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An inequality for moments and its applications to the generalized allocation scheme

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Dedicated to the 100th anniversary of the birthday of Béla Gyires

Abstract. An inequality for conditional moments of centered random variables is proved. This inequality is applied to the generalized allocation scheme. For the generalized allocation scheme an exponential inequality is obtained. The exponential inequality is applied to prove (partial) analogues of the Law of the Iterated Logarithm and an analogue of Prokhorov's Law of Large Numbers.

1. Introduction

Consider the probability measure $\mathbb{P}$ and the conditional probability measure $\mathbb{P}^A$ with respect to the fixed event A. Let $\mathbb{E}^A$ denote the expectation with respect to $\mathbb{P}^A$. We prove the following inequality for centered moments of random variables $\mathbb{E}^A|S - \mathbb{E}^A S|^p \le c_p \frac{\mathbb{E}|S-\mathbb{E}S|^p}{\mathbb{P}(A)}$ (Lemma 2.1). We construct an example showing that it is the best inequality between the ordinary and the conditional moments if $\mathbb{P}(A) \to 0$.

This inequality is applied to the generalized allocation scheme. The random variables $\eta_1, \dots, \eta_N$ satisfy a generalized allocation scheme, if equation (3.1) is true with $\xi_1, \dots, \xi_N$ having distribution (3.4). This scheme is widely studied, see [5], [6], [11].

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Let $\mu_s = \mu_{snN}$ be the number of cells containing s particles when n particles are allocated into N cells. First we apply Theorem 4 of [6] to obtain a lower bound for the denominator of our upper bound (Lemma 3.1). Then we prove an exponential inequality for the random variable $\frac{\mu_s - \mathbb{E}\mu_s}{\sqrt{N}}$ (Lemma 4.1). The method of the proof of the exponential inequality is the following. We estimate the tail probability of $\frac{|\mu_s - \mathbb{E}\mu_s|}{\sqrt{N}}$ by the p-th moment of the centered sum of certain independent indicators which we majorize using Khintchine's inequality. Finally we compute the minimum in p.

The exponential inequality allows us to prove (partial) analogues of the Law of the Iterated Logarithm (Theorem 4.1 and Theorem 4.2) and an analogue of Prokhorov's Law of Large Numbers for random variables μ_s (Theorem 4.3). Moreover, Laws of Large Numbers for weighted sums of μ_s are obtained (Theorem 4.4, Theorem 4.5). We remark that the random variables $\mu_s = \mu_{snN}$ depend on the number of cells (N) and the number of particles (n), so that $n, N \to \infty$. Therefore we have proved analogues of the Law of the Iterated Logarithm and an analogue of the Law of Large Numbers for a two-indexed sequence of random variables with indices varying in a sector.

2. The inequality for moments

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $A \in \mathcal{A}$ be a fixed event such that $\mathbb{P}(A) > 0$. Recall that the conditional probability $\mathbb{P}^A$ is defined by the formula

$$\mathbb{P}^A(B) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}, \quad B \in \mathcal{A}.$$

We will denote by $\mathbb{E}^A$ the expectation with respect to the probability measure $\mathbb{P}^A$.

It is easy to see that for any random variable S and for any $p > 0$ we have $\mathbb{E}^A|S|^p \leq \frac{1}{\mathbb{P}(A)}\mathbb{E}|S|^p$. The following lemma shows that a similar inequality is true for centered absolute moments of S. The statement is simple. However, as we could not find an appropriate reference, we give a proof.

Lemma 2.1. *Let $1 \leq p < \infty$. Then we have*

$$\mathbb{E}^A|S - \mathbb{E}^A S|^p \leq c_p \frac{\mathbb{E}|S - \mathbb{E}S|^p}{\mathbb{P}(A)}. \tag{2.1}$$

Here one can choose $c_p = 4^p/2$.

Proof. Using the definition of the conditional probability and the c_p inequality, we obtain

$$
\begin{aligned}
\mathbb{E}^A|S-\mathbb{E}^A S|^p &= \mathbb{E}\left|S\mathbb{I}_A - \frac{\mathbb{E}(S\mathbb{I}_A)}{\mathbb{P}(A)}\mathbb{I}_A\right|^p \frac{1}{\mathbb{P}(A)} \\
&= \mathbb{E}\left|S\mathbb{I}_A - (\mathbb{E}S)\mathbb{I}_A + (\mathbb{E}S)\mathbb{I}_A - \frac{\mathbb{E}(S\mathbb{I}_A)}{\mathbb{P}(A)}\mathbb{I}_A\right|^p \frac{1}{\mathbb{P}(A)} \\
&\le 2^{p-1}\left(\mathbb{E}|S\mathbb{I}_A - (\mathbb{E}S)\mathbb{I}_A|^p \frac{1}{\mathbb{P}(A)} + \left|\mathbb{E}S - \frac{\mathbb{E}(S\mathbb{I}_A)}{\mathbb{P}(A)}\right|^p \frac{\mathbb{P}(A)}{\mathbb{P}(A)}\right) \\
&= 2^{p-1}\left(\mathbb{E}|(S-\mathbb{E}S)\mathbb{I}_A|^p \frac{1}{\mathbb{P}(A)} + \left|\frac{(\mathbb{E}S)\mathbb{P}(A) - \mathbb{E}(S\mathbb{I}_A)}{\mathbb{P}(A)}\right|^p\right) \\
&= 2^{p-1}\left(\mathbb{E}|(S-\mathbb{E}S)\mathbb{I}_A|^p \frac{1}{\mathbb{P}(A)} + \left|\frac{\mathbb{E}\left((S-\mathbb{E}S)(\mathbb{I}_A - \mathbb{P}(A))\right)}{\mathbb{P}(A)}\right|^p\right). \qquad (2.2)
\end{aligned}
$$

First let $p > 1$. Let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. By Holder's inequality, we have

$$
\begin{aligned}
\left(\frac{|\mathbb{E}\left((S-\mathbb{E}S)(\mathbb{I}_A - \mathbb{P}(A))\right)|}{\mathbb{P}(A)}\right)^p &\le \left(\frac{(\mathbb{E}|S-\mathbb{E}S|^p)^{\frac{1}{p}}(\mathbb{E}|\mathbb{I}_A - \mathbb{P}(A)|^{p'})^{\frac{1}{p'}}}{\mathbb{P}(A)}\right)^p \\
&\le \frac{\mathbb{E}|S-\mathbb{E}S|^p\,(2\mathbb{P}(A))^{\frac{p}{p'}}}{(\mathbb{P}(A))^p} \le 2^{p-1}\frac{\mathbb{E}|S-\mathbb{E}S|^p}{\mathbb{P}(A)}. \qquad (2.3)
\end{aligned}
$$

Moreover

$$\mathbb{E}|(S-\mathbb{E}S)\mathbb{I}_A|^p \le \mathbb{E}|S-\mathbb{E}S|^p. \qquad (2.4)$$

Substituting (2.3) and (2.4) into (2.2), we obtain (2.1). Now let $p = 1$. We have

$$|\mathbb{E}(S-\mathbb{E}S)(\mathbb{I}_A - \mathbb{P}(A))| \le \mathbb{E}|S-\mathbb{E}S|. \qquad (2.5)$$

By (2.5) and (2.2), we obtain (2.1). The proof is complete. □

Remark 2.1. Let $\mathbb{D}^2(S)$ be the variance of S. Since

$$\left(\frac{|\mathbb{E}(S-\mathbb{E}S)(\mathbb{I}_A - \mathbb{P}(A))|}{\mathbb{P}(A)}\right)^p \le \left(\frac{(\mathbb{E}(S-\mathbb{E}S)^2)^{\frac{1}{2}}(\mathbb{E}(I_A - \mathbb{P}(A))^2)^{\frac{1}{2}}}{\mathbb{P}(A)}\right)^p \le \frac{(\mathbb{D}^2(S))^{p/2}}{(\mathbb{P}(A))^{p/2}},$$

from the proof of Lemma 2.1 it follows that

$$\mathbb{E}^A|S-\mathbb{E}^A S|^p \le 2^{p-1}\left(\frac{\mathbb{E}|S-\mathbb{E}S|^p}{\mathbb{P}(A)} + \frac{(\mathbb{D}^2(S))^{p/2}}{(\mathbb{P}(A))^{p/2}}\right). \qquad \square$$

Example 2.1. Let $0 < \mathbb{P}(A) < 1$. Let B_1, B_2 be disjoint subsets of A such that a $\mathbb{P}(B_1) = \mathbb{P}(B_2) = \mathbb{P}(A)/2$. Define the random variable S as follows. $S = 1$ on the set B_1, $S = -1$ on the set B_2, and $S = 0$ on the set A^c. Then $\mathbb{E}S = 0$, $\mathbb{E}^A S = 0$, $\mathbb{E}|S|^p = \mathbb{P}(A)$, $\mathbb{E}^A|S|^p = 1$. Therefore $\mathbb{E}^A|S - \mathbb{E}^A S|^p = 1 = \frac{1}{\mathbb{P}(A)}\mathbb{E}|S - \mathbb{E}S|^p$. Therefore we cannot delete $\mathbb{P}(A)$ from the denominator of the right hand side of (2.1). □

Corollary 2.1. *Let the conditions of Lemma 2.1 be valid. Then we have*

$$\mathbb{E}^A \exp(|S - \mathbb{E}^A S|^p) \le \frac{1}{\mathbb{P}(A)} \mathbb{E}\exp((4|S - \mathbb{E}S|)^p).$$

Proof. Using the monotone convergence theorem and Lemma 2.1, we obtain

$$\begin{aligned}\mathbb{E}^A \exp(|S - \mathbb{E}^A S|^p) &= \sum_{k=0}^{\infty} \frac{\mathbb{E}^A(|S - \mathbb{E}^A S|)^{kp}}{k!} \\ &\le \frac{1}{\mathbb{P}(A)} \sum_{k=0}^{\infty} \frac{\mathbb{E}(4|S - \mathbb{E}S|)^{kp}}{k!} = \frac{1}{\mathbb{P}(A)} \mathbb{E}\exp((4|S - \mathbb{E}S|)^p).\end{aligned}$$

The proof is complete. □

Corollary 2.2. *Let the conditions of Lemma 2.1 be valid. Let* $S = \sum_{i=1}^m \xi_i$, *where* $\xi_1, \dots, \xi_m$ *are independent identically distributed (i.i.d.) random variables. Then we have*

$$\mathbb{E}^A \exp(|S - \mathbb{E}^A S|) \le \frac{1}{\mathbb{P}(A)} b^m$$

where $b = \mathbb{E}\exp((4|\xi_1 - \mathbb{E}\xi_1|))$.

Proof. Apply Corollary 2.1 with $p = 1$. □

3. The generalized allocation scheme

In the generalized scheme of allocations of particles into cells, the distribution of the cell contents is represented as the conditional distribution of independent random variables under the condition that their sum is fixed, see [5], [11]. We can describe the generalized allocation scheme as follows. Let $\eta_1, \dots, \eta_N$ be nonnegative integer-valued random variables (we do not assume the independence of $\eta_1, \dots, \eta_N$). They are considered as certain numerical characteristics of the combinatorial structure of n elements consisting of N components such that

$\eta_1 + \cdots + \eta_N = n$. If there exist independent random variables $\xi_1, \ldots, \xi_N$ such that the joint distribution of $\eta_1, \ldots, \eta_N$ admits the representation

$$\mathbb{P}\{\eta_1 = k_1, \ldots, \eta_N = k_N\} = \mathbb{P}\{\xi_1 = k_1, \ldots, \xi_N = k_N \mid \xi_1 + \cdots + \xi_N = n\}, \quad (3.1)$$

where $k_1, \ldots, k_N$ are arbitrary nonnegative integers, we say that $\eta_1, \ldots, \eta_N$ represent a generalized allocation scheme with parameters n and N, and independent random variables $\xi_1, \ldots, \xi_N$. Throughout the paper we assume that the random variables $\eta_1, \ldots, \eta_N$ and $\xi_1, \ldots, \xi_N$ satisfy (3.1).

In view of independence of the random variables $\xi_1, \ldots, \xi_N$, the study of several questions of the generalized allocation scheme can be reduced to problems of sums of independent random variables. Let μ_s be the number of the random variables $\eta_1, \ldots, \eta_N$ being equal to s, $s = 0, 1, \ldots, n$. Observe that

$$\mu_s = \mu_{snN} = \sum_{i=1}^{N} \mathbb{I}_{\{\eta_i = s\}} \quad (3.2)$$

is the number of cells containing s particles. Details of the theory of generalized allocation schemes can be found in [5], [11].

Let the random variables $\xi_1, \ldots, \xi_N$ be identically distributed. Usually (see, e.g., [6]) the random variables $\xi_1, \ldots, \xi_N$ follow a power-series distribution: $q_k = \mathbb{P}\{\xi_1 = k\} = (b_k \theta^k)/(k! B(\theta))$ where $b_0, b_1, \ldots$ is a certain sequence of non-negative numbers, and $B(\theta) = \sum_{k=0}^{\infty} b_k \theta^k / k!$.

Consider a simple example.

Example 3.1. Let ξ_i have binomial distribution, i.e. $\mathbb{P}(\xi_i = k) = \binom{t}{k} p^k (1-p)^{t-k}$, $k = 0, 1, \ldots, t$. Then

$$\mathbb{P}\{\xi_1 = k_1, \ldots, \xi_N = k_N \mid \xi_1 + \cdots + \xi_N = n\} = \binom{t}{k_1} \cdots \binom{t}{k_N} \Big/ \binom{Nt}{n}$$

if $k_1 + \cdots + k_N = n$. That is $\{\eta_1 = k_1, \ldots, \eta_N = k_N\}$ has poly-hypergeometric distribution. □

Examples considering usual random allocations and random forests can be found in [5], [11] (see also Section 5 of the present paper).

Therefore, in what follows, we consider a sequence of non-negative numbers $b_0, b_1, \ldots$ with $b_0 > 0$, $b_1 > 0$ and assume that the convergence radius R of the series

$$B(\theta) = \sum_{k=0}^{\infty} \frac{b_k \theta^k}{k!} \quad (3.3)$$

is positive. Let us introduce the integer-valued random variable $\xi = \xi(\theta)$ (where $\theta > 0$) with distribution

$$\mathbb{P}\{\xi = k\} = \frac{b_k \theta^k}{k! B(\theta)}, \quad k = 0, 1, 2, \dots . \tag{3.4}$$

By [6], one has

$$m = m(\theta) = \mathbb{E}\xi = \frac{\theta B'(\theta)}{B(\theta)}$$

and

$$\sigma^2 = \sigma^2(\theta) = \mathbb{D}^2\xi = \frac{\theta^2 B^{''}(\theta)}{B(\theta)} + \frac{\theta B'(\theta)}{B(\theta)} - \frac{\theta^2 (B'(\theta))^2}{(B(\theta))^2}. \tag{3.5}$$

The last equality implies that

$$\sigma^2(\theta) = \theta m'(\theta). \tag{3.6}$$

Let $0 < \theta' < \theta^{''} < R$. If $\sigma^2(\theta) = 0$ for some $\theta \in [\theta', \theta^{''}]$, then the random variable $\xi(\theta)$ is a constant. Since $b_0 > 0$, $b_1 > 0$, the random variable $\xi(\theta)$ is not a constant. Therefore $\sigma^2(\theta)$, $\theta \in [\theta', \theta^{''}]$ is a positive continuous function. Consequently,

$$0 < C_1 = \inf_{\theta \in [\theta', \theta'']} \sigma^2(\theta) \le \sup_{\theta \in [\theta', \theta'']} \sigma^2(\theta) = C_2 < \infty. \tag{3.7}$$

By (3.6) and (3.7), we have

$$0 < \frac{C_1}{\theta''} \le \inf_{\theta \in [\theta', \theta'']} m'(\theta) \le \sup_{\theta \in [\theta', \theta'']} m'(\theta) \le \frac{C_2}{\theta'} < \infty.$$

So $m(\theta)$, $\theta \in [\theta', \theta^{''}]$, is a positive, continuous, strictly increasing function. We will denote by m^{-1} the inverse function of m.

We see that the random variable $\xi(\theta)$ has all moments, if $\theta < R$.

Throughout the paper let $\xi_1(\theta), \dots, \xi_N(\theta)$ be independent copies of $\xi(\theta)$ where $\xi(\theta)$ has distribution (3.4). Introduce notation $\alpha = n/N$. In this section we will consider the event

$$A = A_N(n) = \{\omega \in \Omega : \xi_1(\theta_\alpha)(\omega) + \xi_2(\theta_\alpha)(\omega) + \dots + \xi_N(\theta_\alpha)(\omega) = n\}$$

where $\theta_\alpha = m^{-1}(\alpha)$. Let $P_N(n) = \mathbb{P}(A_N(n))$.

Lemma 3.1. *Let $0 < \alpha' < \alpha''$ be such that $m^{-1}(\alpha'') < R$. Let $\alpha = n/N$. Then there exists $N_0 \in \mathbb{N}$ with the following property: if $n, N \in \mathbb{N}$ are such that $N > N_0$ and $\alpha' \le \alpha \le \alpha''$, then we have*

$$P_N(n) > \frac{1}{4\sigma(\theta_\alpha)\sqrt{N}}. \tag{3.8}$$

PROOF. Let $\theta' = m^{-1}(\alpha')$ and $\theta'' = m^{-1}(\alpha'')$. Then $\alpha' \le \alpha \le \alpha''$ if and only if $\theta' \le \theta_\alpha \le \theta''$. Therefore, by Theorem 4 from [6], there exists $N_0 \in \mathbb{N}$ such that if $N > N_0$ and $\alpha' \le \alpha \le \alpha''$, then

$$\sigma(\theta_\alpha)\sqrt{N}P_N(n) - \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(n-m(\theta_\alpha)N)^2}{2\sigma^2(\theta_\alpha)N}\right\} > \frac{1}{4} - \frac{1}{\sqrt{2\pi}}. \tag{3.9}$$

Since $m(\theta_\alpha) = \alpha$, we have $\frac{1}{\sqrt{2\pi}}\exp\big\{-\frac{(n-m(\theta_\alpha)N)^2}{2\sigma^2(\theta_\alpha)N}\big\} = \frac{1}{\sqrt{2\pi}}$. Consequently, (3.9) implies (3.8). $\square$

4. Limit theorems for the generalized allocation scheme

In this section we shall prove limit theorems for $\mu_s = \mu_{snN} = \sum_{i=1}^N \mathbb{I}_{\{\eta_i = s\}}$ defined in (3.2). Recall KHINTCHINE's inequality (see [7], [1], Section 10.3) which we will use. Let $1 \le p < \infty$ and let r_i, $i \in \mathbb{N}$, be the Rademacher functions. Let $\mathbb{E}_r$ denote the expectation with respect to $\{r_i\}$. Then for any $c_i \in \mathbb{R}$, $1 \le i \le l$, we have

$$\left(\mathbb{E}_r\left|\sum_{i=1}^l c_i r_i\right|^p\right)^{1/p} \le C_h\sqrt{p}\left(\sum_{i=1}^l (c_i)^2\right)^{1/2},$$

where C_h does not depend on $\{c_i\}$ and p.

Introduce notation $I_i^{(s)} = \mathbb{I}_{\{\eta_i = s\}}$, $J_i^{(s)} = \mathbb{I}_{\{\xi_i = s\}}$, $\sigma_s^2 = q_s(1-q_s) = \mathbb{D}^2(J^{(s)})$, $q_s = \mathbb{P}\{\xi_i = s\}$ and $p_s = \mathbb{P}\{\eta_i = s\}$.

We will use the following exponential inequality.

Lemma 4.1. *Let $0 < \alpha' < \alpha''$ be such that $m^{-1}(\alpha'') < R$. Suppose that $N_0 \in \mathbb{N}$ is such that the assertion of Lemma 3.1 is valid. Let $n, N \in \mathbb{N}$ are such that $N > N_0$ and $\alpha' \le \alpha = n/N \le \alpha''$. Then for $\varepsilon \ge 4C_h\sqrt{2e}$ we have*

$$\mathbb{P}\left\{\frac{|\mu_s - \mathbb{E}\mu_s|}{\sqrt{N}} \ge \varepsilon\right\} \le 8\sigma_s^2\sqrt{N}\sigma(\theta_\alpha)e^{-\frac{\varepsilon^2}{32eC_h^2}}. \tag{4.1}$$

PROOF. Let $\{J_i^{(ss)}, 1 \le i \le N\}$ be an independent copy of $\{J_i^{(s)}, 1 \le i \le N\}$. We can suppose that $\{J_i^{(ss)}, 1 \le i \le N\}$, $\{J_i^{(s)}, 1 \le i \le N\}$ and $\{r_i, 1 \le i \le N\}$ are independent families. Using (2.1), Lemma 3.1, Jensen's inequality, and Khintchine's inequality for $p \ge 2$, we obtain

$$\mathbb{P}\left\{\frac{|\mu_s - \mathbb{E}\mu_s|}{\sqrt{N}} \ge \varepsilon\right\} = \mathbb{P}\left\{\frac{|\sum_{i=1}^N (I_i^{(s)} - p_s)|}{\sqrt{N}} \ge \varepsilon\right\} \le \frac{1}{\varepsilon^p}\mathbb{E}\left|\frac{\sum_{i=1}^N (I_i^{(s)} - p_s)}{\sqrt{N}}\right|^p$$

$$
\begin{aligned}
&\le \frac{4^p}{\varepsilon^p \mathbb{P}(A)} \mathbb{E}\left| \frac{\sum_{i=1}^N (J_i^{(s)} - q_s)}{\sqrt{N}} \right|^p \le \frac{4\sigma(\theta) N^{1/2} 4^p}{\varepsilon^p} \mathbb{E}\left| \frac{\sum_{i=1}^N (J_i^{(s)} - J_i^{(ss)})}{\sqrt{N}} \right|^p \\
&= \frac{4\sigma(\theta) N^{1/2} 4^p}{\varepsilon^p} \mathbb{E}\mathbb{E}_r \left| \frac{\sum_{i=1}^N r_i (J_i^{(s)} - J_i^{(ss)})}{\sqrt{N}} \right|^p \\
&\le \frac{4\sigma(\theta) N^{1/2} 4^p}{\varepsilon^p} \mathbb{E}\bigg(C_h \sqrt{p} \bigg(\sum_{i=1}^N \frac{(J_i^{(s)} - J_i^{(ss)})^2}{N} \bigg)^{1/2} \bigg)^p \\
&\le \frac{4\sigma(\theta) N^{1/2} 4^p C_h^p p^{p/2}}{\varepsilon^p} \mathbb{E}\bigg(\sum_{i=1}^N \frac{(J_i^{(s)} - J_i^{(ss)})^2}{N} \bigg) \\
&= 2\sigma_s^2 \frac{4\sigma(\theta) N^{1/2} 4^p C_h^p p^{p/2}}{\varepsilon^p}. \qquad (4.2)
\end{aligned}
$$

For $p = \frac{\varepsilon^2}{16eC_h^2}$ from (4.2) follows (4.1). $\square$

Remark 4.1. Actually, the above calculations imply the following analogue of the Kolmogorov exponential inequality ([10]). Let $\zeta_1, \zeta_2, \dots, \zeta_n$ be independent centered random variables such that $\sum_{i=1}^n (\zeta_i)^2 \le c^2$ almost surely. Then for $\varepsilon \ge 2cC_h\sqrt{e}$ we have

$$
\mathbb{P}\bigg\{ \bigg| \sum_{i=1}^n \zeta_i \bigg| \ge \varepsilon \bigg\} \le \frac{1}{2c^2} \left(\sum_{i=1}^n \mathbb{D}^2(\zeta_i) \right) e^{-\frac{\varepsilon^2}{8e(cC_h)^2}}.
$$

If, moreover, ζ_i, $1 \le i \le n$, are symmetric, then for $\varepsilon \ge cC_h\sqrt{e}$ we have

$$
\mathbb{P}\left\{ \bigg| \sum_{i=1}^n \zeta_i \bigg| \ge \varepsilon \right\} \le \frac{1}{c^2} \left(\sum_{i=1}^n \mathbb{D}^2(\zeta_i) \right) e^{-\frac{\varepsilon^2}{2e(cC_h)^2}}. \qquad \square
$$

In the following theorems we will assume that the random variables considered are defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Let n_k, N_k be positive integer numbers such that $N_k < N_{k+1}$, $k \in \mathbb{N}$. Let $\alpha_k = n_k/N_k$, $\theta_k = m^{-1}(\alpha_k)$ and

$$
\mu_{sk} = \sum_{i=1}^{N_k} \mathbb{I}_{\{\eta_i = s\}}. \qquad (4.3)
$$

Now we will prove two analogues of the Law of the Iterated Logarithm.

Theorem 4.1. *Let $0 < \alpha' < \alpha''$ be such that $m^{-1}(\alpha'') < R$. Suppose that $0 < \alpha' \le \alpha_k \le \alpha''$ for all $k \in \mathbb{N}$. Then we have*

$$
\limsup_{k \to \infty} \frac{|\mu_{sk} - \mathbb{E}\mu_{sk}|}{\sqrt{N_k \ln(N_k)}} \le 4C_h\sqrt{3e} \quad \textit{almost surely.} \qquad (4.4)
$$

Proof. Let N_0 be the value from Lemma 3.1. Let $t > 4C_h\sqrt{3e}$. Then $\frac{t^2}{32eC_h^2} - \frac{1}{2} > 1$. Therefore, by (4.1), we obtain for $N > N_0$

$$\sum_{k=N_0+1}^{\infty} \mathbb{P}\Big\{\frac{|\mu_{sk} - \mathbb{E}\mu_{sk}|}{\sqrt{N_k \ln(N_k)}} \geq t\Big\} = \sum_{k=N_0+1}^{\infty} \mathbb{P}\left\{\frac{|\mu_{sk} - \mathbb{E}\mu_{sk}|}{\sqrt{N_k}} \geq \sqrt{\ln(N_k)}t\right\}$$
$$\leq \sum_{k=N_0+1}^{\infty} 8\sigma_s^2\sqrt{N_k}\sigma(\theta_k)e^{-\frac{\left(\sqrt{\ln(N_k)}t\right)^2}{32eC_h^2}} \leq 8\sigma_s^2\sqrt{C_2}\sum_{k=N_0+1}^{\infty} N_k^{-\frac{t^2}{32eC_h^2}+\frac{1}{2}} < \infty.$$

Consequently, by the Borel–Cantelli lemma, for all $t > 4C_h\sqrt{3e}$ we have $\limsup_{k\to\infty} \frac{|\mu_{sk} - \mathbb{E}\mu_{sk}|}{\sqrt{N_k \ln(N_k)}} \leq t$ almost surely. This implies (4.4). □

Theorem 4.2. *Let $0 < \alpha' < \alpha''$ be such that $m^{-1}(\alpha'') < R$. Then, for μ_{snN} defined in* (3.2), *we have*

$$\limsup_{n,N\to\infty,\, \alpha'<\alpha<\alpha''} \frac{|\mu_{snN} - \mathbb{E}\mu_{snN}|}{\sqrt{N\ln(N)}} \leq 4C_h\sqrt{5e}, \quad \textit{almost surely.} \tag{4.5}$$

Proof. First we remark that, for any two index sequence $l_{n,N}$, $\limsup_{n,N\to\infty,\, \alpha'<\alpha<\alpha''} l_{n,N}$ exists and it is unique because of $0 < \alpha' < \alpha'' < \infty$. Let $t > 4C_h\sqrt{5e}$. Then $\frac{t^2}{32eC_h^2} - \frac{1}{2} - 1 > 1$. Therefore, by (4.1), we obtain

$$\sum_{N=N_0+1}^{\infty} \sum_{N\alpha'\leq n\leq\alpha''N} \mathbb{P}\Big\{\frac{|\mu_{snN} - \mathbb{E}\mu_{snN}|}{\sqrt{N\ln(N)}} \geq t\Big\}$$
$$= \sum_{N=N_0+1}^{\infty} \sum_{N\alpha'\leq n\leq\alpha''N} \mathbb{P}\Big\{\frac{|\mu_{snN} - \mathbb{E}\mu_{snN}|}{\sqrt{N}} \geq \sqrt{\ln(N)}t\Big\}$$
$$\leq \sum_{N=N_0+1}^{\infty} 8\sigma_s^2(\alpha'' - \alpha')N\sqrt{N}\sqrt{C_2}e^{-\frac{\left(\sqrt{\ln(N)}t\right)^2}{32eC_h^2}}$$
$$\leq 8\sigma_s^2\sqrt{C_2}(\alpha'' - \alpha')\sum_{N=N_0+1}^{\infty} N^{-\frac{t^2}{32eC_h^2}+\frac{1}{2}+1} < \infty.$$

Consequently, by the Borel–Cantelli lemma, we have $\limsup_{n,N\to\infty,\alpha'<\alpha<\alpha''} \frac{|\mu_{snN} - \mathbb{E}\mu_{snN}|}{\sqrt{N\ln(N)}} \leq t$ almost surely, for all $t > 4C_h\sqrt{5e}$. This implies (4.5). □

Now we turn to an analogue of Prokhorov's Law of Large Numbers (see [9]).

Theorem 4.3. *Let $0<\alpha'<\alpha''$ be such that $m^{-1}(\alpha'')<R$. Let $\sqrt{N\ln(N)}=o(b_N)$ as $N\to\infty$. Then we have*

$$\lim_{n,N\to\infty,\ \alpha'<\alpha<\alpha''}\frac{\mu_{snN}-\mathbb{E}\mu_{snN}}{b_N}=0 \quad \textit{almost surely.} \tag{4.6}$$

Proof. Let $\varepsilon>0$. Choose $N'>N_0$ such that $\frac{b_N}{\sqrt{N\ln(N)}}\varepsilon>4C_h\sqrt{5e}$ for all $N\geq N'$. Then $\frac{(b_N\varepsilon)^2}{\left(\sqrt{N\ln(N)}\right)^2}\frac{1}{32eC_h^2}-\frac{1}{2}-1>1$ for all $N\geq N'$. Therefore, by (4.1), we obtain

$$\begin{aligned}
&\sum_{N=N'}^{\infty}\sum_{N\alpha'\leq n\leq\alpha''N}\mathbb{P}\left\{\frac{|\mu_{snN}-\mathbb{E}\mu_{snN}|}{b_N}\geq\varepsilon\right\}\\
&\quad=\sum_{N=N'}^{\infty}\sum_{N\alpha'\leq n\leq\alpha''N}\mathbb{P}\left\{\frac{|\mu_{snN}-\mathbb{E}\mu_{snN}|}{\sqrt{N}}\geq\frac{b_N\sqrt{\ln(N)}}{\sqrt{N\ln(N)}}\varepsilon\right\}\\
&\quad\leq\sum_{N=N'}^{\infty}8\sigma_s^2(\alpha''-\alpha')N\sqrt{N}\sqrt{C_2}e^{-\frac{\left(b_N\sqrt{\ln(N)}\varepsilon\right)^2}{\left(\sqrt{N\ln(N)}\right)^2}\frac{1}{32eC_h^2}}\\
&\quad\leq 8\sigma_s^2\sqrt{C_2}(\alpha''-\alpha')\sum_{N=N'}^{\infty}N^{-\frac{(b_N\varepsilon)^2}{\left(\sqrt{N\ln(N)}\right)^2}\frac{1}{32eC_h^2}+\frac{1}{2}+1}<\infty.
\end{aligned}$$

Consequently, for all $\varepsilon>0$ we have $\limsup_{n,N\to\infty,\,\alpha'<\alpha<\alpha''}\frac{|\mu_{snN}-\mathbb{E}\mu_{snN}|}{\sqrt{N\ln(N)}}\leq\varepsilon$ almost surely. This implies (4.6). $\square$

Lemma 4.2. *Let $0<\alpha'<\alpha''<R$. For all $0\leq r<\infty$, as $N,n\to\infty$ uniformly for $\alpha'<\alpha<\alpha''$ we have*

$$\mathbb{E}\left(\frac{1}{N}\mu_{rnN}\right)=\frac{b_r\theta_\alpha^r}{r!B(\theta_\alpha)}\cdot\sqrt{\frac{N}{N-1}}\cdot\frac{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(n-r-\frac{n}{N}(N-1))^2}{2\sigma^2(\theta_\alpha)(N-1)}\right\}+o(1)}{\frac{1}{\sqrt{2\pi}}+o(1)}.$$

Proof. By Theorem 4 of [6], as $N,n\to\infty$ uniformly for $\alpha'<\alpha<\alpha''$ we have

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{N}\mu_{rnN}\right)=\mathbb{P}\{\eta_1=r\}\\
&\quad=\frac{\mathbb{P}\{\xi_1(\theta_\alpha)=r\}\mathbb{P}\{\xi_2(\theta_\alpha)+\xi_3(\theta_\alpha)+\cdots+\xi_N(\theta_\alpha)=n-r\}}{\mathbb{P}\{\xi_1(\theta_\alpha)+\xi_2(\theta_\alpha)+\cdots+\xi_N(\theta_\alpha)=n\}}\\
&\quad=\mathbb{P}\{\xi_1(\theta_\alpha)=r\}\frac{\sqrt{N}\sigma(\theta_\alpha)\sqrt{N-1}\mathbb{P}\{\xi_2(\theta_\alpha)+\xi_3(\theta_\alpha)+\cdots+\xi_N(\theta_\alpha)=n-r\}}{\sqrt{N-1}\sigma(\theta_\alpha)\sqrt{N}\mathbb{P}\{\xi_1(\theta_\alpha)+\xi_2(\theta_\alpha)+\cdots+\xi_N(\theta_\alpha)=n\}}
\end{aligned}$$

$$= \frac{b_r \theta_\alpha^r}{r! B(\theta_\alpha)} \cdot \sqrt{\frac{N}{N-1}} \cdot \frac{\frac{1}{\sqrt{2\pi}} \exp\left\{ - \frac{(n-r-\frac{n}{N}(N-1))^2}{2\sigma^2(\theta_\alpha)(N-1)} \right\} + o(1)}{\frac{1}{\sqrt{2\pi}} + o(1)}.$$

The proof is finished. □

Recall the Rosenthal inequality ([14], [12], Theorem 2.9) which we will use in the next theorem. Let $\{\zeta_i\}$ be independent centered random variables. Let $p \geq 2$. Then we have

$$\mathbb{E}\left| \sum_{i=1}^{n} \zeta_i \right|^p < c_p \left(\sum_{i=1}^{n} \mathbb{E}|\zeta_i|^p + \left(\sum_{i=1}^{n} \mathbb{D}^2(\zeta_i) \right)^{p/2} \right), \tag{4.7}$$

where c_p depends on p only.

Let

$$S_{nN} = \frac{1}{N} \sum_{r=0}^{\infty} \beta_r \mu_{rnN} \tag{4.8}$$

where μ_{rnN} is defined by (3.2).

Lemma 4.3. *Let $0 < \alpha' < \alpha'' < R$. Let $\beta_r \in \mathbb{R}$ be such that $\sum_{r=0}^{\infty} |\beta_r|^p \frac{b_r(\theta'')^r}{r!} < \infty$ for some $2 < p$ and $m(\theta'') = \alpha''$. Let $\zeta_{\alpha i} = \sum_{r=0}^{\infty} \beta_r \mathbb{I}_{\{\xi_i(\theta_\alpha)=r\}}$, $\zeta_{\alpha i}^c = \zeta_{\alpha i} - \mathbb{E}\zeta_{\alpha i}$. Then there exists $N_0 \in \mathbb{N}$ such that*

$$\mathbb{E}\left| S_{nN} - \mathbb{E}S_{nN} \right|^p \leq 4^{p+1} \sigma(\theta_\alpha) c_p \left(\frac{\mathbb{E}|\zeta_{\alpha i}^c|^p}{N^{p-1-1/2}} + \frac{(\mathbb{E}(\zeta_{\alpha i}^c)^2)^{p/2}}{N^{p/2-1/2}} \right). \tag{4.9}$$

for all $N > N_0$ and $\alpha' < \alpha < \alpha''$.

Proof. First we remark that $\mathbb{E}|\zeta_{\alpha,i}|^p = \sum_{r=0}^{\infty} |\beta_r|^p \frac{b_r \theta^r}{r! B(\theta)} < C < \infty$. By Lemma 3.1, there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$ and $\alpha' < \alpha < \alpha''$ it holds

$$\mathbb{P}\{\xi_1(\theta_\alpha) + \xi_2(\theta_\alpha) + \cdots + \xi_N(\theta_\alpha) = n\} \geq \frac{1}{4\sigma(\theta_\alpha)\sqrt{N}}.$$

Therefore, by Lemma 2.1, Lemma 3.1 and (4.7), we obtain for $N > N_0$

$$\begin{aligned}
\mathbb{E}\left| S_{nN} - \mathbb{E}S_{nN} \right|^p &\leq \frac{4^p}{N^p} \frac{\mathbb{E}\left| \sum_{i=1}^{N} \zeta_{\alpha i}^c \right|^p}{\mathbb{P}\{\xi_1(\theta_\alpha) + \xi_2(\theta_\alpha) + \cdots + \xi_N(\theta_\alpha) = n\}} \\
&\leq 4^{p+1} \sigma(\theta_\alpha) \sqrt{N} c_p \left(\frac{N \mathbb{E}|\zeta_{\alpha i}^c|^p}{N^p} + \left(\frac{N \mathbb{E}(\zeta_{\alpha i}^c)^2}{N^2} \right)^{p/2} \right) \\
&\leq 4^{p+1} \sigma(\theta_\alpha) c_p \left(\frac{\mathbb{E}|\zeta_{\alpha i}^c|^p}{N^{p-1-1/2}} + \frac{(\mathbb{E}(\zeta_{\alpha i}^c)^2)^{p/2}}{N^{p/2-1/2}} \right).
\end{aligned}$$

The proof is finished. □

Theorem 4.4. *Let $N, n \to \infty$ such that $n/N = \alpha \to \alpha_0$ and $m(\theta_{\alpha_0}) = \alpha_0$ for some $0 < \theta_{\alpha_0} < R$. Let $\beta_1, \beta_2, \dots$ be a real sequence such that $\sum_{r=0}^{\infty} |\beta_r|^p \frac{b_r(\theta^{''})^r}{r!} < \infty$ for some $5 < p$ and some $\theta^{''}$ such that $\theta_{\alpha_0} < \theta^{''} < R$. Then for S_{nN} defined in (4.8) we have*

$$S_{nN} \to S, \text{ as } n, N \to \infty \text{ such that } n/N = \alpha \to \alpha_0 \tag{4.10}$$

almost surely, where $S = \sum_{r=0}^{\infty} \beta_r \frac{b_r \theta_{\alpha_0}^r}{r! B(\theta_{\alpha_0})}$.

Proof. Let $0 < \theta' < \theta_{\alpha_0}$. We use the notation $\alpha' = m(\theta')$ and $\alpha'' = m(\theta^{''})$. Since $N, n \to \infty$ so that $n/N = \alpha \to \alpha_0$, we can assume that $\alpha' < \alpha < \alpha''$ for all n, N. By condition $5 < p$, using (4.9), we obtain

$$\sum_{N=1}^{\infty} \sum_{\alpha' N \le n \le \alpha'' N} \mathbb{P}\left\{|S_{nN} - \mathbb{E}S_{nN}| > \varepsilon\right\}$$
$$\le \frac{1}{\varepsilon^p} \sum_{N=1}^{\infty} \sum_{\alpha' N \le n \le \alpha'' N} \mathbb{E}\,|S_{nN} - \mathbb{E}S_{nN}|^p < \infty,$$

for all $\varepsilon > 0$. Therefore, as $N, n \to \infty$ so that $\alpha \to \alpha_0$, we have

$$S_{nN} - \mathbb{E}S_{nN} \to 0 \tag{4.11}$$

almost surely. From Lemma 4.2 it follows that

$$\mathbb{E}S_{nN} \to S, \text{ as } N, n \to \infty \text{ so that } \alpha \to \alpha_0. \tag{4.12}$$

Since

$$S_{nN} = S_{nN} - \mathbb{E}S_{nN} + \mathbb{E}S_{nN},$$

from (4.11) and (4.12) we obtain (4.10). The proof is finished. □

For $\mathbb{M} \subset \mathbb{N}$ let

$$\widetilde{\mu}_{\mathbb{M}nN} = \frac{1}{N} \sum_{s \in \mathbb{M}} \mu_{snN}.$$

Observe that $\widetilde{\mu}_{\mathbb{M}nN}$ is the average of the numbers of boxes containing s balls with $s \in \mathbb{M}$. Using Theorem 4.4 with $\beta_r = \mathbb{I}_{\mathbb{M}}(r)$, we obtain the following corollary.

Corollary 4.1. *Let $N, n \to \infty$ so that $\alpha \to \alpha_0$ and $m(\theta_{\alpha_0}) = \alpha_0$ for some $0 < \theta_{\alpha_0} < R$. Then*

$$\widetilde{\mu}_{\mathbb{M}nN} \to \widetilde{\mu}_{\mathbb{M}}$$

almost surely, where $\widetilde{\mu}_{\mathbb{M}} = \sum_{r \in \mathbb{M}} \frac{b_r \theta_{\alpha_0}^r}{r! B(\theta_{\alpha_0})}$. □

Let

$$\nu_{\mathbb{M}nN} = \frac{1}{N} \sum_{s \in \mathbb{M}} s \mu_{snN}.$$

We see that ν_{MnN} is the average of the numbers of balls belonging to boxes containing s balls with $s \in \mathbb{M}$. Using Theorem 4.4 with $\beta_r = r\mathbb{I}_{\mathbb{M}}(r)$, we obtain the following corollary.

Corollary 4.2. *Let $N, n \to \infty$ so that $\alpha \to \alpha_0$ and $m(\theta_{\alpha_0}) = \alpha_0$ for some $0 < \theta_{\alpha_0} < R$. Then*

$$\nu_{\mathbb{M}nN} \to \nu_{\mathbb{M}}$$

almost surely, where $\nu_{\mathbb{M}} = \sum_{r \in \mathbb{M},\ r \geq 1} \frac{b_r \theta_{\alpha_0}^r}{(r-1)! B(\theta_{\alpha_0})}$. □

Theorem 4.5. *Let $N_k, n_k \to \infty$ as $k \to \infty$ such that $n_k/N_k = \alpha_k \to \alpha$ and $m(\theta) = \alpha$ for some $0 < \theta < R$. Let $\beta_1, \beta_2, \ldots$ be a real sequence such that $\sum_{r=0}^{\infty} |\beta_r|^p \frac{b_r (\theta'')^r}{r!} < \infty$ for some $3 < p$ and some θ'' such that $\theta < \theta'' < R$. Let*

$$S_k = \frac{1}{N_k} \sum_{r=0}^{\infty} \beta_r \mu_{rk}$$

where μ_{rk} is defined by (4.3). Then

$$S_k \to S, \text{ as } k \to \infty \tag{4.13}$$

almost surely, where $S = \sum_{r=0}^{\infty} \beta_r \frac{b_r \theta_\alpha^r}{r! B(\theta_\alpha)}$.

Proof. Let $0 < \theta' < \theta$. We use the notation $\alpha' = m(\theta')$ and $\alpha'' = m(\theta'')$. We can assume that $\alpha' < \alpha_k < \alpha''$ for all $k \in \mathbb{N}$. Condition $3 < p$ and (4.9) imply

$$\sum_{k=1}^{\infty} \mathbb{P}\{|S_k - \mathbb{E}S_k| > \varepsilon\} \leq \frac{1}{\varepsilon^p} \sum_{k=1}^{\infty} \mathbb{E}\,|S_k - \mathbb{E}S_k|^p < \infty$$

for all $\varepsilon > 0$. Therefore, as $k \to \infty$, it holds that

$$S_k - \mathbb{E}S_k \to 0 \tag{4.14}$$

almost surely. From Lemma 4.2 it follows that

$$\mathbb{E}S_k \to S, \text{ as } k \to \infty \text{ so that } \alpha_k \to \alpha. \tag{4.15}$$

Since $S_k = S_k - \mathbb{E}S_k + \mathbb{E}S_k$, from (4.14) and (4.15) we obtain (4.13). The proof is finished. □

Remark 4.2. In several papers generalizations and improvements of the Rosenthal inequality were obtained. Consider the following inequality from [15]. Let $p \geq 2$ and let $\zeta_1, \dots, \zeta_n$ be independent centered random variables. Then

$$\mathbb{E}\left|\sum_{i=1}^{n} \zeta_i\right|^p < K^p \left(\frac{p}{\ln p}\right)^p \left(\mathbb{E} \max_{1 \leq i \leq n} |\zeta_i|^p + \left(\sum_{i=1}^{n} \mathbb{D}^2(\zeta_i)\right)^{p/2}\right) \tag{4.16}$$

where K does not depend on p and $\zeta_1, \dots, \zeta_n$. In the proof of Lemma 4.1, using (4.16) instead of Khintchine's inequality and choosing $p = \frac{\varepsilon}{4eK\sigma_s}$, we obtain

$$\mathbb{P}\left\{\frac{|\mu_s - \mathbb{E}\mu_s|}{\sqrt{N}} > \varepsilon\right\} \leq 4\sigma(\theta)\sqrt{N}\left(e \ln\left(\frac{\varepsilon}{4eK\sigma_s}\right)\right)^{-\frac{\varepsilon}{4eK\sigma_s}} \left[1 + (\sigma_s\sqrt{N})^{-\frac{\varepsilon}{4eK\sigma_s}}\right]$$

where $\varepsilon \geq 8eK\sigma_s$.

5. Applications

Example 5.1. Let ξ_i have Poisson distribution, i.e. $\mathbb{P}(\xi_i = k) = \frac{\lambda^k}{k!}e^{-\lambda}$, $k = 0, 1, \dots$. Then

$$\mathbb{P}\{\xi_1 = k_1, \dots, \xi_N = k_N \mid \xi_1 + \dots + \xi_N = n\} = \frac{n!}{k_1! \dots k_N!}\left(\frac{1}{N}\right)^n$$

if $k_1 + \dots + k_N = n$. That is $\{\eta_1 = k_1, \dots, \eta_N = k_N\}$ has polynomial distribution. Now $\{\eta_1 = k_1, \dots, \eta_N = k_N\}$ means that the cell contents are $k_1, \dots, k_N$ after allocating n particles into N cells considering the usual allocation procedure.

We see that the parameter θ used in the generalized allocation scheme is the same as λ in the above usual allocation procedure. As $m(\lambda) = \mathbb{E}\xi_1 = \lambda$, the parameter $n/N = \alpha$ coincides with λ.

We see that μ_{rnN} is the number of cells with r particles (after allocating n particles into N cells). Now, by Theorem 4.4, if $n, N \to \infty$ so that $n/N \to \lambda_0$, then

$$\frac{\mu_{rnN}}{N} \to \frac{\lambda_0^r}{r!}e^{-\lambda_0}$$

almost surely. (For a direct proof of the above result, see [2].) □

Example 5.2. Let $\mathcal{T}_{n,N}$ denote the set of forests containing N labelled roots and n labelled non-root vertices. By Cayley's theorem, $\mathcal{T}_{n,N}$ has $N(n+N)^{n-1}$

elements. Consider uniform distribution on $\mathcal{T}_{n,N}$. Let η_i denote the number of the non-root vertices of the ith tree. Then

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \frac{n!}{k_1! \dots k_N!} \frac{(k_1+1)^{k_1-1} \dots (k_N+1)^{k_N-1}}{N(N+n)^{n-1}}.$$

Now let ξ_i have Borel distribution (see [4], [8]) $\mathbb{P}(\xi_i = k) = \frac{\lambda^k (1+k)^{k-1}}{k!} e^{-(k+1)\lambda}$, $k = 0, 1, \dots$, $0 < \lambda < 1$. Then

$$\begin{aligned} \mathbb{P}\{\xi_1 = k_1, \dots, \xi_N = k_N \mid \xi_1 + \dots + \xi_N = n\} \\ = \frac{n!}{k_1! \dots k_N!} \frac{(k_1+1)^{k_1-1} \dots (k_N+1)^{k_N-1}}{N(N+n)^{n-1}} \end{aligned}$$

if $k_1 + \dots + k_N = n$. See [5], [3], [11]. Therefore $\eta_1, \dots, \eta_N$ satisfy (3.1).

Now we apply our general results for forests. First we remark that

$$\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (ae^{-a})^k = a,$$

see [13]. Using this fact we can see that $\sum_{k=1}^{\infty} \frac{\lambda^k (k+1)^{k-1}}{k!} e^{-(k+1)\lambda} = 1$, moreover

$$m(\lambda) = \mathbb{E}\xi_1 = \sum_{k=1}^{\infty} k \frac{\lambda^k (k+1)^{k-1}}{k!} e^{-(k+1)\lambda} = \frac{\lambda}{1-\lambda}.$$

So $m^{-1}(\alpha) = \frac{\alpha}{1+\alpha}$. Therefore the relation between $\alpha = n/N$ and $\lambda_\alpha = m^{-1}(\alpha)$ is $\lambda_\alpha = \frac{\alpha}{1+\alpha}$. We see that μ_{rnN} is the number of trees with r non-root vertices. Now, by Theorem 4.4, if $n, N \to \infty$ so that $n/N \to \alpha_0$, then

$$\frac{\mu_{rnN}}{N} \to \frac{\lambda_0^r (r+1)^{r-1}}{r!} e^{-(r+1)\lambda_0}$$

almost surely, where $\lambda_0 = \frac{\alpha_0}{1+\alpha_0}$. (For a direct proof of the above result, see [3].) We remark that for the generalized allocation scheme we used parameter θ while for random forests we used parameter λ. The relation of the parameters is $\theta = \lambda e^\lambda$, $\lambda \in (0, 1)$. $\square$

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