{2z^{2/3}}^\infty \varphi(u) du \leq c \exp(-2z^{4/3}),$$

so I_1 and I_3 are negligible compared to (1.3). For I_2 we can use (2.2) and hence

$$I_2 \sim \frac{8}{\pi} \int_{z^{2/3}/2}^{2z^{2/3}} \frac{z}{\sqrt{u}} \exp\left(-\frac{z^2}{2u} - \frac{u^2}{2}\right) du$$

$$= \frac{16z^{4/3}}{\pi} \int_{1/\sqrt{2}}^{\sqrt{2}} \exp\left(-\frac{z^{4/3}}{2} \left(\frac{1}{v^2} + v^4\right)\right) dv.$$

The asymptotic value of this integral can be obtained by Laplace's method (cf., e.g., de BRUIJN [3])

$$\int_a^b \exp(-\lambda h(v)) dv \sim \frac{\sqrt{2\pi} e^{-\lambda h(v_0)}}{\sqrt{\lambda h''(v_0)}}, \quad \lambda \rightarrow \infty,$$

where v_0 is the place of the minimum of h in (a, b) , i.e., $h'(v_0) = 0$. Applying this, a straightforward calculation leads to (1.3).

To see (1.4), we have similarly

$$P(\Upsilon^*(t) < zt^{1/4}) = 2 \int_0^\infty G\left(\frac{z}{\sqrt{u}}\right) \varphi(u) du = 4z^2 \int_0^\infty \frac{G(s)}{s^3} \varphi\left(\frac{z^2}{s^2}\right) ds.$$

This integral is finite, since

$$G(s) \sim c \exp\left(-\frac{2j_1^2}{s^2}\right), \quad s \rightarrow 0,$$

where j_1 is the smallest positive zero of the Bessel function $J_0(\cdot)$ (cf. CSÁKI and FÖLDES [9]).

Since $\varphi(z^2/s^2) \leq \varphi(0)$, we have

$$P(\Upsilon^*(t) < xt^{1/4}) \sim 4z^2\varphi(0) \int_0^\infty \frac{G(s)}{s^3} ds, \quad z \rightarrow 0$$

by the dominated convergence theorem. This completes the proof of Theorem 1.1. \square

3. Proof of Theorem 1.2

From SHI [13] we have the following result.

Lemma A. *Let f be a function as in Theorem 1.2. Put $T_1 = 1$,*

$$T_{k+1} = T_k \left(1 + \frac{1}{f_k^{4/3}}\right), \quad k = 1, 2, \dots,$$

where $f_k = f(T_k)$. Then $I(f) < \infty$ if and only if

$$\sum_{k=1}^\infty f_k^{2/3} \exp\left(-\frac{3}{2^{5/3}} f_k^{4/3}\right) < \infty.$$

First we prove the convergence part of Theorem 1.2. Assume that $I(f) < \infty$ and define the events

$$A_k = \{\Upsilon^*(T_{k+1}) > T_k^{1/4} f_k\}.$$

It follows from Theorem 1.1 that

$$P(A_k) \leq c f_k^{2/3} \exp\left(-\frac{3}{2^{5/3}} \left(1 + \frac{1}{f_k^{4/3}}\right)^{-1/3} f_k^{4/3}\right).$$

Using the inequality

$$(1 + u)^{-1/3} \geq 1 - \frac{u}{3}, \quad 0 \leq u \leq 1,$$

with $u = f_k^{-4/3}$, we obtain further

$$P(A_k) \leq c f_k^{2/3} \exp\left(-\frac{3}{2^{5/3}} f_k^{4/3}\right),$$

which is summable by Lemma A. Hence $P(A_k \text{ i.o.}) = 0$, i.e., for large k we have almost surely

$$\Upsilon^*(T_{k+1}) \leq T_k^{1/4} f(T_k).$$

But for $T_k \leq t \leq T_{k+1}$, i.e., for large t

$$\Upsilon^*(t) \leq \Upsilon(T_{k+1}) \leq T_k^{1/4} f(T_k) \leq t^{1/4} f(t),$$

proving the convergence part.

For the divergence part, we follow the proof in [5]. Without loss of generality we may assume

$$(\log \log t)^{3/4} \leq f(t) \leq (2 \log \log t)^{3/4}$$

and, as easily seen,

$$(\log k/2)^{3/4} \leq f_k \leq (2 \log k)^{3/4}.$$

In the proof we also use the inequality

$$\frac{T_k}{T_\ell} \leq \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k)}, \quad k < \ell.$$

Now assume that $I(f) = \infty$, and define the events

$$B_k = \{T_k^{1/4} f_k \leq \Upsilon^*(T_k) < T_{k+1}^{1/4} f_k\},$$

where $f_k = f(T_k)$. It follows from Theorem 1.1 that

$$P(B_k) \geq c f_k^{2/3} \exp\left(-\frac{3 f_k^{4/3}}{2^{5/3}}\right) \left[1 - \left(\frac{T_{k+1}}{T_k}\right)^{1/6} \exp\left(-\frac{3 f_k^{4/3}}{2^{5/3}} \left(\left(\frac{T_{k+1}}{T_k}\right)^{1/3} - 1\right)\right)\right].$$

It is readily seen that $\lim_{k \rightarrow \infty} T_{k+1}/T_k = 1$, and

$$\lim_{k \rightarrow \infty} f_k^{4/3} \left(\left(\frac{T_{k+1}}{T_k}\right)^{1/3} - 1\right) = \frac{1}{3},$$

so there is a positive constant c such that

$$P(B_k) \geq c f_k^{2/3} \exp\left(-\frac{3 f_k^{4/3}}{2^{5/3}}\right),$$

and hence by Lemma A we have $\sum_k P(B_k) = \infty$.

Next we estimate $P(B_k B_\ell)$. Let $k < \ell$ and

$$\Upsilon^*(T_k, T_\ell) = \sup_{x \in \mathbb{R}} (\eta_1(x, \eta_2(0, T_\ell)) - \eta_1(x, \eta_2(0, T_k))).$$

Then, similarly to the proof in [5],

$$\Upsilon^*(T_k, T_\ell) \leq \Upsilon^*(T_\ell) \leq \Upsilon^*(T_k) + \Upsilon^*(T_k, T_\ell)$$

and

$$\begin{aligned} P(B_k B_\ell) &\leq P(T_k^{1/4} f_k \leq \Upsilon^*(T_k) < T_{k+1}^{1/4} f_k, \Upsilon^*(T_\ell) - \Upsilon^*(T_k) \geq T_\ell^{1/4} f_\ell - T_{k+1}^{1/4} f_k) \\ &\leq P(B_k) P(T_\ell^{1/4} f_\ell - T_{k+1}^{1/4} f_k \leq \Upsilon^*(T_k, T_\ell) \leq T_{\ell+1}^{1/4} f_\ell). \end{aligned}$$

But $\Upsilon^*(T_k, T_\ell)$ has the same distribution as $\Upsilon^*(T_\ell - T_k)$, or $(T_\ell - T_k)^{1/4} \Upsilon^*(1)$, hence

$$\begin{aligned} P(B_k B_\ell) &\leq P(B_k) P\left(\Upsilon^*(1) \geq \frac{f_\ell T_\ell^{1/4} - f_k T_{k+1}^{1/4}}{(T_\ell - T_k)^{1/4}}\right) \\ &\leq P(B_k) P\left(\Upsilon^*(1) \geq f_\ell \frac{T_\ell^{1/4} - T_{k+1}^{1/4}}{(T_\ell - T_k)^{1/4}}\right) \\ &\leq c P(B_k) f_\ell^{2/3} H_{k,\ell}^{2/3} \exp\left(-\frac{3f_\ell^{4/3} H_{k,\ell}^{4/3}}{2^{5/3}}\right), \end{aligned} \tag{3.1}$$

where

$$H_{k,\ell} = \frac{T_\ell^{1/4} - T_{k+1}^{1/4}}{(T_\ell - T_k)^{1/4}}.$$

Using the inequality

$$\frac{(1-u)^{3/4}}{4} \leq \frac{1-u^{1/4}}{(1-u)^{1/4}} \leq 1, \quad 0 < u < 1,$$

we get

$$\frac{1}{4} \left(1 - \frac{T_k}{T_\ell}\right)^{3/4} \frac{T_\ell^{1/4} - T_{k+1}^{1/4}}{T_\ell^{1/4} - T_k^{1/4}} \leq H_{k,\ell} \leq 1.$$

For $k + 2 \leq \ell$ we have, by straightforward calculation,

$$\frac{T_\ell^{1/4} - T_{k+1}^{1/4}}{T_\ell^{1/4} - T_k^{1/4}} \geq \frac{T_{k+2}^{1/4} - T_{k+1}^{1/4}}{T_{k+2}^{1/4} - T_k^{1/4}} \sim \frac{1}{1 + \left(\frac{f_{k+1}}{f_k}\right)^{4/3}},$$

from which

$$c \left(1 - \frac{T_k}{T_\ell}\right)^{3/4} \leq H_{k,\ell} \leq 1$$

with certain constant $c > 0$. Consequently,

$$P(B_k B_\ell) \leq cP(B_k) f_\ell^{2/3} \exp\left(-c_1 f_\ell^{4/3} \left(1 - \frac{T_k}{T_\ell}\right)\right).$$

Now, for fixed k , let

$$\begin{aligned} L_1 &= \{\ell : k + 2 \leq \ell \leq k + f_\ell^{4/3}\}, \\ L_2 &= \left\{\ell : k + f_\ell^{4/3} < \ell \leq k + 4f_\ell^{4/3} \log f_\ell^{4/3}\right\}, \\ L_3 &= \left\{\ell : k + 4f_\ell^{4/3} \log f_\ell^{4/3} < \ell\right\}. \end{aligned}$$

If $\ell \in L_1$, then

$$1 - \frac{T_k}{T_\ell} \geq 1 - \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k)} \geq \frac{\ell - k}{2f_\ell^{4/3}},$$

i.e.,

$$P(B_k B_\ell) \leq cP(B_k) f_\ell^{2/3} e^{-c_2(\ell-k)},$$

consequently

$$\sum_{\ell \in L_1} P(B_k B_\ell) \leq KP(B_k). \quad (3.2)$$

If $\ell \in L_2$, then

$$1 - \frac{T_k}{T_\ell} \geq 1 - \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k)} \geq c$$

with some $c > 0$. We have

$$\begin{aligned} P(B_k B_\ell) &\leq cP(B_k) f_\ell^{2/3} e^{-c_0 f_\ell^{4/3}} \leq cP(B_k) (\log \ell)^{1/2} \ell^{-c_0/2} \\ &\leq cP(B_k) (\log k)^{1/2} k^{-c_0/2}. \end{aligned}$$

But

$$\ell - k \leq 4f_\ell^{4/3} \log f_\ell^{4/3} \leq \frac{\ell}{2},$$

i.e., $\ell \leq 2k$, hence

$$\ell - k \leq 4f_{2k}^{4/3} \log f_{2k}^{4/3}.$$

Consequently,

$$\sum_{\ell \in L_2} P(B_k B_\ell) \leq cP(B_k)(\log k)^{1/2} k^{-c_0/2} f_{2k}^{4/3} \log f_{2k}^{4/3} \leq cP(B_k). \quad (3.3)$$

If $\ell \in L_3$, then

$$\frac{T_\ell^{1/4} - T_{k+1}^{1/4}}{(T_\ell - T_k)^{1/4}} \geq 1 - \left(\frac{T_{k+1}}{T_\ell}\right)^{1/4} \geq 1 - \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k-1)/4}.$$

Hence, using (3.1),

$$P(B_k B_\ell) \leq cP(B_k) f_\ell^{2/3} \exp\left(-\frac{3f_\ell^{4/3}}{2^{5/3}} \left(1 - \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k-1)/4}\right)^{4/3}\right).$$

It can be seen that

$$\begin{aligned} & \frac{3f_\ell^{4/3}}{2^{5/3}} \left(\left(1 - \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k-1)/4}\right)^{4/3} - 1 \right) \sim -2^{1/3} f_\ell^{4/3} \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k-1)/4} \\ & = -2^{1/3} f_\ell^{4/3} \exp\left(-\frac{\ell-k-1}{4} \log\left(1 + \frac{1}{f_\ell^{4/3}}\right)\right) \\ & \sim -2^{1/3} f_\ell^{4/3} \exp\left(-\frac{\ell-k-1}{4f_\ell^{4/3}}\right) \geq -2^{1/3} f_\ell^{4/3} \exp(-\log f_\ell^{4/3}) \geq -2^{1/3}. \end{aligned}$$

It follows that

$$P(B_k B_\ell) \leq cP(B_k) f_\ell^{2/3} \exp\left(-\frac{3f_\ell^{4/3}}{2^{5/3}}\right) \leq cP(B_k)P(B_\ell). \quad (3.4)$$

On using (3.2), (3.3), (3.4) together with $P(B_k B_\ell) \leq P(B_k)$ for $\ell = k, k + 1$, we obtain

$$\liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sum_{\ell=1}^n P(B_k B_\ell)}{(\sum_{k=1}^n P(B_k))^2} > 0,$$

hence from Borel–Cantelli lemma and 0-1 law we obtain $P(B_k \text{ i.o.}) = 1$, completing the proof of Theorem 1.2. \square

4. Proof of Theorem 1.3

First assume that $J(g) < \infty$. Let $t_k = e^k$ and define the events

$$B_k = \{\Upsilon^*(t_k) < t_{k+1}^{1/4}g(t_{k+1})\}.$$

Then

$$P(B_k) \leq cg^2(t_{k+1}),$$

which is well-known to be summable if $J(g) < \infty$. Hence for large k we have almost surely

$$\Upsilon^*(t_k) \geq t_{k+1}^{1/4}g(t_{k+1}),$$

and for $t_k \leq t < t_{k+1}$

$$\Upsilon^*(t) \geq \Upsilon^*(t_k) \geq t_{k+1}^{1/4}g(t_{k+1}) \geq t^{1/4}g(t),$$

proving the convergence part.

Now assume that $J(g) = \infty$. Put $t_k = 2^k$ and define the events

$$A_k = \{\eta_2(0, t_k) \leq t_k^{1/2}g^2(t_k)\},$$

$$B_k = \{\eta_1^*(t_k^{1/2}g^2(t_k)) \leq t_k^{1/4}g(t_k)\}.$$

Then $P(A_k \text{ i.o.}) = 1$ (cf. CSÁKI [4], the proof of the divergent part of Theorem 2.1 (i) on p. 211) and, by scaling property, $P(B_k) = p > 0$, independently of k . It follows from Lemma 3.1 of CSÁKI et al. [7] that $P(A_k B_k \text{ i.o.}) \geq p$. Consequently, $P(\Upsilon^*(t_k) \leq t_k^{1/4}g(t_k) \text{ i.o.}) \geq p > 0$. Now the proof of the divergence part is complete by 0 – 1 law. □

5. Simple random walk on 2-dimensional comb

We consider a simple random walk $\mathbf{C}(n)$ on the 2-dimensional comb lattice \mathbb{C}^2 that is obtained from \mathbb{Z}^2 by removing all horizontal lines off the x -axis.

A formal way of describing a simple random walk $\mathbf{C}(n)$ on the above 2-dimensional comb lattice \mathbb{C}^2 can be formulated via its transition probabilities as follows: for $(x, y) \in \mathbb{Z}^2$

$$P(\mathbf{C}(n+1) = (x, y \pm 1) \mid \mathbf{C}(n) = (x, y)) = \frac{1}{2}, \quad \text{if } y \neq 0, \tag{5.1}$$

$$\begin{aligned} &P(\mathbf{C}(n+1) = (x \pm 1, 0) \mid \mathbf{C}(n) = (x, 0)) \\ &= P(\mathbf{C}(n+1) = (x, \pm 1) \mid \mathbf{C}(n) = (x, 0)) = \frac{1}{4}. \end{aligned} \tag{5.2}$$

Unless otherwise stated, we assume that $\mathbf{C}(0) = \mathbf{0} = (0, 0)$. The coordinates of the just defined vector valued simple random walk $\mathbf{C}(n)$ on \mathbb{C}^2 will be denoted by $C_1(n), C_2(n)$, i.e., $\mathbf{C}(n) := (C_1(n), C_2(n))$.

For a recent review of some related literature concerning this simple random walk we refer to BERTACCHI [1] and CSÁKI et al. [8]. In the latter paper we established a strong approximation for the random walk $\mathbf{C}(n) = (C_1(n), C_2(n))$ that reads as follows.

Theorem B. *On an appropriate probability space for the random walk*

$$\{\mathbf{C}(n) = (C_1(n), C_2(n)); n = 0, 1, 2, \dots\}$$

on \mathbb{C}^2 , one can construct two independent standard Wiener processes $\{W_1(t); t \geq 0\}, \{W_2(t); t \geq 0\}$ so that, as $n \rightarrow \infty$, we have with any $\varepsilon > 0$

$$n^{-1/4}|C_1(n) - W_1(\eta_2(0, n))| + n^{-1/2}|C_2(n) - W_2(n)| = O(n^{-1/8+\varepsilon}) \quad \text{a.s.},$$

where $\eta_2(0, \cdot)$ is the local time process at zero of $W_2(\cdot)$.

Define now the local time process $\Xi(\cdot, \cdot)$ of the random walk $\{\mathbf{C}(n); n = 0, 1, \dots\}$ on the 2-dimensional comb lattice \mathbb{C}^2 by

$$\Xi(\mathbf{x}, n) := \#\{0 < k \leq n : \mathbf{C}(k) = \mathbf{x}\}, \quad \mathbf{x} \in \mathbb{C}^2, n = 1, 2, \dots \quad (5.3)$$

We now introduce our next result that concludes a strong approximation of the just introduced local time process $\Xi((x, 0), n)$.

Theorem 5.1. *On a suitable probability space we can define a simple random walk on \mathbb{C}^2 and two independent Wiener local times $\eta_1(\cdot, \cdot), \eta_2(\cdot, \cdot)$ such that as $n \rightarrow \infty$, we have for any $\varepsilon > 0$*

$$\sup_{x \in \mathbb{Z}} |\Xi((x, 0), n) - 2\eta_1(x, \eta_2(0, n))| = O(n^{1/8+\varepsilon}) \quad \text{a.s.} \quad (5.4)$$

PROOF. As in [8], start with two independent simple symmetric random walks on the line

$$\{S_1(n), S_2(n); n = 0, 1, \dots\}$$

with respective local times

$$\xi_i(x, n) := \#\{j : 1 \leq j \leq n, S_i(j) = x\}, \quad i = 1, 2, \quad x \in \mathbb{Z}, \quad n = 1, 2, \dots$$

and inverse local times

$$\rho_i(N) := \min\{j > \rho_{N-1} : S_i(j) = 0\}, \quad i = 1, 2, \quad N = 1, 2, \dots$$

with $\rho_i(0) = 0$. Assume that on the same probability space we have an i.i.d. sequence of random variables G_1, G_2, \dots with geometric distribution,

$$P(G_1 = k) = \frac{1}{2^{k+1}}, \quad k = 0, 1, 2, \dots,$$

that is independent of $S_1(\cdot), S_2(\cdot)$. We may construct a simple random walk on the 2-dimensional comb lattice \mathbb{C}^2 as follows. Put $T_N = G_1 + G_2 + \dots + G_N$, $N = 1, 2, \dots$. For $n = 0, \dots, T_1$, let $C_1(n) = S_1(n)$ and $C_2(n) = 0$. For $n = T_1 + 1, \dots, T_1 + \rho_2(1)$, let $C_1(n) = C_1(T_1)$, $C_2(n) = S_2(n - T_1)$. In general, for $T_N + \rho_2(N) < n \leq T_{N+1} + \rho_2(N)$, let

$$C_1(n) = S_1(n - \rho_2(N)), \quad C_2(n) = 0,$$

and, for $T_{N+1} + \rho_2(N) < n \leq T_{N+1} + \rho_2(N + 1)$, let

$$C_1(n) = C_1(T_{N+1} + \rho_2(N)) = S_1(T_{N+1}), \quad C_2(n) = S_2(n - T_{N+1}).$$

Then it can be seen that, in terms of these definitions for $C_1(n)$ and $C_2(n)$, $\mathbf{C}(n) = (C_1(n), C_2(n))$ is a simple random walk on the 2-dimensional comb lattice \mathbb{C}^2 .

First we approximate the local time $\Xi((x, 0), n)$ by iterated simple symmetric random walk local time.

Proposition 5.1. *On a suitable probability space we can define a simple random walk \mathbf{C} on \mathbb{C}^2 with local time Ξ and two simple random walks S_1, S_2 on \mathbb{Z} with local times ξ_1, ξ_2 such that as $n \rightarrow \infty$, we have for any $\varepsilon > 0$*

$$\sup_{x \in \mathbb{Z}} |\Xi((x, 0), n) - 2\xi_1(x, \xi_2(0, n))| = O(n^{1/8+\varepsilon}) \quad a.s. \tag{5.5}$$

PROOF. Introduce the following notations. For the random walk $\mathbf{C}(\cdot)$ let $H(n)$ be the horizontal steps on the x -axis up to time n and let $V(n)$ be the number of vertical steps up to time n . Moreover, let $B(n)$ be the number of vertical visits to the x -axis up to time n . Put

$$\begin{aligned} &\Xi^{(h)}((x, 0), n) \\ &:= \#\{0 < k \leq n : \mathbf{C}(k) = (x, 0), |C_1(k) - C_1(k - 1)| > 0, C_2(k - 1) = 0\} \end{aligned}$$

and

$$\Xi^{(v)}((x, 0), n) = \Xi((x, 0), n) - \Xi^{(h)}((x, 0), n),$$

i.e., the horizontal, resp. vertical, visits to the point $(x, 0)$ up to time n . Then, we have clearly

$$\begin{aligned} \Xi^{(h)}((x, 0), n) &= \xi_1(x, H(n)), \\ B(n) = \xi_2(0, V(n)) &= \xi_2(0, n - H(n)) = O(n^{1/2+\varepsilon}) \quad \text{a.s.}, \\ H(n) = G_1 + G_2 + \dots + G_{B(n)} &= O(B(n)) = O(n^{1/2+\varepsilon}) \quad \text{a.s.}, \\ |H(n) - B(n)| &= |G_1 + G_2 + \dots + G_{B(n)} - B(n)| \\ &= O((B(n))^{1/2+\varepsilon}) = O(n^{1/4+\varepsilon}) \quad \text{a.s.}, \end{aligned}$$

as $n \rightarrow \infty$. Using the increment property of simple symmetric random walk local time (cf. RÉVÉSZ [12], Theorem 11.15), we get

$$\xi_2(0, n) - \xi_2(0, n - H(n)) = O((H(n))^{1/2+\varepsilon}) \quad \text{a.s.}, \quad n \rightarrow \infty,$$

and

$$\begin{aligned} \Xi^{(h)}((x, 0), n) &= \xi_1(x, H(n)) = \xi_1(x, B(n) + O(B(n)^{1/2+\varepsilon})) \\ &= \xi_1(x, B(n)) + O(B(n)^{1/4+\varepsilon}) \\ &= \xi_1(x, \xi_2(0, n - H(n))) + O((\xi_2(0, n - H(n)))^{1/4+\varepsilon}) \\ &= \xi_1(x, \xi_2(0, n)) + O((H(n))^{1/4+\varepsilon}) = \xi_1(x, \xi_2(0, n)) + O(n^{1/8+\varepsilon}), \end{aligned}$$

almost surely, where we used that $H(n) = O(n^{1/2+\varepsilon})$ a.s., $n \rightarrow \infty$.

Now we show that $\Xi^{(h)}$ and $\Xi^{(v)}$ are close to each other.

Lemma 5.1. *As $n \rightarrow \infty$, we have almost surely*

$$\sup_{x \in \mathbb{Z}} |\Xi^{(h)}((x, 0), n) - \Xi^{(v)}((x, 0), n)| = O(n^{1/8+\varepsilon}). \quad (5.6)$$

PROOF. By the law of the iterated logarithm we have $C_1(n) = O(n^{1/4+\varepsilon})$ almost surely, as $n \rightarrow \infty$, and hence it suffices to show

$$\sup_{|x| \leq n^{1/4+\varepsilon}} |\Xi^{(h)}((x, 0), n) - \Xi^{(v)}((x, 0), n)| = O(n^{1/8+\varepsilon}) \quad \text{a.s.} \quad (5.7)$$

as $n \rightarrow \infty$.

Let $\kappa(x, 0)$ be the time of the first horizontal visit of $\mathbf{C}(\cdot)$ to $(x, 0)$, and for $\ell \geq 1$ let $\kappa(x, \ell)$ denote the time of the ℓ -th horizontal return of $\mathbf{C}(\cdot)$ to $(x, 0)$. Then

$$\Xi^{(v)}((x, 0), \kappa(x, \ell)) = \sum_{j=1}^{\ell} \left(\Xi^{(v)}((x, 0), \kappa(x, j)) - \Xi^{(v)}((x, 0), \kappa(x, j - 1)) \right),$$

which is a sum of i.i.d. random variables with geometric distribution

$$P(\Xi^{(v)}((x, 0), \kappa(x, j)) - \Xi^{(v)}((x, 0), \kappa(x, j - 1)) = i) = \frac{1}{2^{i+1}}, \quad i = 0, 1, 2, \dots$$

By exponential Kolmogorov inequality (see TÓTH [14])

$$P(\max_{\ell \leq m} |\Xi^{(v)}((x, 0), \kappa(x, \ell)) - \ell| > u) \leq 2 \exp\left(-\frac{u^2}{8m}\right).$$

Hence, we have also

$$P(\max_{|x| \leq m} \max_{\ell \leq m} |\Xi^{(v)}((x, 0), \kappa(x, \ell)) - \ell| > u) \leq 2m \exp\left(-\frac{u^2}{8m}\right).$$

Putting $u = m^{1/2+\varepsilon}$, Borel–Cantelli lemma implies

$$\max_{|x| \leq m} \max_{\ell \leq m} |\Xi^{(v)}((x, 0), \kappa(x, \ell)) - \ell| = O(m^{1/2+\varepsilon}) \quad \text{a.s.}$$

as $m \rightarrow \infty$.

Since

$$\Xi^{(h)}((x, 0), n) = O(n^{1/4+\varepsilon}) \quad \text{a.s.,} \quad n \rightarrow \infty,$$

with $m = n^{1/4+\varepsilon}$, we have the Lemma. □

This also completes the proof of the Proposition. □

Now Theorem 5.1 follows from strong invariance principle for local time (cf. RÉVÉSZ [11]) quoted as Theorem C below, and increment results for Wiener local time (cf. RÉVÉSZ [12], Theorem 11.11).

Theorem C. *On a suitable probability space one can define a Wiener process with local time η and a simple symmetric random walk on \mathbb{Z} with local time ξ such that as $n \rightarrow \infty$, for any $\varepsilon > 0$ we have almost surely*

$$\sup_{x \in \mathbb{Z}} |\xi(x, n) - \eta(x, n)| = O(n^{1/4+\varepsilon}).$$

The proof of Theorem 5.1 is complete. □

Theorems 1.2, 1.3 and 5.1 imply the following Corollary.

Corollary 5.1. *Let $a(n)$ be a non-decreasing sequence of positive numbers.*

Then

$$P(\sup_{x \in \mathbb{Z}} \Xi((x, 0), n) > n^{1/4} a(n) \text{ i.o.}) = 0 \text{ or } 1$$

according as

$$\sum_{n=1}^{\infty} \frac{a^2(n)}{n} \exp\left(-\frac{3a^{4/3}(n)}{2^{5/3}}\right) < \infty \text{ or } = \infty.$$

Let $b(n)$ be a non-increasing sequence of positive numbers. Then

$$P(\sup_{x \in \mathbb{Z}} \Xi((x, 0), n) < n^{1/4} b(n) \text{ i.o.}) = 0 \text{ or } 1$$

according as

$$\sum_{n=1}^{\infty} \frac{b^2(n)}{n} < \infty \text{ or } = \infty.$$

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