Publ. Math. Debrecen
44 / 3-4 (1994), 205–213

A common fixed point theorem of Gregus type for compatible mappings and its applications

By H. K. PATHAK (Bhilai) and R. GEORGE (Bhilai)

Abstract. Let T and I be two compatible self maps of a closed, convex bounded subset C of a Normed space X such that $I(C) \supseteq (1-k)\cdot I(C)+k\cdot T(C)$ where $0<k<1$ is fixed and $\|Tx-Ty\|^p \le a\cdot\|Ix-Iy\|^p+(1-a)\cdot\max[\|Tx-Ix\|^p,\|Ty-Iy\|^p]$ for all $x, y \in C$, where $0<a<1$ and $p>0$. If for some $x_0 \in C$, the sequence $\langle x_n\rangle$ defined by $Ix_{n+1}=(1-k)\cdot Ix_n+k\cdot Tx_n$, for all $n\ge 0$, converges to a point z in C and if I is continuous at z then T and I have a unique common fixed point. Further if I is continuous at Tz then T and I have a unique common fixed point at which T is continuous. We have also applied this result to obtain iterative solution of certain variational inequalities.

Let T and I be two mappings of a metric space (X,d) into itself. Sessa [11] defined T and I to be weakly commuting if $d(TIx,ITx)\le d(Tx,Ix)$ for any $x\in X$. Clearly two commuting mappings weakly commute, but two weakly commuting mappings in general do not commute. Refer to Example 1 in Diviccaro et al. [5]. Gerald Jungck [10] defined T and I to be compatible mappings, if $d(Tx,Ix)\to 0$ implies $d(TIx,ITx)\to 0$. Clearly, two weakly commuting mappings are compatible, but two compatible mappings are in general not weakly commuting. For example refer to Jungck [10].

Recently Diviccaro et al. [5], established the following result.

Theorem A. *Let T and I two weakly commuting mappings of a closed, convex subset C of a Banach space X into itself satisfying the inequality*

$$\|Tx-Ty\|^p \le a\cdot\|Ix-Iy\|^p+(1-a)\cdot\max[\|Tx-Ix\|^p,\|Ty-Iy\|^p]$$

for all x, y in C, where $0<a<1/2^{p-1}$ and $p\ge 1$. If I is linear, nonexpansive in C and such that $I(C)$ contains $T(C)$, then T and I have a unique common fixed point at which T is continuous.

1980 *Mathematics Subject Classification* (1985 *Revision*): AMS (MOS) 1985 (Revision) Subject Classification: Primary 54H 25, Secondary 47H 10 Research supported by U.G.C New Delhi.

The object of the present paper is to replace linearity and nonexpansiveness, of the map I, and proof of Theorem A is made under considerably weaker conditions of mappings, i.e. replacing weakly commuting pair of maps (T, I) with compatible maps, and using the iteration method of Mann type. Also the range of p has been extended to the case when $0 < p < 1$. The technique used in the proof of our theorem is different from that of Diviccaro et al [5]. Further we have used our main theorem to obtain iterative solution of certain variational inequalities.

Main results

Theorem. *Let T and I be two compatible self maps of a closed convex bounded subset C of a Normed space X satisfying the following*

$$\|Tx-Ty\|^p \le a \cdot \|Ix-Iy\|^p + (1-a) \cdot \max\left[\|Tx-Ix\|^p, \|Ty-Iy\|^p\right] \tag{1}$$

$$I(C) \supseteq (1-k) \cdot I(C) + k \cdot T(C) \tag{2}$$

$\forall x, y \in C$ where $0 < a < 1$, $p > 0$ and for some fixed k such that $0 < k < 1$. If for some $x_0 \in C$, the sequence $\langle x_n \rangle$ defined by

$$Ix_{n+1} = (1-k) \cdot Ix_n + k \cdot Tx_n, \quad \forall n \ge 0 \tag{3}$$

converges to a point z of C and if I is continuous at z then T and I have a unique common fixed point. Further if I is continuous at Tz then T and I have a unique commom fixed point at which T is continuous.

Proof. First we are going to prove that $Tz = Iz$. We have,

$$\begin{aligned} \|Iz - Tz\|^p &= \|Iz - Ix_{n+1} + Ix_{n+1} - Tz\|^p \le \\ &\le \left[\|Iz - Ix_{n+1}\| + \|Ix_{n+1} - Tz\|\right]^p . \end{aligned} \tag{4}$$

Now, from (3) we have

$$\begin{aligned} \|Ix_{n+1} - Tz\|^p &= \|(1-k)Ix_n + kTx_n - Tz\|^p = \\ &= \|(1-k)(Ix_n - Tz) + k(Tx_n - Tz)\|^p \le \\ &\le \left[(1-k)\|Ix_n - Tz\| + k\|Tx_n - Tz\|\right]^p = \\ &= \left[(1-k)\|Ix_n - Tz\| + k \cdot (\|Tx_n - Tz\|^p)^{1/p}\right]^p \end{aligned} \tag{5}$$

From (1) we have

$$\|Tx_n - Tz\|^p \le a \cdot \|Ix_n - Iz\|^p + (1-a) \cdot \max\left[\|Tx_n - Ix_n\|^p, \|Tz - Iz\|^p\right].$$

Now since I is continuous at z, from Heine's definition of continuity we see that $Ix_n \to Iz$ as $n \to \infty$. Also from (3) we have $\|Tx_n - Ix_n\| \to 0$

as $n \to \infty$. Therefore,

$$\|Tx_n - Tz\|^p \le (1-a)\cdot\|Tz - Iz\|^p + \varepsilon, \tag{6}$$

if n is large enough. Hence from (4) (5) and (6) we have

$$\|Iz - Tz\|^p \le \|Iz - Tz\|^p \cdot \left[(1-k) + k\cdot(1-a)^{1/p}\right]^p \tag{7}$$

which is a cotradiction. Therefore $Iz = Tz$. Now since T and I are compatible $TIz = ITz$. Hence, by using (1)

$$\begin{gathered}\|T^2z - Tz\|^p \le \\ \le a\cdot\|ITz - Iz\|^p + (1-a)\cdot\max\left[\|T^2z - ITz\|^p, \|Tz - Iz\|^p\right] = \\ = a\|T^2z - Tz\|^p,\end{gathered}$$

whence $T^2z = Tz$, ie. Tz is a fixed point of T. Now $ITz = TIz = TTz = T^2z = Tz$, i.e. Tz is also a fixed point of I.

Now, let $\langle y_n\rangle$ be a sequence of points of C, with limit $Tz = z_1$. Thus, using condition (1), we have

$$\|Ty_n - Tz_1\|^p \le a\cdot\|Iy_n - Iz_1\|^p + (1-a)\cdot\max\left[\|Ty_n - Iy_n\|^p, \|Tz_1 - Iz_1\|^p\right].$$

Since I is continuous at $Tz = z_1$, we have,

$$\|Ty_n - Tz_1\|^p \le (1-a)\cdot\|Ty_n - Iz_1\|^p + \varepsilon,$$

if n is large enough. Again, since $ITz = TIz = TTz = Tz_1$, we have

$$\|Ty_n - Tz_1\|^p \le (1-a)\cdot\|Ty_n - Tz_1\|^p + \varepsilon,$$

if n is large enough, i.e. $\lim\limits_{n\to\infty}\|Ty_n - Tz_1\| = 0$ and this means that T is continuous at Tz. Proof of uniqueness follows from that of Diviccaro et al. [5].

Example 1. Let X be the reals and $C = [0,2]$. Let T and I be self maps of C such that

$$Tx = \begin{cases} x^8/128 & 0 \le x \le 1/2^{1/4} \\ x^4/128 & 1/2^{1/4} < x \le 2 \end{cases} \quad \text{and} \quad Ix = x^4/8.$$

Clearly I is not linear and $\|I1 - I2\| = 15/8 > \|1 - 2\|$. Therefore I is not nonexpansive. For $1/2^{1/4} < x \le 2$, $\|Tx - Ix\| \to 0$ iff $x \to 0$ and $\|TIx - ITx\| \to 0$ iff $x \to 0$. For $0 \le x \le 1/2^{1/4}$ we see that $TIx = ITx$. Therefore T and I are compatible maps.

Now, for $0 \leq x \leq 1/2^{1/4}$

$$\|Tx - Ty\|^p = \|x^8/128 - y^8/128\|^p = (1/128^p)\|x^8 - y^8\|^p =$$
$$= (1/128^p)\|(x^4 - y^4)(x^4 + y^4)\|^p \leq (1/128^p) \cdot \|x^4 - y^4\|^p =$$
$$= (1/16^p) \cdot \|Ix - Iy\|^p = a \cdot \|Ix - Iy\|^p$$

where $a = 1/16^p \in (0,1)$

For $1/2^{1/4} < x \leq 2$

$$\|Tx - Ty\|^p = (1/128^p) \cdot \|x^4 - y^4\|^p = (1/16^p) \cdot \|Ix - Iy\|^p =$$
$$= a \cdot \|Ix - Iy\|^p \quad \text{where } a = 1/16^p \in (0,1).$$

Hence we see that for all x in C condition (1) is satisfied. Setting $k = 1/3$, and for any $x_0 \in C$, we see that the sequence $\langle x_n \rangle$ of elemants of C, such that $Ix_{n+1} = (1-k)Ix_n + kTx_n$ for $n \geq 1$, converges to the point 0. Clearly $T0 = 0$ is a fixed point of T and I.

Remark 1. If $p = 1$ we obtain a result or Fisher and Sessa [8] with appreciably weaker conditions of the space X.

Assuming I to be the identity map of X we have the following

Corollary 1. *Let T be a self map of a closed convex bounded subset C of a Normed space X satisfying*

$$\text{(8)} \quad \|Tx - Ty\|^p \leq a \cdot \|x - y\|^p + (1-a) \cdot \max\left[\|Tx - x\|^p, \|Ty - y\|^p\right]$$

and $C \supseteq (1-k) \cdot C + k \cdot T(C)$ for all x, y in C, where $0 < a < 1$ and $p > 0$ and fixed k such that $0 < k < 1$. If for some $x_0 \in C$, the sequence $\langle x_n \rangle$ defined by $x_{n+1} = (1-k)x_n + kT_n$, $\forall n \geq 0$ converges to a point z of C then T has a unique fixed point at which T is continuous.

Remark 2. Delbosco et al. [4], generalizing the result of Gregus [9], considered the inequality

$$\text{(9)} \qquad \|Tx - Ty\|^p \leq a \cdot \|x - y\|^p + b \cdot \|Tx - x\|^p + c \cdot \|Ty - y\|^p$$

for all x, y in C, where $0 < a < 1/2^{p-1}$, $p \geq 1$, $b \geq 0$, $c \geq 0$ and $a+b+c = 1$. Due to symmetry, one may suppose $b = c$ and clearly (8) is more general than (9) and involves wider range of p than that of Diviccaro et al. [5].

Remark 3. For $p = 1$, the result of Corollary 1 was established by Fisher [7].

The condition that T and I are compatible maps is necessary in our theorem as shown in the following

Example 2. Let X be the reals and $C = [0,2]$. Let T and I be two self maps of C such that, $Tx = (x+1)/4$, and $Ix = x/2$ for all x in C. We have

$$\|Tx - Ty\|^p = (1/4^p) \cdot \|x - y\|^p = 1/2^p \cdot \|Ix - Iy\|^p = a \cdot \|Ix - Iy\|^p$$

where $a = 1/2^p$ and for all x, y in C. Hence condition (1) of our theorem is satisfied. We see that T and I are not compatible maps since $\|Tx - Ix\| \to 0$ when $x \to 1$ but $\|TIx - ITx\|$ does not tend to zero when $x \to 1$.

On the other hand, T and I do not have common fixed points.

Remark 4. It is not known whether the condition $I(C)$ contains $T(C)$ of Diviccaro et al. [5] is necessary in our Theorem.

Remark 5. The proof of inequality (7) can also be obtained by using the technique of binomial expansion as follows. If p is a positive integer,

$$\|Iz - Tz\|^p \le \|Iz - Ix_{n+1}\|^p + {}^pC_1 \|Iz - Ix_{n+1}\|^{p-1} \|Ix_{n+1} - Tz\| + \\ + \cdots + \|Ix_{n+1} - Tz\|^p.$$

Also

$$\|Ix_{n+1} - Tz\|^p \le (1-k)^p \|Ix_n - Tz\|^p + \\ + {}^pC_1 \cdot (1-k)^{p-1} \|Ix_n - Tz\|^{p-1} k \cdot (\|Tx_n - Tz\|^p)^{1/p} + \\ + {}^pC_2 \cdot (1-k)^{p-2} \|Ix_n - Tz\|^{p-2} k^2 \cdot (\|Tx_n - Tz\|^p)^{2/p} + \cdots + \\ + k^p \|Tx_n - Tz\|^p.$$

Hence using (1) and the fact that $\|Tx_n - Ix_n\| \to 0$ as $n \to \infty$, we get (7). If p is a positive fraction then

$$\text{(a)} \qquad \|Iz - Tz\|^p \le \|Ix_{n+1} - Tz\|^p \left[1 + \frac{\|Iz - Ix_{n+1}\|}{\|Ix_{n+1} - Tz\|}\right]^p.$$

But using (3) we get

$$\|I_{x+1} - Tz\|^p = \|(1-k) \cdot (Ix_n - Tz) + k \cdot (Tx_n - Tz)\|^p \le$$

$$\text{(b)} \qquad \le (1-k)^p \cdot \|Ix_n - Tz\|^p \cdot \left[1 + \frac{k \cdot (\|Tx_n - Tz\|^p)^{1/p}}{(1-k) \cdot \|Ix_n - Tz\|}\right]^p.$$

As in the proof of our main theorem we see that

$$\text{(c)} \qquad \|Tx_n - Tz\|^p \le (1-a) \cdot \|Tz - Iz\|^p + \varepsilon,$$

if n is large enough.

Also using the continuity of I, we see that expression in the parenthesis of (a) tends to 1 as $n \to \infty$.

Hence using (a), (b) and (c) we get (7).

We conclude exhibiting the following

Corollary 2. *Let T and I be two compatible self maps of a closed, convex bounded subset C of a Normed space X satisfying $I(C) \supseteq (1-k)\cdot I(C) + k\cdot T(C)$ and*

(9A) $\|Tx-Ty\| \leq a\cdot\|Ix-Iy\| + 1/2\cdot(1-a)\cdot\max\left[\|Tx-Iy\|, \|Ty-Ix\|\right]$

for all x, y in C, where $0 < a < 1$ and fixed k such that $0 < k < 1$. For an arbitrary $x_0 \in C$, consider the sequence $\langle x_n \rangle$ such that $Ix_{n+1} = (1-k)Ix_n + kTx_n$, $\forall n \geq 0$. If $\langle x_n \rangle$ converges to a point z of C and if I is continuous at z then T and I have a unique common fixed point. Further if I is continuous at Tz then T and I have a unique common fixed point at which T is continuous.

Proof. The proof follows from Corollary 2 of Diviccaro et al. [5] and our main theorem for $p = 1$.

Applications

Drawing inspiration from a recent work of Belbas et al. [1], we apply our theorem to prove the existence of solutions of variational inequalities.

Variational inequalities arise in optimal stochastic control [2], as well as in other problems in mathematical physics, e.g. deformation of elastic bodies stretched over solid obstacles, elasto-plastic torsion, etc. [6]. The iterative methods for solution of discrete V.I's are very suitable for implementation on parallel computers with single instruction, multiple-data architecture, particularly on massively parallel processors.

The variational inequality problem is to find a function u such that

(10) $$\max\{Lu - f, u - \phi\} = 0 \quad \text{on } \Omega : u = 0 \quad \text{on} \quad \partial\Omega$$

where Ω is a bounded, open subset of R^n, L is an elliptic operator defined on Ω by $L = -a_{ij}(x)\partial^2/\partial x_i \partial x_j + b_i(x)\partial/\partial x_i + c(x)\cdot I$ where summation with respect to repeated indices is implied; $c(x) \geq 0$, $[a_{ij}(x)]$ is a strictly positive definition matrix, uniformly in x, for $x \in \Omega$; f and ϕ are smooth functions defined in Ω and ϕ satisfies the condition $\phi(x) \geq 0$ for $x \in \Omega$.

A problem related to (10) is the two-obstacle variational inequality. Given two functions ϕ and μ defined on Ω, and satisfying $\phi \leq \mu$ in Ω, $\phi \leq 0 \leq \mu$ on Ω, the corresponding variational inequality is

(11) $$\max\{\min\{Lu - f, u - \phi\}, u - \mu\} = 0 \text{ in } \Omega : u = 0 \text{ on } \partial\Omega.$$

The problem (11) arises in stochastic game theory. In this situation, two players are trying to control a diffusion process by stopping the process; the first player is trying to maximize a cost functional, and the second player is trying to minimize a similar functional. Here, f represents the continuous rate of cost for both players, ϕ is the stopping cost for the maximizing player, and μ is the stopping cost for the minimizing player.

Let A be an $N \times N$ matrix corresponding to the finite difference discretizations of the operator L.

We shall make the following assumptions about the matrix A:

$$A_{ij} = 1, \quad \sum_{j:j\neq i} A_{ij} > -1, \quad A_{ij} < 0 \quad \text{for } i \neq j. \tag{12}$$

These assumption is related to the definition of "M-matrices"; matrices arising from the finite difference discretization of continuous elliptic operators will have property (12) under appropriate conditions; see [3, 12].

Let $B = I - A$. Then the corresponding property for the B matrix will be

$$B_{ij} = 0 \sum_{j:j\neq i} B_{ij} < 1, \quad B_{ij} > 0 \text{ for } i \neq j. \tag{13}$$

Let $q = \max\limits_{i} \sum\limits_{j} B_{ij}$, and A^* be an $N \times N$ matrix such that $A^*{}_{ij} = 1 - q$ and $A^*{}_{ij} = -q$, for $i \neq j$. $B^* = I - A^*$.

Iterative solution of variational inequalities

Consider the following discrete variational inequality:

$$\max\Big[\min\big[A(x - A^* \cdot \|x - Tx\|) - f,\ x - A^* \cdot \|x - Tx\| - \phi\big], \\ x - A^* \cdot \|x - Tx\| - \mu\Big] = 0 \tag{14}$$

where, T is implicitly defined by

$$Tx = \min\Big[\max\big[Bx + A \cdot (1 - B^*) \cdot \|x - Tx\| + f, \\ (1 - B^*) \cdot \|x - Tx\| + \phi\big], (1 - B^*) \cdot \|x - Tx\| + \mu\Big]. \tag{15}$$

Then (14) is equivalent to the fixed point problem

$$x = Tx. \tag{16}$$

Theorem. *Under assumptions* (12), (13) *a solution exists for* (16).

Proof. For any x, y and i

If $(Ty)_i = [(1 - B^*{}_{ij})\|y_j - Ty_j\| + \mu_i]$, then, since

$$(Tx)_i \leq [(1 - B^*{}_{ij}\|x_j - Tx_j\| + \mu_i]\,, \text{ we have}$$
$$(Tx)_i - (Ty)_i \leq (1 - B^*{}_{ij}) \cdot [\|x_i - Tx_j\| - \|y_j - Ty_j\|]\,, \text{ or}$$

$$(17) \qquad (Tx)_i = (Ty)_i \le (1 - {B^*}_{ij}) \cdot \max\left[\|x_j - Tx_j\|, \|y_j - Ty_j\|\right].$$

If $(Ty)_i = \max\left[B_{ij}y_j + A_{ij}(1-{B^*}_{ij})\|y_j - Ty_j\| + f_i, (1-{B^*}_{ij})\|y_j - Ty_j\| + \right.$ $\left. +\phi_i\right]$ then we introduce the one sided operator

$$\bar{T}x = \max\left[Bx + A(1 - B^*)\|x - Tx\| + f, (1 - B^*)\|x - Tx\| + \phi\right].$$

Then $(Ty)_i = (\bar{T}y)_i$. Now since $(Tx)_i \le (\bar{T}x)_i$, we have

$$(18) \qquad (Tx)_i - (Ty)_i \le (\bar{T}x)_i - (\bar{T}y)_i.$$

Now, if $(\bar{T}x)_i = B_{ij}x_j + A_{ij}(1 - {B^*}_{ij})\|x_j - Tx_j\| + f_i$, then since $(\bar{T}y)_i \ge B_{ij}y_j + A_{ij}(1 - {B^*}_{ij})\|y_j - Ty_j\| + f_i$, then by using (12), we find that

$$(19) \quad (\bar{T}x)_i - (\bar{T}y)_i \le$$
$$\le B_{ij}\|x_j - y_j\| + (1 - {B^*}_{ij}) \cdot \max\left[\|x_i - Tx_i\| \cdot \|y_i - Ty_j\|\right].$$

If $(\bar{T}y)_i = (1 - {B^*}_{ij})\|x_i - Tx_j\| + \phi_i$, then since $(\bar{T}y)_i \ge (1 - {B^*}_{ij})\|y_i - Ty_i\| + \phi_i$, we find that

$$(20) \qquad (\bar{T}x)_i - (\bar{T}y)_i \le (1 - {B^*}_{ij}) \cdot \max\left[\|x_j - Tx_j\|,\ \|y_j - Ty_j\|\right].$$

Hence, from (17), (18), (19) and (20) we have,

$$(21) \quad (\bar{T}x)_i - (Ty)_i \le q \cdot \|x - y\| + (1 - q) \cdot \max\left[\|x - Tx\|, \|y - Ty\|\right].$$

Since x and y are arbitrarily choosen, we have by interchanging x and y

$$(22) \quad (Ty)_i - (Tx)_i \le q \cdot \|x - y\| + (1 - q) \cdot \max\left[\|x - Tx\|, \|y - Ty\|\right].$$

Therefore, from (21) and (22) it follows that

$$\|Tx - Ty\| \le q \cdot \|x - y\| + (1 - q) \cdot \max\left[\|x - Tx\|,\ \|y - Ty\|\right].$$

Hence we see that condition (8) is satisfied for $p = 1$.

Therefore, Corollary 1 ensures the existance of a solution of (16).

Acknowledgement. The authors are highly indebted to the learned refferee for the valuable suggestions provided towards the improvement of this paper.

References

[1] S. A. Belbas and I. D. Mayergoyz, Applications of fixed point methods, *Number. Math.* **51** (1987), 631–654.

[2] A. Bensoussan and J. L. Lions, Applications des Inequations variationnels en controle strochastique, *Dunod*, *Paris*, 1978.

[3] A. Berman and R. J. Plemmons, Non negative matrices in mathematical sciences, *Academic Press*, *New York*, 1979.

[4] D. Delbesco, O. Ferrero and F. Rossati, Teoremi di punto fisso per applicazioni negli spazi di Banach, *Boll. Un. Mat. Ital* **(6) 2-A** (1983), 297–303.

[5] M. L. Diviccaro, B. Fisher and S. Sessa, A common fixed point theorem of Gregus type, *Publ. Math.* **34** (1987), 83–89.
[6] G. Duvaut and J. L. Lions, Inequalities in mechanics and physics, *Springer, Berlin, Heidelberg, New York*, 1976.
[7] B. Fisher, Common fixed points on a Banach space, *Chung Yuan J.* **11** (1982), 12–15.
[8] B. Fisher and S. Sessa, On a fixed point theorem of Gregus, *Internat. J. Math. and Math. Sci.* **2** (1986), 23–28.
[9] M. Gregus Jr., A fixed point theorem in Banach space, *Boll. Un. Mat. Ital.* **(5) 17-A** (1980), 193–198.
[10] Gerald Jungck, Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.* **103 (3)** (1988), 977-983.
[11] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.* **36(46)** (1982), 149–153.
[12] R. S. Varga, Matrix iterative analysis, *Englewood Cliff, NJ, Prentice Hall*, 1962.

H. K. PATHAK
DEPARTMENT OF MATHEMATICS
KALYAN MAHAVIDYALAYA
BHILAI NAGAR, DURG, (M.P)
INDIA

RENY GEORGE
DEPARTMENT OF MATHEMATICS
ST. THOMAS COLLEGE, RUABANDHA
BHILAI NAGAR, DURG, (M.P)
INDIA

(Received June 17, 1991; revised April 6, 1993)